A modular equality for m-ovoids of elliptic quadrics

Alexander Gavrilyuk

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(joint work with Klaus Metsch and Francesco Pavese)

September 2, 2022

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generators: subspaces of $\mathcal{P}$ of max dimension $(r-1)$

Polar spaces and (m-)ovoids

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and up to taking the complement:

- $\mathcal{P} \setminus \mathcal{O}$ is $\big(|\mathrm{PG}(r-1, q)| - m\big)$-ovoid of $\mathcal{P}$.

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Let $\mathcal{O}$ be an m-ovoid of $\mathcal{P}$ and
$\mathcal{P} \in \{\mathcal{H}(2r, q^2), \mathcal{Q}^-(2r+1, q), \mathcal{W}(2r-1, q)\}$

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Theorem (B. Segre, 1965)

If $\mathcal{O}$ is a proper m-ovoid of $\mathcal{Q}^-(5, q)$, then q is odd and $m = (q+1)/2$.

m-ovoids of $\mathcal{Q}^-(2r+1, q)$, $r = 2$

Quite a few hemisystems were constructed in the past few decades:

- A. Cossidente, T. Penttila, Hemisystems on the Hermitian surface, *LMS*, 2005.
- J. Bamberg, M. Giudici, G.F. Royle, Every flock generalized quadrangle has a hemisystem, *BLMS*, 2010.
- J. Bamberg, M. Giudici, G.F. Royle, Hemisystems of small flock generalized quadrangles, *DCC*, 2013.
- A. Cossidente, F. Pavese, Intriguing sets of quadrics in $\mathrm{PG}(5, q)$, *Adv. Geom.*, 2017.
- J. Bamberg, M. Lee, K. Momihara, Q. Xiang, A new infinite family of hemisystems of the Hermitian surface, *Combinatorica*, 2018.
- G. Korchmáros, G.P. Nagy, P. Speziali, Hemisystems of the Hermitian surface, *JCTA*, 2019.

m-ovoids of $\mathcal{Q}^-(2r+1, q)$, $r > 2$

Consider $V = \mathbb{F}_{q^e}^n$ as the vector space $\mathbb{F}_q^{en}$:

$$\text{a point of } \mathrm{PG}(n-1, q^e) \mapsto \text{a set of points of } \mathrm{PG}(en-1, q)$$
$$\text{compose a form } f \text{ on } V \text{ with the trace map } \mathbb{F}_{q^e} \to \mathbb{F}_q$$

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$\bar{\mathcal{P}} \subset \mathrm{PG}(n-1,q^e)$	$\to$	$\mathcal{P} \subset \mathrm{PG}(en-1,q)$
m-ovoid of $\mathcal{Q}^-(2r+1,q^e)$	$\to$	$\left(m\frac{q^e-1}{q-1}\right)$-ovoid of $\mathcal{Q}^-(2e(r+1)-1,q)$
m-ovoid of $\mathcal{H}(2r,q^{2e})$	$\to$	$\left(m\frac{q^{2e}-1}{q-1}\right)$-ovoid of $\mathcal{Q}^-(2e(2r+1)-1,q)$

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- hemisystem of $\mathcal{Q}^-(5,q^e)$, $e\geq 2$,
- all points of $\mathcal{H}(2r,q^{2e})$, $r\geq 1$, $e\geq 2$.

Main result

Theorem (A.G., K. Metsch, F. Pavese)

Let $\mathcal{O}$ be an m-ovoid of $\mathcal{Q}^-(2r+1,q)$, $r \geq 2$. Then

$$m^2 - m \equiv 0 \pmod{q+1}$$

if r is odd, and

$$\begin{aligned} m^2 &\equiv 0 \pmod{q+1} \quad \text{when } q \text{ is even, or} \\ m^2 + \frac{q+1}{2}m &\equiv 0 \pmod{q+1} \quad \text{when } q \text{ is odd,} \end{aligned}$$

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The characteristic function of an m-ovoid

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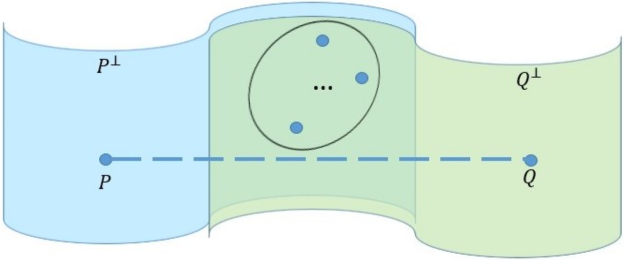

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Assuming that $P_0 \notin \mathcal{O}$:

$$\begin{aligned} m^2(q-1)^2 + m\left(q^r + q\right) &= \|\mu^{\downarrow}_{P_0}\|^2 \\ &\equiv \begin{cases} -2qm^2 + (q+1)(q^{r-1}+1)m & \text{if } r \text{ is odd} \\ (q^2+1)m^2 & \text{if } r \text{ is even} \end{cases} \end{aligned}$$

Computing modulo $q+1$ and simplifying gives the result.

Main result

Theorem (A.G., K. Metsch, F. Pavese)

Let $\mathcal{O}$ be an m-ovoid of $\mathcal{Q}^-(2r+1,q)$, $r \geq 2$. Then

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THANK YOU