Steiner triple systems with a given automorphism group

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(joint work with Bill Kantor)

- Steiner triple system (STS) of order v
  $= S(2,3, v) = 2\text{-}(v,3,1)$ design.
  Existence ⇔ $v = 1$ or $3 \pmod 6$ ⇔ v admissible.

- Babai 1980: Almost all STSs are rigid
  (no automorphism, except the identity).

- Mendelsohn 1978: Any finite group is the automorphism group of some STS of order $v = 2^n - 1$.

Problem: Given a finite abstract group G, for which integers v is there an STS of order v whose full automorphism group is isomorphic to G ?

Theorem 1 (J.D. and B.K. 2022).
Given a finite group G, there is an integer N_G such that, for every admissible $v \geq N_G$, there is an STS V of order v for which Aut $V \cong G$.

If $|G|=1$, $N_G = 15$ (Lindner and Rosa 1975)

If $|G|=2$, is $N_G = 15$?

$v=15$	80 (White, Cole and Cummings 1919)
$\|G\|=1$:	36
$\|G\|=2$:	6
$v=19$	11,084,874,829 (Kaski & Östergärd 2004)
$\|G\|=1$:	11,084,710,071
$\|G\|=2$:	149,522

Our proof produces the bound $N_G = 2^{\mathcal{O}(|G|)}$

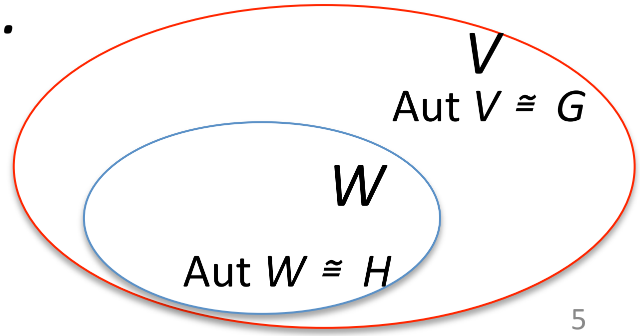
Theorem 2 (J.D. and B.K. 2022).

If W is an STS, there is an integer N_W such that ,
for every admissible $v \geq N_W$, there is an STS V of order v
having W as an Aut $V-$invariant subsystem such that
Aut $V \cong$ Aut W and Aut V induces Aut W on W.

Theorem 2 (J.D. and B.K. 2022).
If *W* is an STS, there is an integer N_W such that, for every admissible $v \geq N_W$, there is an STS *V* of order *v* having *W* as an Aut *V* − invariant subsystem such that Aut $V \cong$ Aut *W* and Aut *V* induces Aut *W* on *W*.

Theorem or Fake News ?

Given any two finite groups *G* and *H*, there is an integer $N_{G,H}$ such that, for every admissible $v \geq N_{G,H}$, there is an STS *V* of order *v* having a subsystem *W* such that Aut $V \cong G$ and Aut $W \cong H$.




FAKE NEWS!

What about other Steiner systems S(t,k,v) ?

Theorem (Kantor 2019).

Given G, there are infinitely many integers v such that there is an $S(3,4,v)$ V for which Aut $V \cong G$.

Same conclusion for the systems $S(2,k,v)$ where $k = q$ or $q+1$ (q a prime power >2) .

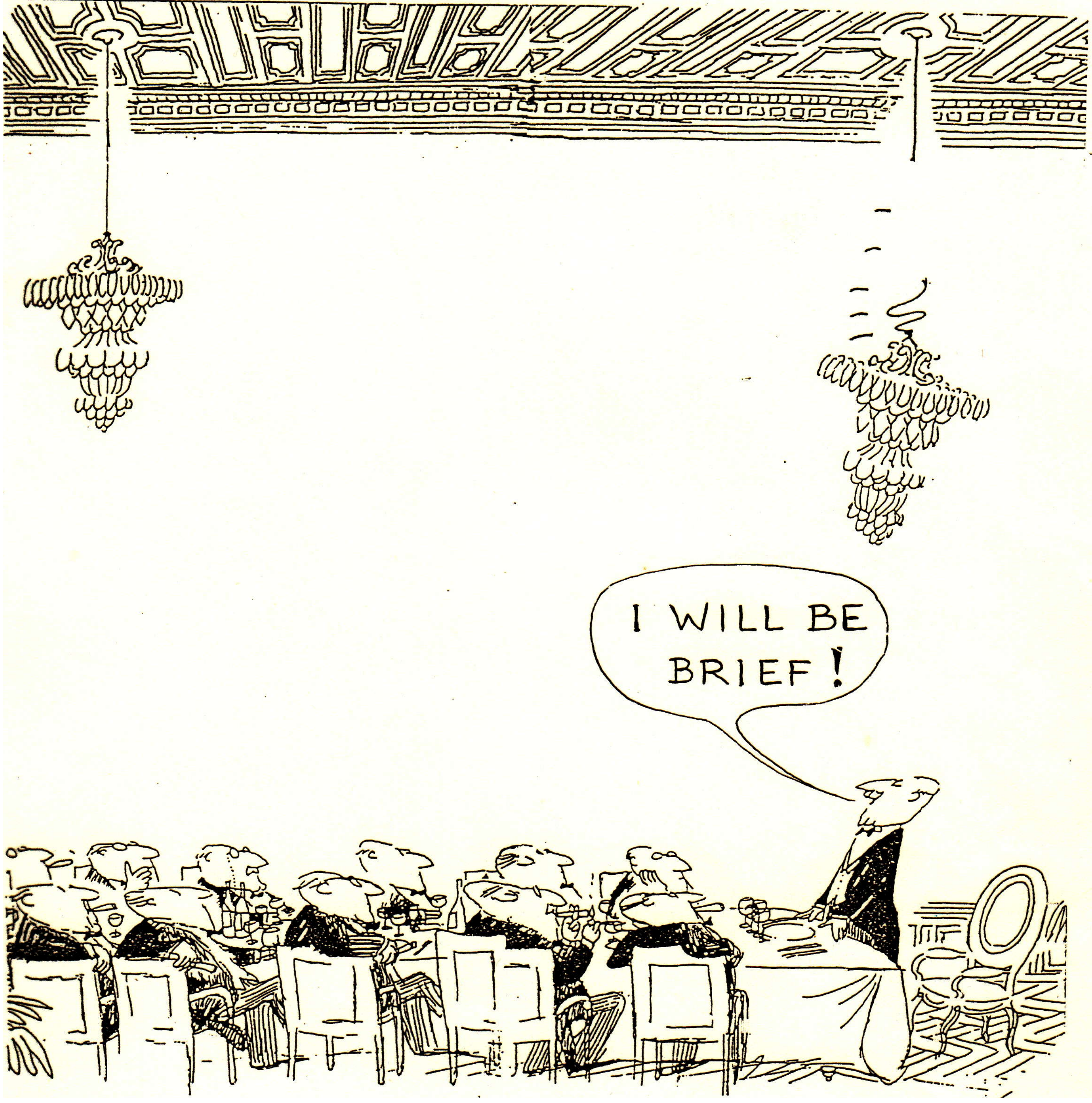
What about graphs?

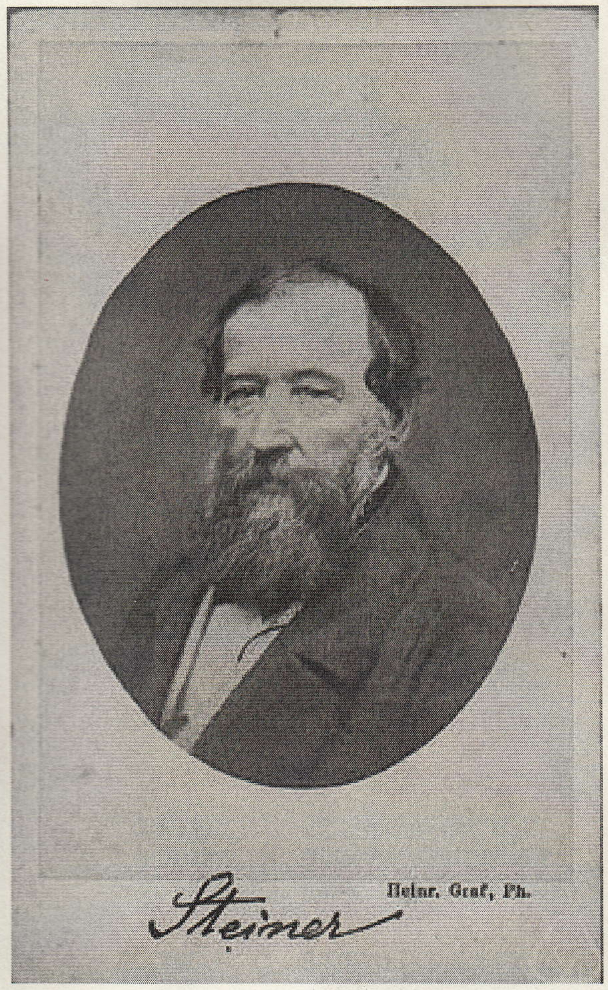
Exercise (folklore)

If $|G| > 5$, then for every integer $v \geq 2|G|+2$, there is a finite connected undirected graph V on v vertices such that Aut $V \cong G$.

Hint :

Start with such a graph on $v = 2|G|$ vertices (exists by Babai 1974) and try to enlarge it without changing its automorphism group.


I WILL BE BRIEF!

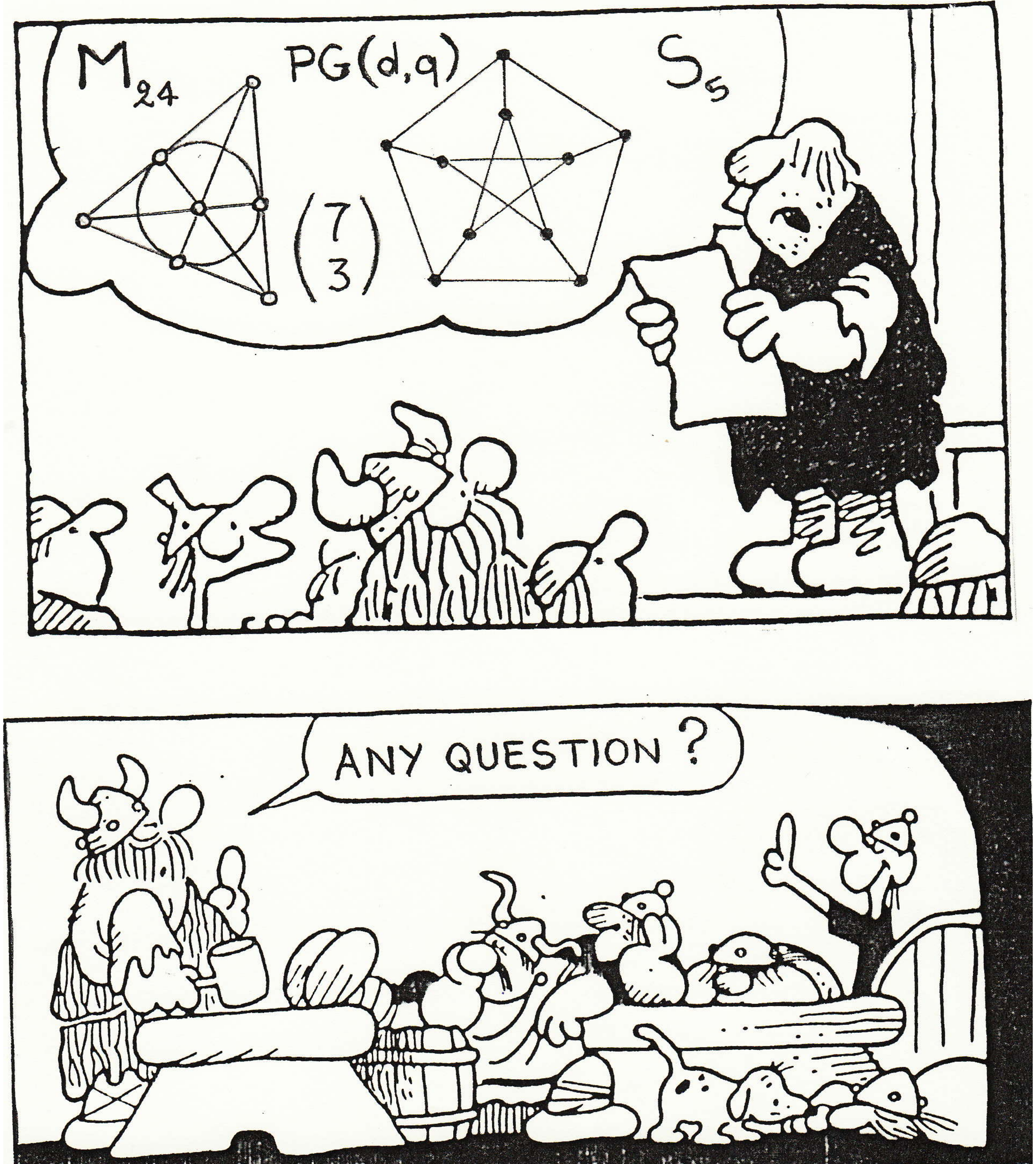


Jakob Steiner

1796 to 1863


euroblan
eb
STEINER
SYSTEM


COMMONWEALTH OF INDEPENDENT STATES
OBSERVERS


M_{24}
PG(d,q)
S_5
$\binom{7}{3}$
ANY QUESTION ?