A modular equality for Cameron-Liebler line classes in projective and affine spaces of odd dimension

joint work with Jonathan Mannaert (VUB)

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Jan De Beule

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History

1982: P.J. Cameron and R.A. Liebler (1982), Tactical decompositions and orbits of projective groups, *Linear Algebra Appl.*, 46, 91–102.

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Tactical Decompositions and Orbits of Projective Groups

P. J. Cameron
Merton College
Oxford, England

and

R. A. Liebler
Colorado State University
Fort Collins, Colorado

Submitted by N. Biggs

ABSTRACT

We consider decompositions of the incidence structure of points and lines of $PG(n,q)$ $(n \geq 3)$ with equally many point and line classes. Such a decomposition, if line-tactical, must also be point-tactical. (This holds more generally in any 2-design.) We conjecture that such a tactical decomposition with more than one class has either a singleton point class, or just two point classes, one of which is a hyperplane. Using the previously mentioned result, we reduce the conjecture to the case $n = 3$, and prove it when q^2+q+1 is prime and for very small values of q. The truth of the conjecture would imply that an irreducible collineation group of $PG(n,q)$ $(n \geq 3)$ with equally many point and line orbits is line-transitive (and hence known).

1. INTRODUCTION

It is well known that a collineation group of a finite projective space $PG(n,q)$ has at least as many orbits on lines as on points. This paper reports on an attempt to determine which collineation groups have equally many point orbits and line orbits. For $n = 2$, any group has this property, and the problem is simply the determination of all subgroups of $P\Gamma L(3,q)$; we ignore this case. However, for $n > 2$, the position is very different. We conjecture that such a group is line-transitive, or fixes a hyperplane and acts line-transitively on it, or (dually) fixes a point and acts line-transitively on the quotient space. (Note that all line-transitive collineation groups have been determined:

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- Let $G \leqslant \mathrm{P\Gamma L}(n+1, q)$, having equally many orbits on the points as on the lines of $\mathrm{PG}(n, q)$ (property *)

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- Conjecture: G is line transitive or fixes a hyperplane and acts transitively on the lines of the hyperplanes, or, dually, fixes a point and acts transitively on the lines through the fixed point.

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- Characterize the line orbits of G?

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(i) The vectors $A^T\chi_{O_i}$, $i = 1, \ldots, s$, are linear combinations of the vectors $\chi_{O'_j}$.

(ii) If A has full row rank, then $s \leq t$. If $s = t$, then all vectors $\chi_{O'_j}$ are linear combinations of the vectors $A^T\chi_{O_i}$, hence $\chi_{O'_j} \in \mathrm{Im}(A^T)$.

Cameron-Liebler line classes

Consider $\mathrm{PG}(n,q)$, $n \geq 3$. Let A be a $0/1$-matrix,

- rows indexed by the points of $\mathrm{PG}(n,q)$;
- columns indexed by the lines of $\mathrm{PG}(n,q)$;
- $A_{x,l} = 1$ if and only if $x \in l$, otherwise 0.

Definition

A *Cameron-Liebler line class of* $\mathrm{PG}(n,q)$ is a set $\mathcal{L}$ of lines with characteristic vector $\chi_{\mathcal{L}} \in \mathrm{Im}(A^T)$.

Theorem (due to Block's lemma)

The orbits of a group satisfying property * are Cameron-Liebler line classes.

Cameron-Liebler line classes

Definition

A *line spread of* $\mathrm{PG}(n,q)$ is a set $\mathcal{S}$ of lines of $\mathrm{PG}(n,q)$ partitioning the point set of $\mathrm{PG}(n,q)$.

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Theorem (Cameron-Liebler 1982)

Let n be odd. A line set $\mathcal{L}$ is a Cameron-Liebler line set of $\mathrm{PG}(n, q)$ if there exists a constant $x \in \mathbb{N}$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for any line spread $\mathcal{S}$ of $\mathrm{PG}(n, q)$.

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Remark

An orbit of a group satisfying property * is a Cameron-Liebler line set, but the converse is not true (see Examples)

Groups with property *

Theorem (Bamberg and Penttila (2008), Cameron (1985))

Let $G \leqslant \mathrm{P\Gamma L}(n+1,q)$ be a group having equally many orbits on the points as on the lines of $\mathrm{PG}(n,q)$. Then G

1. stabilizes a hyperplane π and acts line-transitively on it, or (dually),
2. fixes a point P and acts line-transitively on the quotient space, or,
3. is line-transitive. In this case, there are three possibilities,
   a. G contains $\mathrm{PSL}(n+1,q)$
   b. $G = A_7 \leqslant \mathrm{PGL}(4,2)$
   c. G is the normalizer in $\mathrm{PGL}(5,2)$ of a Singer cyclic group of $\mathrm{PG}(4,2)$.

Examples of Cameron-Liebler line classes

Examples of Cameron-Liebler line classes in $\mathrm{PG}(3, q)$

1. The set of lines through a point P
2. The set of lines in a hyperplane π
3. The union of (1) and (2) if $P \notin \pi$.
4. The complements of (1), (2) and (3) in the set of lines.

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4. The complements of (1), (2) and (3) in the set of lines.

These examples are all *trivial*.

Theorem (Cameron 1985, Pavese 2019)

Example (3) is not an orbit of a group satisfying *.

Non-trivial examples of CL line classes

Examples of non-trivial Cameron-Liebler line classes in $\mathrm{PG}(3, q)$

(a) $x = \frac{q^2+1}{2}$, q odd (Drudge (1998), Bruen and Drudge (1999)).

(b) $x = \frac{q^2-1}{2}$, $q \equiv 5, 9 \mod 12$ (DB, Demeyer, Metsch, Rodgers (2016), and independently Feng, Momihara, Xiang (2015))

(c) $x = \frac{q^2+1}{2}$, $q > 7$ odd (Cossidente and Pavese (2019)) and $q \equiv 1 \mod 4$, $q \geq 9$ (Cossidente and Pavese (2019))

(d) $x = \frac{(q+1)^2}{3}$, $q \equiv 2 \mod 3$ (Feng, Momihara, Rodgers, Xiang, Zou (2021))

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Non-trivial examples are rare.

Restriction on the parameter x

Theorem (Gavrilyuk and Metsch (2014))

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter x of $\mathrm{PG}(3, q)$. Then for every plane and every point of $\mathrm{PG}(3, q)$,

$$\binom{x}{2} + m(m - x) \equiv 0 \mod (q + 1),$$

where m is the number of lines of $\mathcal{L}$ in the plane, respectively through the point.

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Definition (D'haeseleer, Mannaert, Storme, Švob)

A line set $\mathcal{L}$ is a Cameron-Liebler line set of $\mathrm{AG}(n, q)$ if there exists a constant $x \in \mathbb{N}$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for any line spread $\mathcal{S}$ of $\mathrm{AG}(n, q)$.

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Equivalent definition (D'haeseleer, Mannaert, Storme, Švob)

Let A be the point-line incidence matrix of $\mathrm{AG}(n, q)$. A *CL line class of* $\mathrm{AG}(n, q)$ is a set $\mathcal{L}$ of lines with characteristic vector $\chi_{\mathcal{L}} \in \mathrm{Im}(A^T)$

Cameron-Liebler line classes in $\mathrm{AG}(3, q)$

Theorem (D'haeseleer, Mannaert, Storme, Švob)

Suppose that $\mathcal{L}$ is a CL line class in $\mathrm{PG}(3, q)$ of parameter x. Then $\mathcal{L}$ defines a CL line class in $\mathrm{AG}(3, q)$ with the same parameter x if and only if $\mathcal{L}$ is disjoint to the set of lines in the plane at infinity of $\mathrm{AG}(3, q)$.

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Examples (b) and (d) of CL line classes in $\mathrm{PG}(3, q)$ turn out to be affine.

Cameron-Liebler line classes in $\mathrm{AG}(3, q)$

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Examples (b) and (d) of CL line classes in $\mathrm{PG}(3, q)$ turn out to be affine.

Theorem (D'haeseleer, Mannaert, Storme, Švob)

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{AG}(3, q)$ with parameter x. Then

$$x(x-1) \equiv 0 \mod 2(q+1).$$

Cameron-Liebler line classes in $\mathrm{PG}(n, q)$

Definition

Let $\mathcal{L}$ be a Cameron-Lieber line class in $\mathrm{PG}(n, q)$, $n \geq 3$. Then its parameter x is defined as

$$\frac{|\mathcal{L}|}{q^{n-1} + \ldots + q + 1}.$$

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Let $\mathcal{L}$ be a Cameron-Lieber line class in $\mathrm{PG}(n, q)$, $n \geq 3$. Then its parameter x is defined as

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Lemma

The parameter is always an integer if and only if n is odd, i.e. if and only if $\mathrm{PG}(n, q)$ admits line spreads, in which case $x = |\,\mathcal{S} \cap \mathcal{L}\,|$ for $\mathcal{S}$ any line spread of $\mathrm{PG}(n, q)$, in which case $\mathcal{L}$ is characterized by its constant intersection property with line spreads.

Cameron-Liebler line classes in $\mathrm{PG}(n, q)$

Theorem

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(n, q)$, $n > 3$. Then for every i-dimensional subspace π, $i > 2$, the set $\mathcal{L} \cap [\pi]_1$ is a Cameron-Liebler line class of a certain parameter x_π in π.

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Lemma (Blokhuis, De Boeck, D'haeseleer (2019))

Suppose that $\mathcal{L}$ is a Cameron- Liebler line class with parameter x in $\mathrm{PG}(n, q)$, $n \geq 3$. If ℓ is an arbitrary line in $\mathrm{PG}(n, q)$ then there are in total $q^2 \frac{q^{n-2}-1}{q-1}(x - \chi(\ell))$ lines of $\mathcal{L}$ skew to ℓ, $\chi(\ell) = 1$ if $\ell \in \mathcal{L}$ or 0 otherwise.

Modular equalities

Theorem (DB, Mannaert (2022))

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{AG}(n, q)$, $n \geq 3$ odd, with parameter x, then

$$x(x-1) \equiv 0 \mod 2(q+1).$$

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Theorem (DB, Mannaert, Storme (2022))

Let $n \geq 4$. Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{AG}(n, q)$ with parameter x, different from a point-pencil. Then

$$x \geq 2\left(\frac{q^{n-1}-1}{q^2-1}\right)+1.$$

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$$x \geq 2\left(\frac{q^{n-1}-1}{q^2-1}\right)+1\,.$$

Example

Let $n = 5$, $q = 7$, then there remain only 274 possibilities for x apart from $x \in \{0,1,2400,2401\}$.

Modular equalities

Theorem (DB, Mannaert (2022))

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter x in $\mathrm{PG}(n, q)$, with $n \geq 7$ odd. Then for any point p,

$$x(x-1) + 2\overline{m}(\overline{m} - x) \equiv 0 \mod (q+1)\,,$$

where $\overline{m}$ is the number of lines of $\mathcal{L}$ through p.

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