Quadratic sets on the Klein quadric

Bart De Bruyn

Finite Geometries 2022, Sixth Irsee Conference

Quadratic sets of quadrics

Let Q be a nonsingular quadric of Witt index at least 3 in a projective space.

A quadratic set of Q is a set of points meeting each subspace π of Q in a possibly singular quadric of π.

It suffices to demand this condition for maximal subspaces π.

Classical (standard) examples: Intersections of Q with quadrics of the ambient projective space of Q.

Question 1: Are all quadratic sets classical?

Question 2: If not, are there certain families of quadratic sets all whose members are classical?

The case of Witt index 3

Let Q be a finite nonsingular quadric of Witt index 3, i.e. $Q^+(5,q)$, $Q(6,q)$ or $Q^-(7,q)$. A set $X \subset Q$ is a quadratic set if it meets each plane π of Q in a singleton (type S), a line (type L), an irreducible conic (type C), a pencil of two lines (type P) or the whole of π (type W).

Theorem (Mou Gao and BDB (DCC, 2022))

All quadratic sets of $Q^+(5,2)$, $Q(6,2)$ and $Q^-(7,2)$ are classical.

Not true for $q \geq 3$!

$Q^+(5,2)$: 2^{20} examples, 131 nonisomorphic ones.

$Q(6,2)$: 2^{27} examples.

$Q^-(7,2)$: 2^{35} examples.

What are the interesting quadratic sets?

Criteria:

- Possibility to obtain classification results about them.
- Connection with other geometric objects.

Good quadratic sets: Among the five possible plane intersections, at most two occur.

Type (X): All plane intersections have type X.

Type (XY): All plane intersections have type X or type Y.

Good quadratic sets of the Klein quadric $Q^+(5, q)$

There are 15 possible types:

- (S), (L), (C), (P), (W)
- (SL), (SC), (SP), (SW), (LC), (LP), (LW), (CP), (CW), (PW)

Type (W): the whole Klein quadric $Q^+(5, q)$

Type (S): ovoids of $Q^+(5, q)$, images under the Klein correspondence of line spreads of $\mathrm{PG}(3, q)$. Such an ovoid is a classical quadratic set if and only if it is a $Q^-(3, q)$-quadric.

Classification and (non)-existence results (BDB, 2022)

- Nonexistence for types (SP), (SW) and (CW).
- Complete classification for types (L), (P), (SL), (LP), (LW) and (PW).
- Construction of at least one infinite family for each of the types (SC), (LC), (CP), both for q even and q odd.
- Six families of type (LC): examples of $(q+1)$-ovoids.

Type C: unique example for $q = 2$
Open problem: What about $q \geq 3$? (no classical example for $q = 3$)

Complete classification for type (L)

- $Q(4,q)$-quadric (intersection with non-tangent hyperplane)

Complete classification for type (LW)

- $pQ^+(3, q)$-quadric (intersection with tangent hyperplane)

Complete classification for type (SL)

- $Q^+(3, q)$-quadric
- $\bigcup_{L \in \mathcal{L}} L$, where $\mathcal{L}$ is a set of lines through a point $x \in Q^+(5, q)$ forming an ovoid in the local polar space in x (which is a $(q+1) \times (q+1)$-grid)

Complete classification for type (P)

- union of two $Q(4,q)$-quadrics intersecting in a $Q^-(3,q)$-quadric
- extra example for $q=2$

Complete classification for type (LP)

- union of two $Q(4,q)$-quadrics intersecting in a $Q^+(3,q)$-quadric
- union of two $Q(4,q)$-quadrics intersecting in a $xQ(2,q)$-quadric

Complete classification for type (PW)

- union of an $xQ^+(3,q)$-quadric and an $x'Q^+(3,q)$-quadric, with x and x' noncollinear on $Q^+(5,q)$.
- union of an $xQ^+(3,q)$-quadric and a $Q(4,q)$-quadric with x belonging to the $Q(4,q)$-quadric.

Quadratic sets of type (SC), q odd

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_1^2 + a_{33}a_{44}X_2^2 + a_{33}X_3^2 + a_{44}X_4^2 = 0,$$

where a_{33}, a_{44} are non-squares in $\mathbb{F}_q$.

Quadratic sets of type (SC), q even

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_2^2+X_3^2+X_4^2+X_5^2+X_6^2+\lambda X_3X_5+\lambda X_3X_6+\lambda X_4X_5+\lambda X_4X_6+\lambda^2X_5X_6 = 0,$$

where $\lambda \in \mathbb{F}_q$ such that the polynomial $X^2 + \lambda X + 1 \in \mathbb{F}_q[X]$ is irreducible.

Quadratic sets of type (LC), q odd

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_2X_5 + d_1X_2X_6 + a_{33}X_3^2 + 2a_{33}d_2X_3X_4 + a_{33}d_2^2X_4^2 = 0,$$

where $a_{33}, d_1, d_2 \in \mathbb{F}_q^*$ with $-d_1d_2$ a non-square in $\mathbb{F}_q$.

Quadratic sets of type (LC), q even

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_2X_5 + a_{26}X_2X_6 + a_{33}X_3^2 + a_{44}X_4^2 + a_{66}X_6^2 = 0,$$

where $a_{26}, a_{33}, a_{44}, a_{66} \in \mathbb{F}_q^*$ with $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$.

$Tr(x) = x + x^2 + \cdots + x^{q/2}$

Quadratic sets of type (CP), q odd

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_2^2+a_{35}X_3X_5+a_{36}X_3X_6+a_{45}X_4X_5+a_{46}X_4X_6+a_{56}X_5X_6+a_{66}X_6^2 = 0,$$

where $a_{35}, a_{36}, a_{45}, a_{46}, a_{56}, a_{66} \in \mathbb{F}_q^*$ such that $a_{46} = \frac{a_{36}a_{45}}{a_{35}}$, $a_{66} = \frac{a_{36}a_{56}}{a_{35}}$ and $a_{56}^2 - 4a_{36}a_{45}$ is a nonsquare in $\mathbb{F}_q$.

Quadratic sets of type (CP), q even

The intersection of the Klein quadric $X_1X_2 + X_3X_4 + X_5X_6 = 0$ with the quadric having and equation of the form

$$X_2^2 + a_{35}X_3X_5 + a_{36}X_3X_6 + a_{45}X_4X_5 + a_{46}X_4X_6 + a_{56}X_5X_6 + a_{66}X_6^2 = 0,$$

where $a_{35}, a_{36}, a_{45}, a_{46}, a_{56}, a_{66} \in \mathbb{F}_q^*$ with

$$a_{46} = \frac{a_{36}a_{45}}{a_{35}}, \qquad a_{66} = \frac{a_{36}a_{56}}{a_{35}}, \qquad \mathrm{Tr}\left(\frac{a_{36}a_{45}}{a_{56}^2}\right) = 1.$$

Application 1: Line sets in $PG(3, q)$

Theorem (Sahu and Pradhan, 2020)

Let $\mathcal{S}$ be a set of lines in $\mathrm{PG}(3, q)$, $q \geq 7$ odd, for which the following hold:

- *There are $\frac{1}{2}q(q+1)$ or q^2 lines of $\mathcal{S}$ through each point of $\mathrm{PG}(3, q)$ and both cases occur.*
- *Precisely one of the following holds for every plane π:*
  - *every pencil of lines in π contains 0 or q lines of $\mathcal{S}$ (with both cases occurring);*
  - *every pencil of lines in π contains $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of $\mathcal{S}$.*

Then $\mathcal{S}$ is one of the following:

1. *The set of secant lines with respect to a hyperbolic quadric;*
2. *A hypothetical family consisting of $\frac{q^4+q^3+2q^2}{2}$ lines.*

Application 1: Line sets in $PG(3, q)$

Problem further investigated by Sahu, Pradhan, Sahoo and BDB.

- Result remains valid for $q \in \{3, 5\}$.
- The line sets are related to quadratic sets.

Suppose q odd. Take a quadratic set X of type (LC) for which the following hold:

- The planes of type (L) are precisely the planes through a given point $x^* \in Q^+(5, q)$.
- If A is the set of all points that are exterior with respect to some conic intersection $\pi \cap X$ where π is a plane of type (C), then every plane π' through an element $x \in A$ has type (C) and x is exterior with respect to the conic $\pi' \cap X$.

Application 1: Line sets in $PG(3, q)$

Theorem (Pradhan, Sahoo, Sahu, BDB)

- *The line set* $\kappa^{-1}(((x^*)^\perp \setminus X) \cup A)$ *belongs to the hypothetical family, with* κ *denoting the Klein correspondence.*
- *If q is distinct from 5 and 9, then every line set in the hypothetical family can be obtained in this way.*
- *The quadratic sets of type (LC) mentioned earlier satisfy the above conditions and so give rise to line sets in the hypothetical family.*
- *There exists up to isomorphism a unique example for* $q = 3$.

Application 2: Hyperovals of polar spaces

Suppose q even. Take a quadratic set X of type (SC).

A_1: Set of all points $x \in X$ that are contained in plane of type (S).

A_2: Set of the kernels of all conics of the form $\pi \cap X$ for planes π of type (C).

Suppose the following hold:

- Every plane π_1 through a point $x_1 \in A_1$ has type (S).
- Every plane π_2 through a point $x_2 \in A_2$ has type (C) and x_2 is the kernel of the conic $\pi_2 \cap X$.

Application 2: Hyperovals of polar spaces

Then $(X \setminus A_1) \cup A_2$ is a hyperoval of $Q^+(5,q)$, i.e. a nonempty set of points meeting each line in either 0 or 2 points, or equivalently, each plane in either the empty set or a hyperoval of that plane.

The family of type (SC) mentioned above satisfies this property and so gives rise to an infinite family of hyperovals.

Seems to be first family for $q \geq 3$ and Witt index at least 3.

D. V. Pasechnik. Extending polar spaces of rank at least 3. *J. Combin. Theory Ser. A* 72 (1995), 232–242.

- Computer classification (backtrack) of all hyperovals of $Q^+(5,4)$: two nonisomorphic examples.
- Open problem: Computer-free constructions of these hyperovals.