Notes on multiple blocking sets of $\mathrm{PG}(2, q)$

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Π_q: A finite projective plane of order q

$\mathrm{PG}(2, q)$: The finite Desarguesian projective plane of order q

$\mathrm{PG}(2, q) = \mathrm{AG}(2, q) \cup \ell_\infty$, for me ℓ_∞ has equation $Z = 0$

Definition

- A **blocking set** of Π_q is a set of points meeting each line of Π_q in at least 1 point.
- A blocking set is called **trivial** if it contains a line.
- A ***t*-fold blocking set** of Π_q is a set of points meeting each line of Π_q in at least t points.
- A t-fold blocking set $\mathcal{B}$ is called **minimal** if for each point P of $\mathcal{B}$ the point set $\mathcal{B} \backslash \{P\}$ is not a t-fold blocking set.

- If q is a square then $\mathrm{PG}(2,q)$ can be partitioned into $q-\sqrt{q}+1$ Baer subplanes.
- The union of t pairwise disjoint Baer subplanes is a minimal t-fold blocking set of size

$$t(q+\sqrt{q}+1).$$

- If q is a square, t is "small" w.r.t. q and $\mathcal{B}$ is a "small" t-fold blocking set in $\mathrm{PG}(2,q)$ then $\mathcal{B}$ contains t pairwise disjoint Baer subplanes (Blokhuis–Storme–Szőnyi 1999 + Lovász 2007)
- If $q=t^4$ then there is a minimal t-fold blocking set of size

$$t(q+\sqrt{q}+1)$$

which is not the union of t Baer subplanes (Ball-Blokhuis-Lavrauw 2000).

Definition

A blocking set of size less than $3(q+1)/2$ is called small.

Theorem (Lunardon 1999)

Assume $q = s^n$ for some prime power s.

Let U be an $(n+1)$-dimensional $\mathbb{F}_s$-subspace of $V = \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$.

The set of points

$$L_U = \{\langle \mathbf{v} \rangle_{\mathbb{F}_q} : \mathbf{v} \in U \setminus \{\mathbf{0}\}\}$$

is a minimal blocking set of $\mathrm{PG}(2,q)$ of size at most

$$\frac{|U|-1}{s-1} = q + \frac{q-1}{s-1}.$$

*L_U is called an $\mathbb{F}_s$-**linear** (or simply **linear**) **blocking set**.*
If $\langle U \rangle_{\mathbb{F}_q} = V$ then L_U is non-trivial.

Linearity Conjecture, planar case (Sziklai 2008)

If $\mathcal{B}$ is a small minimal blocking set, then $\mathcal{B}$ is a linear blocking set.

- If $\mathcal{B}$ is a non-trivial blocking set of Π_q then for every line ℓ:

$$|\mathcal{B} \setminus \ell| \geqslant q.$$

- If $|\mathcal{B} \setminus \ell| = q$ then $\mathcal{B}$ is of **Rédei type** and ℓ is the **Rédei line** of $\mathcal{B}$.

- If $\mathcal{U}$ is a set of q affine points then the **set of directions determined by** $\mathcal{U}$ is

$$D_{\mathcal{U}} = \{P \in \ell_\infty : P \in \langle R, Q \rangle \text{ for some } R, Q \in \mathcal{U}\}.$$

- $\mathcal{U} \cup D_{\mathcal{U}}$ is a blocking set of Rédei type of size

$$q + |D_{\mathcal{U}}|.$$

- If $D_{\mathcal{U}} = \ell_\infty$ then we obtain a trivial blocking set.
- If $D_{\mathcal{U}} \neq \ell_\infty$ then $\mathcal{U}$ is equivalent to the graph of some $\mathbb{F}_q \to \mathbb{F}_q$ function f:

$$\mathcal{U} \cong \mathcal{U}_f = \{(x : f(x) : 1) : x \in \mathbb{F}_q\} \subseteq \mathrm{AG}(2, q)$$

$$D_f := D_{\mathcal{U}_f} = \left\{ \left(1 : \frac{f(x) - f(y)}{x - y} : 0\right) : x \neq y, x, y \in \mathbb{F}_q \right\} \subseteq \ell_\infty$$

- If $|D_f| < (q+3)/2$ then $U_f \cup D_f$ is a small blocking set.

Theorem (Part of Ball–Blokhuis–Brouwer–Storme–Szőnyi 1999 and Ball 2003)

Let f be an $\mathbb{F}_q \to \mathbb{F}_q$ function, $q = p^n$, p prime, such that

$$|D_f| \leqslant \frac{q+1}{2}.$$

Then $f(x) = c + \sum_{i=0}^{n-1} \alpha_i x^{p^i}$ and $U_f \cup D_f$ is a linear blocking set.

- The linearity conjecture states that blocking sets of size less than

$$q + \frac{q+3}{2}$$

are linear.

- If $|D_f| \leqslant \frac{q+1}{2}$, then the Rédei type blocking set $U_f \cup D_f$ is linear.
- We do not know whether there is a small non-linear Rédei type blocking set of size

$$q + \frac{q}{2} + 1.$$

Theorem (BCs)

If $|D_f| = \frac{q}{2} + 1$ then parallel lines meeting U_f in at least one point meet U_f in the same number of points

(This is a property which holds for every additive function and when q is odd then it implies additivity.)

When q is even then there are **non-additive** functions such that parallel lines meeting U_f in at least one point meet U_f in the same number of points:

- Functions such that $U_f \cup (\ell_\infty \setminus D_f)$ is a non-translation hyperoval
- Functions such that $U_f \cup (\ell_\infty \setminus D_f)$ is a non-translation Korchmáros–Mazzocca arc
- There is another example where U_f is contained in a $\sqrt{q} \times \sqrt{q}$ grid

Problem 1

Is there an $\mathbb{F}_q \to \mathbb{F}_q$ function f not of the form $x \mapsto c + \sum_{i=0}^{n-1} \alpha_i x^{p^i}$ but determining $\frac{q}{2} + 1$ directions?

2-fold blocking sets

- Union of the sides of a triangle is a 2-fold blocking set of size $3q$
- If q is a prime then it is difficult to go below $3q$.
  There are examples of size $3q-1$ when
  - $q = 13$, Braun–Kohnert–Wassermann 2005
  - $q \in \{19, 31, 37, 43\}$, BCs–Héger 2019
- The $q = s^n$, $n > 1$ case
  - Bacsó–Héger–Szőnyi 2013:

    Construction of two disjoint linear blocking sets of Rédei type.
    Their union is a 2-fold blocking set of size at most

$$2\left(q + \frac{q-1}{s-1}\right)$$

The $q = s^n$, $n > 1$ case

- Similar constructions by De Beule, Héger, Szőnyi, Van de Voorde 2015, and also the following:

  If $\mathcal{B}$ is blocking set of $\mathrm{PG}(2,q)$, $|\mathcal{B}| \leqslant \frac{3}{2}\left(q - \frac{q}{p}\right)$, $p > 5$, then there is a small linear Rédei type blocking set of size $q + \frac{q}{p} + 1$ disjoint from $\mathcal{B}$.

- Bartoli, Cossidente, Marino, Pavese 2020:

  They find two disjoint copies of $\mathrm{PG}(3,q)$ in $\mathrm{PG}(3,q^3)$ and a point P such that there is no line through P meeting both subgeometries.

  The projection of the two subgeometries to a plane of $\mathrm{PG}(3,q^3)$ from P is the union of two disjoint small linear blocking sets and hence a small 2-fold blocking set.

Corollary of BBBSSz

Assume

$$|D_f| \leqslant \frac{q+1}{2} \quad \text{and} \quad |D_g| \leqslant \frac{q+1}{2}.$$

Then

$$(U_f \cup D_f) \cap (U_g \cup D_g) \neq \varnothing.$$

Proof.

$$f = F(x) + \alpha \quad \text{and} \quad g = G(x) + \beta \quad \text{where } F \text{ and } G \text{ are additive}$$

$$U_f \cap U_g = \varnothing \quad \Rightarrow \quad (F(x)+\alpha) - (G(x)+\beta) = 0 \text{ has no root in } \mathbb{F}_q$$

$$\Rightarrow F(x) - G(x) \quad \text{is additive and it is not a permutation of } \mathbb{F}_q$$

Indeed, if it was a permutation, then we could solve

$$F(x) - G(x) = \beta - \alpha.$$

$\Rightarrow$ The set of roots of $F(x) - G(x)$ in $\mathbb{F}_q$ is a non-trivial $\mathbb{F}_p$-subspace

$\Rightarrow$ There exists $c \in \mathbb{F}_q \setminus \{0\}$ such that $F(c) = G(c)$

$$\Downarrow$$

the line joining $(0 : \alpha : 1), (c : F(c) + \alpha : 1) \in U_f$

and

the line joining $(0 : \beta : 1), (c : G(c) + \beta : 1) \in U_g$

meet ℓ_∞ at the same point:

$$\left(1 : \frac{F(c)}{c} : 0\right) = \left(1 : \frac{G(c)}{c} : 0\right)$$

$$\Downarrow$$

$$D_f \cap D_g \neq \varnothing$$

Corollary

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are **disjoint** *small linear Rédei type blocking sets then they have different Rédei lines.*

Problem 2

Is it possible to find two $\mathbb{F}_q \to \mathbb{F}_q$ functions f and g such that only one of them is additive and

$$(U_f \cup D_f) \cap (U_g \cup D_g) = \emptyset?$$

$\mathcal{B} := (U_f \cup D_f) \cup (U_g \cup D_g)$ wouldn't be a very small 2-fold blocking set but maybe it is easier to find a third small blocking set disjoint from $\mathcal{B}$.

To construct small 3-fold blocking sets, one can try to find 3 pairwise disjoint small Rédei type blocking sets.

The Rédei lines have to be different so they can form a triangle or they can be concurrent.

Computations with computer show that there are examples but I could not find an explicit description.

Problem 3

Find for each prime p and infinitely many odd n, 3 pairwise disjoint small Rédei type blocking sets in $\mathrm{PG}(2, p^n)$.

Example

$$s := 137 \quad q := 137^{15}$$

$$\omega : \text{ a primitive element of } \mathbb{F}_q$$

$$\mathcal{B}_1 := \{(x : \omega x^s : y) : x \in \mathbb{F}_q,\ y \in \mathbb{F}_s\}$$

$$\mathcal{B}_2 := \{(y : x : \omega x^{s^2}) : x \in \mathbb{F}_q,\ y \in \mathbb{F}_s\}$$

$$\mathcal{B}_3 := \{(\omega x^{s^{12}} : y : x) : x \in \mathbb{F}_q,\ y \in \mathbb{F}_s\}$$

Then $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are pairwise disjoint linear blocking sets of Rédei type.
The Rédei lines form the base triangle.

Let φ be a collineation of order 3 and let $\mathcal{B}$ be a blocking set.

Assume

$$\mathcal{B} \cap \varphi(\mathcal{B}) = \emptyset. \tag{1}$$

Then also

$$\varphi(\mathcal{B}) \cap \varphi^2(\mathcal{B}) = \emptyset \quad \text{and} \quad \varphi^2(\mathcal{B}) \cap \mathcal{B} = \emptyset$$

$$\Downarrow$$

$\mathcal{B} \cup \varphi(\mathcal{B}) \cup \varphi^2(\mathcal{B})$ is a 3-fold blocking set.

Thus it is enough to verify (1) in order to find a 3-fold blocking set.

Theorem (BCs)

Put $s = 3^n$, *with* $\gcd(21, n) = 1$.

In $\mathrm{PG}(2, s^3)$ *there exist* 3 *pairwise disjoint* $\mathbb{F}_s$*-linear blocking sets.*

Their union is a 3*-fold blocking set of size at most*

$$3(1 + s + s^2 + s^3).$$

Sketch of Proof.

ω: a root of $x^3 - x - 1 \in \mathbb{F}_3[x]$

$$U := \{(a + \omega b, b + \omega^2 a, c + d\omega) : a, b, c, d \in \mathbb{F}_s\} \subseteq \mathbb{F}_{s^3} \times \mathbb{F}_{s^3} \times \mathbb{F}_{s^3}$$

$$\mathcal{B} := L_U = \{(a + \omega b : b + \omega^2 a : c + d\omega) : a, b, c, d \in \mathbb{F}_s\} \subseteq \mathrm{PG}(2, s^3)$$

φ: the collineation of order 3 mapping $(x : y : z)$ to $(z : x : y)$

We claim that $\mathcal{B} \cap \varphi(\mathcal{B}) = \emptyset$.

- Assume $P \in \mathcal{B} \cap \varphi(\mathcal{B})$, so for some $a, b, c, d, a', b', c', d' \in \mathbb{F}_s$

$$(a + \omega b : b + \omega^2 a : c + \omega d) = (c' + \omega d' : a' + \omega b' : b' + \omega^2 a')$$

- It is easy to see that P cannot have a zero-coordinate so

$$\frac{a + \omega b}{b + \omega^2 a} = \frac{c' + \omega d'}{a' + \omega b'} \tag{2}$$

and

$$\frac{a + \omega b}{c + \omega d} = \frac{c' + \omega d'}{b' + \omega^2 a'} \tag{3}$$

- Applying $\omega^3 = \omega + 1$, (2) and (3) are equivalent with

$$(aa' - bc' - ad') + \omega(a'b + ab' - ad' - bd') + \omega^2(bb' - ac') = 0$$

and

$$(ab' + a'b - cc') + \omega(a'b + bb' - c'd - cd') + \omega^2(aa' - dd') = 0.$$

- Recall $s = 3^n$ and $3 \nmid n$. It follows that $\mathbb{F}_s(\omega) = \mathbb{F}_{s^3}$, so $\{1, \omega, \omega^2\}$ are $\mathbb{F}_s$-linearly independent and this gives

$$aa' - bc' - ad' = a'b + ab' - ad' - bd' = bb' - ac' = 0$$

and

$$ab' + a'b - cc' = a'b + bb' - c'd - cd' = aa' - dd' = 0.$$

- This system of equations leads to

$$(b'^3 - b' + 1)(b'^7 + b'^6 - b'^3 + b'^2 - b' + 1) = 0.$$

Both factors are irreducible over $\mathbb{F}_3$, thus

$$b' \in (\mathbb{F}_{3^3} \cup \mathbb{F}_{3^7}) \backslash \mathbb{F}_3.$$

But $b' \in \mathbb{F}_s = \mathbb{F}_{3^n}$, $\gcd(21, n) = 1$, a contradiction.

$$\Downarrow$$

$$\mathcal{B} \cap \varphi(\mathcal{B}) = \varnothing$$

$$\Downarrow$$

$\mathcal{B}, \varphi(\mathcal{B}), \varphi^2(\mathcal{B})$ are pairwise disjoint

□

Problem 4

For every non-square, non-prime q, find 3 pairwise disjoint small linear blocking sets in $\mathrm{PG}(2, q)$.

Problem 5

For every non-square, non-prime q, find the maximum number of pairwise disjoint small linear blocking sets in $\mathrm{PG}(2, q)$.

THANK YOU FOR YOUR ATTENTION

1. **Problem 1:** Is there an $\mathbb{F}_q \to \mathbb{F}_q$ function f not of the form $x \mapsto c + \sum_{i=0}^{n-1} \alpha_i x^{p^i}$ but determining $\frac{q}{2} + 1$ directions?
2. **Problem 2:** Is it possible to find two $\mathbb{F}_q \to \mathbb{F}_q$ functions f and g such that one of them is additive and
$$(U_f \cup D_f) \cap (U_g \cup D_g) = \emptyset?$$
3. **Problem 3:** Find for each prime p and infinitely many odd n, 3 pairwise disjoint small Rédei type blocking sets in $\mathrm{PG}(2, p^n)$.
4. **Problem 4:** For every non-square, non-prime q, find 3 pairwise disjoint small linear blocking sets in $\mathrm{PG}(2, q)$.
5. **Problem 5:** For every non-square, non-prime q, find the maximum number of pairwise disjoint small linear blocking sets in $\mathrm{PG}(2, q)$.