Minimal blocking sets in small Desarguesian projective planes

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Preliminaries

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A blocking set is called *minimal* if no proper subset is a blocking set.

The *weight* of a line is the amount of points of the blocking set on it.

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Research

1. Generation
2. Description

Research

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   - isomorph-free backtracking
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2. Description
   - describing the set and its automorphism group

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   - isomorph-free backtracking
2. Description
   - describing the set and its automorphism group. . . if possible

PG(2,5)

$\vert B\vert$	9		10		11		12	
#	1		5		1		1	
	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#
	24	1	2	2	20	1	96	1
			8	1				
			12	1				
			20	1				

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RESEARCH ARTICLE

WILEY

Classification of minimal blocking sets in small Desarguesian projective planes

Kris Coolsaet | Arne Botteldoorn | Veerle Fack

PG(2,7)

$\vert B\vert$	12		13		14		15		16		17		18		19	
#	2		9		227		446		702		38		7		1	
	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#	$\vert G\vert$	#
	54	**1**	2	1	1	154	1	402	1	642	1	28	2	3	**57**	**1**
	216	**1**	4	2	2	54	2	25	2	49	2	2	6	1		
			6	3	3	6	3	12	3	5	3	4	9	1		
			8	1	4	5	4	5	4	4	6	4	12	1		
			12	**1**	6	4	6	1	6	2			**216**	**1**		
			24	**1**	12	1	12	1								
					18	1										
					24	**1**										
					42	**1**										

PG(2,8)

$\vert B\vert$	13			15			16			17			18		
#PGL	1			33			2498			18392			66873		
#PΓL	1			17			852			6156			22353		
	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#
	288	**96**	**1**	1	1	1	1	1	757	1	1	6004	1	1	22098
				2	2	4	2	2	64	2	2	102	2	2	127
				3	1	2	3	1	27	3	1	31	3	1	81
				4	4	1	3	3	1	4	4	12	3	3	32
				6	2	2	4	4	1	6	2	4	4	4	1
				6	6	1	6	2	1	12	4	1	6	2	8
				21	**7**	**1**	**42**	**14**	**1**	24	8	1	6	6	2
				24	8	2				**96**	**32**	**1**	9	3	1
				24	**24**	**1**							12	4	1
				42	**14**	**1**							18	6	1
				168	**56**	**1**							36	12	1

$\vert B\vert$	19			20			21			22			23		
#PGL	42141			6584			125			3			1		
#PΓL	14053			2204			49			1			1		
	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#
	1	1	13973	1	1	2188	1	1	38	**1**	**1**	**1**	**21**	**7**	**1**
	2	2	67	2	2	2	3	1	10						
	3	1	7	3	1	12	**882**	**294**	**1**						
	4	4	3	6	2	2									
	6	1	1												
	8	8	1												
	12	4	1												

PG(2,9)

$\lvert B\rvert$	13			15			16			17			18		
#PGL	1			2			3			132			30726		
#PΓL	1			2			3			91			15855		
	$\lvert\Gamma\rvert$	$\lvert G\rvert$	#	$\lvert\Gamma\rvert$	$\lvert G\rvert$	#	$\lvert\Gamma\rvert$	$\lvert G\rvert$	#	$\lvert\Gamma\rvert$	$\lvert G\rvert$	#	$\lvert\Gamma\rvert$	$\lvert G\rvert$	#
	11232	**5616**	**1**	**120**	**60**	**1**	6	3	1	1	1	14	1	1	14263
				192	**96**	**1**	**16**	**8**	**1**	2	1	20	2	1	795
							72	**36**	**1**	2	2	24	2	2	570
										4	2	17	3	3	15
										4	4	2	4	2	144
										6	6	1	4	4	11
										8	4	5	6	3	11
										12	6	2	6	6	11
										16	8	3	8	4	15
										24	12	1	8	8	1
										32	16	1	12	6	9
										48	**24**	**1**	16	8	4
													18	9	1
													24	12	2
													32	16	1
													48	**24**	**1**
													144	**72**	**1**

PG(2,9)

$\lvert B \rvert$	19			20			21			22		
#PGL	524394			4544050			12508783			10899207		
#PΓL	263904			2276093			6259366			5453644		
	$\lvert\Gamma\rvert$	$\lvert G \rvert$	#	$\lvert\Gamma\rvert$	$\lvert G \rvert$	#	$\lvert\Gamma\rvert$	$\lvert G \rvert$	#	$\lvert\Gamma\rvert$	$\lvert G \rvert$	#
	1	1	259106	1	1	2263708	1	1	6247527	1	1	5442318
	2	1	3255	2	1	7850	2	1	9820	2	1	7794
	2	2	1320	2	2	4218	2	2	1704	2	2	3019
	3	3	32	3	3	7	3	3	164	3	3	196
	4	2	110	4	2	261	4	2	93	4	2	219
	4	4	30	4	4	22	4	4	19	4	4	26
	6	3	21	6	3	4	6	3	20	6	3	34
	6	6	2	6	6	1	6	6	2	6	6	3
	8	4	21	8	4	11	8	4	8	8	4	21
	12	6	2	10	10	1	8	8	1	12	6	5
	16	8	2	12	6	4	12	6	1	16	8	4
	18	9	1	16	8	4	**14**	**7**	**1**	18	18	1
	36	18	1	24	12	2	16	8	3	32	16	1
	192	**96**	**1**				32	16	1	36	18	1
							42	**21**	**1**	**48**	**24**	**1**
							336	**168**	**1**	**64**	**32**	**1**

PG(2,9)

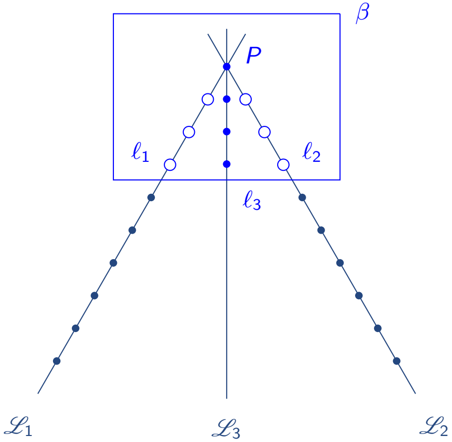
$\|B\|$	23			24			25			26			28		
$\#_{\text{PGL}}$	2252493			65702			195			6			2		
$\#_{\text{P}\Gamma\text{L}}$	1127161			33011			100			5			2		
	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#
	1	1	1125123	1	1	32551	1	1	86	4	4	1	**216**	**108**	**1**
	2	1	1800	2	1	299	2	1	2	8	4	2	**12096**	**6048**	**1**
	2	2	202	2	2	88	2	2	2	24	12	1			
	4	2	23	3	3	47	3	3	7	**48**	**24**	**1**			
	4	4	7	4	2	9	4	2	2						
	8	4	5	4	4	1	6	3	1						
	16	8	1	6	3	8									
				6	6	4									
				12	6	2									
				24	12	1									
				768	**384**	**1**									

Derived from a Baer subplane

Example

In $\mathrm{PG}(2, q^2)$, let β denote a Baer subplane, in which lies a point P. Let ℓ_1, ℓ_2, ℓ_3 denote three lines of the Baer subplane through P. Let $\mathscr{L}_i$ denote the extension of ℓ_i to $\mathrm{PG}(2, q^2)$.

Then $B = (\mathscr{L}_1 \setminus \ell_1) \cup (\mathscr{L}_2 \setminus \ell_2) \cup (\ell_3)$ is a minimal blocking set of $\mathrm{PG}(2, q^2)$ of size $2q^2 - q + 1$.



Derived from a Baer subplane

Example

In $\mathrm{PG}(2, q^2)$, let β denote a Baer subplane, in which lies a point P. Let ℓ_1, ℓ_2, ℓ_3 denote three lines of the Baer subplane through P. Let $\mathscr{L}_i$ denote the extension of ℓ_i to $\mathrm{PG}(2, q^2)$.

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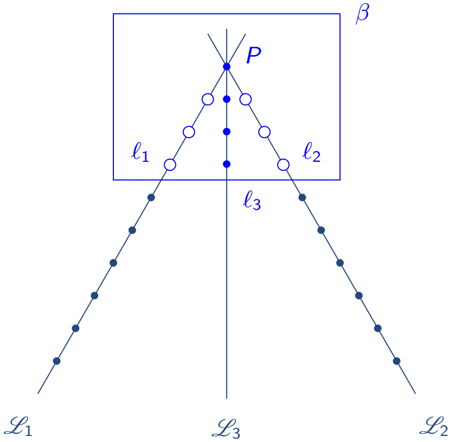
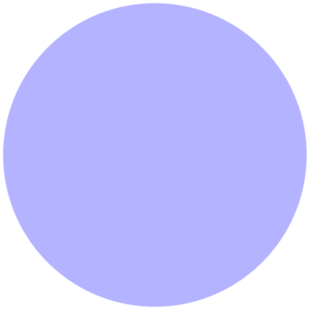
For $q > 2$, the collineation group of B has size $4q^2(q - 1)$.



From a Hermitian curve



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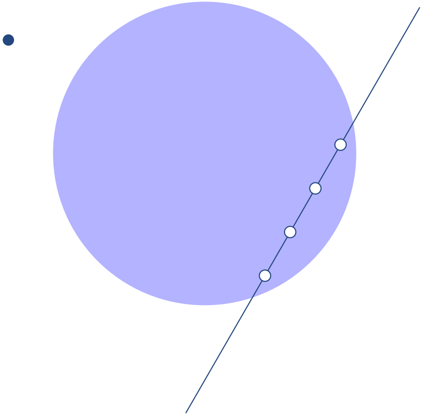

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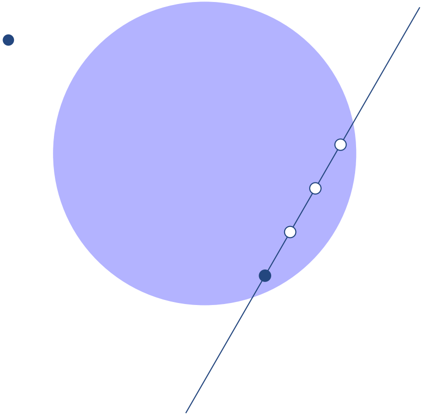

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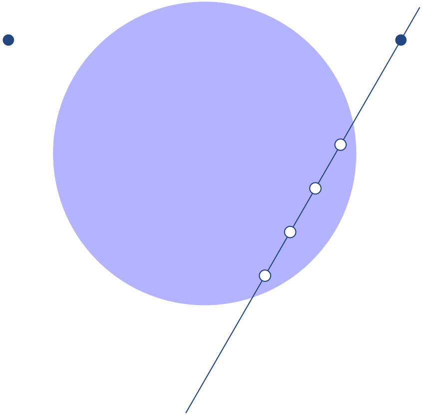

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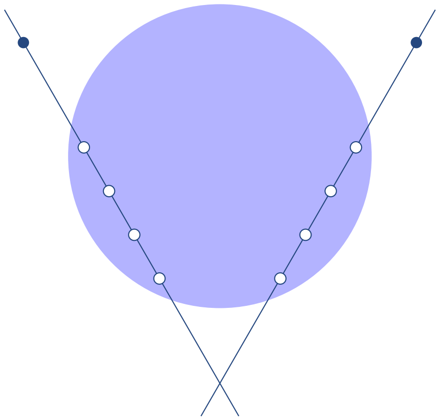

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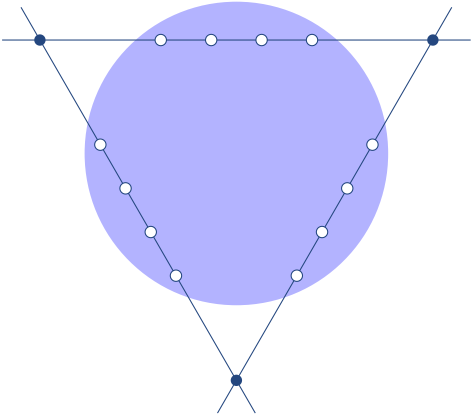

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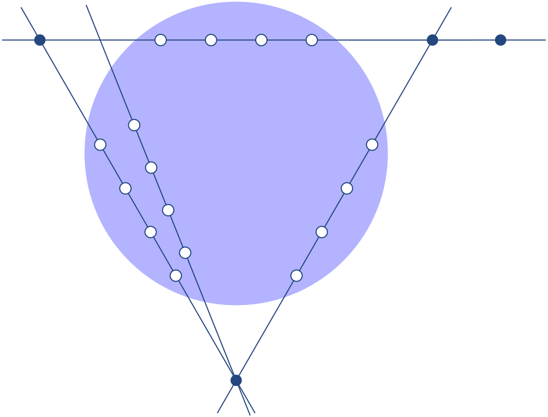

From a perfect difference set in $\mathrm{PG}(2,9)$

Use a perfect difference set representation of the plane, for example:
$L_0 = \{0, 1, 3, 9, 27, 81, 61, 49, 56, 77\}$
(points are represented by the integers mod 91, lines by the sets $L_i = L_0 - i$.)

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- $K_r = \{P_i | i \equiv r \mod 13\}$ is a complete 7-arc for each $r \in \{0, \ldots, 12\}$
- The union of the four 7-arcs K_0, K_2, K_5 and K_6 form a Hermitian curve.

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From a perfect difference set in $\mathrm{PG}(2,9)$

Constructing minimal blocking sets by taking the union of 3 such arcs:

- $K_0 \cup K_2 \cup K_4$ has a projective automorphism group of size 7
- $K_0 \cup K_2 \cup K_8$ has a projective automorphism group of size 21
- $K_0 \cup K_2 \cup K_7$ has a projective automorphism group of size 168

From a perfect difference set in $\mathrm{PG}(2,9)$

Let $B = K_{10} \cup K_{12} \cup K_4$, which is equivalent to $K_0 \cup K_2 \cup K_7$.

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B can be partitioned into polar triangles (w.r.t. Herm. curve $K_0 \cup K_2 \cup K_5 \cup K_6$):

	δ_1			δ_2	
K_4	K_{10}	K_{12}	K_4	K_{10}	K_{12}
4	10	64	30	10	12
17	23	77	43	23	25
30	36	90	56	36	38
43	49	12	69	49	51
56	62	25	82	62	64
69	75	38	4	75	77
82	88	51	17	88	90

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69	75	38	4	75	77
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Define the incidence geometry $\mathscr{G} = (\mathscr{P}, \mathscr{L}, I)$ with

- $\mathscr{P}$ the polar triangles of type δ_1;
- $\mathscr{L}$ the polar triangles of type δ_2;
- the natural incidence.

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$\mathscr{G}$ is a Fano plane.

From a perfect difference set in $\mathrm{PG}(2,9)$

Lemma (Polster - Van Maldeghem (2001))

Let $\mathscr{U}$ be a Hermitian unital in $\mathrm{PG}(2,9)$. Let $\mathscr{P}$ be the set of 63 points off $\mathscr{U}$ and let $\mathscr{L}$ be the set of 63 polar triangles with respect to $\mathscr{U}$.
Then $(\mathscr{P}, \mathscr{L}, I_{nat})$ is a generalized hexagon of order $(2,2)$ isomorphic to the dual of $\mathbf{H}(2)$.

Future work

- Generation

Future work

- Generation
  - $\mathrm{PG}(2, 11)$

Future work

- Generation
  - $\mathrm{PG}(2, 11)$
  - Specific blocking sets?