# On the number of rational points of curves over a surface in $\mathbb{P}^3$

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## Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with $q$ elements and $\mathbb{P}^n_{\mathbb{F}_q}$ the projective space.

An algebraic projective variety $X$ defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ irreducible over $\mathbb{F}_q$:

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**Today:** algebraic varieties of dimension one (curves $C$) and two (surfaces $S$) in $\mathbb{P}^3$.

## Existing bounds

### Theorem [Hasse–Weil, 1948]

If $C$ is an absolutely irreducible smooth curve of genus $g$ defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$.

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**Theorem [Stöhr–Voloch, 1986]**

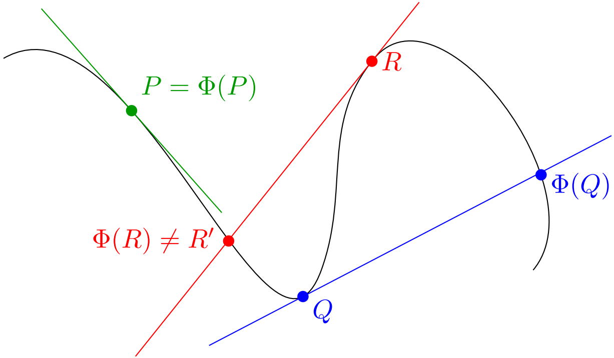
Let $C/\mathbb{F}_q$ be an irreducible smooth curve of genus $g$ and degree $\delta$ in $\mathbb{P}^n$. Let $\nu_1, \ldots, \nu_{n-1}$ be its Frobenius orders (generically $\nu_i = i$). Then

$$\#C(\mathbb{F}_q) \leq \frac{1}{n}\left((\nu_1 + \cdots + \nu_{n-1})(2g-2) + (q+n)\delta\right).$$

## Stöhr and Voloch's strategy for plane curves

Take $C$ a plane curve of deg. $\delta$ defined by $f = 0$ over $\mathbb{F}_q$. Write $\Phi$ for the $q$–Frobenius morphism.

$$C(\mathbb{F}_q) = \{P \in C \mid \Phi(P) = P\} \subseteq \{P \in C \mid \Phi(P) \in T_P C\} \stackrel{\text{def}}{=} \mathcal{Z}.$$



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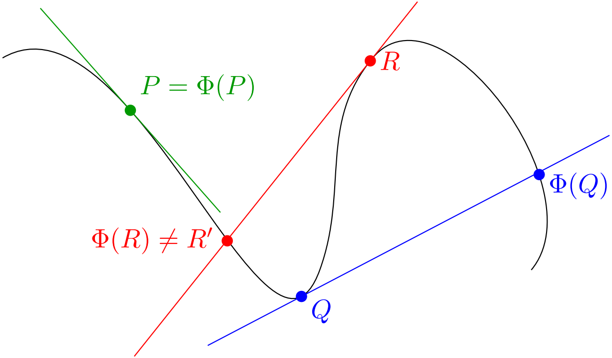

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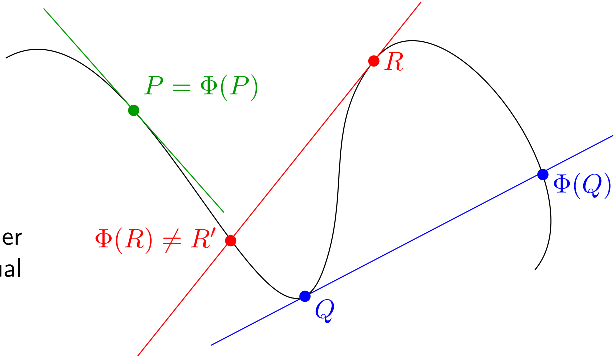


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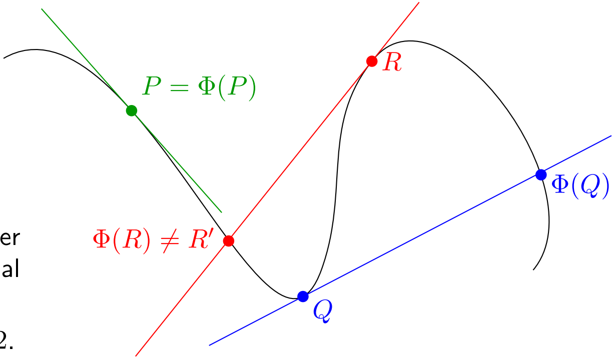


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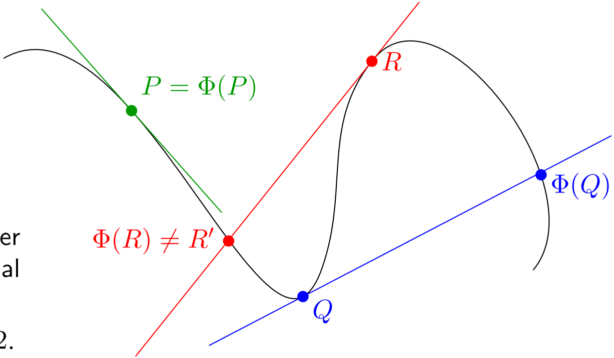


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### Theorem [Stöhr–Voloch, 1986]

If $C$ has at least a non–flex point $(\Rightarrow \dim \mathcal{Z} = 0)$, then $\#C(\mathbb{F}_q) \leq \frac{1}{2}\delta(\delta + q - 1)$.

## Ideas & Motivations

Let $C \subset S \hookrightarrow \mathbb{P}^n$ (via a very ample divisor).

**Goal:** bounding $\#C(\mathbb{F}_q)$ in terms of the embedding.

(features of the surface $S$ and the ambient $\mathbb{P}^n$)

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|---|---|---|
| Bounding the minimum distance of a code from a surface $S$ | $\rightsquigarrow$ | Bounding $\#C(\mathbb{F}_q)$ for the irreducible curves $C$ on $S$ |
| Better lower bound for the minimum distance | $\Longleftrightarrow$ | Better upper bound for $\#C(\mathbb{F}_q)$ |

## Strategy ($n = 3$)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree $d$ defined $\mathbb{F}_q$.

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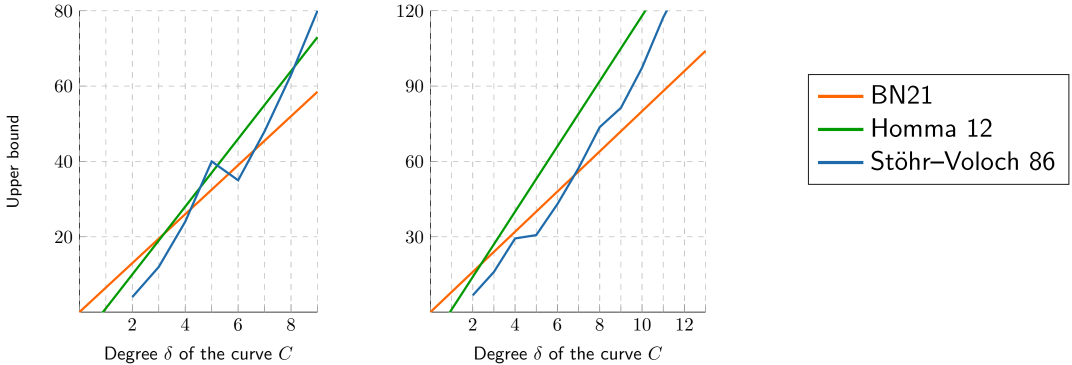
If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

## Comparisons with pre–existing bounds




(a) $q = 9$ and $d = 5$ (b) $q = 13$ and $d = 4$

Figure: Bounds on the number of $\mathbb{F}_q$–points on a non–plane curve $C$ on a degree $d$ surface $S \subset \mathbb{P}^3$.

$\rightarrow$ It is worth working on this bound!

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**Aim:** understanding the components of the curve $C_\Phi^S$ for a Frobenius classical surface.

## Osculating spaces and $P$-orders (Stöhr–Voloch theory 1)

Let $C \subset \mathbb{P}^3$ be an absolutely irreducible projective curve defined over $\mathbb{F}_q$. Fix $P \in C$.
An integer $j$ is a $P$–order if there exists a plane intersecting the curve $C$ with multiplicity $j$ at $P$.
If $C$ is non–plane and $P$ is non–singular, there are exactly four distinct $P$–orders:

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**Osculating spaces:** $T_P^{(i)}C = \bigcap \{\text{planes } H \text{ s.t. } m_P(C,H) \geq j_{i+1}\}$.

Equation of the osculating plane $T_P^{(2)}C$ :
$$\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$$

where $D_t^{(j)}$ are the *Hasse derivatives* with respect to a a local parameter $t$ at $P$ defined by

$$D_t^{(i)} t^k = \binom{k}{i} t^{k-i}.$$

## Frobenius orders (Stöhr–Voloch theory 2)

Fix $P \in C \subset \mathbb{P}^3$ with $P$–orders $(0, j_1, j_2, j_3)$. Then $\Phi(P) \in T_P^{(2)}C$ if and only if

$$\Delta(j_1, j_2) \overset{\text{def}}{=} \begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$$

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### Theorem [Stöhr–Voloch, 1986]

There exist integers $\nu_1 < \nu_2$ s.t. $\Delta(\nu_1, \nu_2)$ is a nonzero function.

### Definition

The integers $\nu_0 = 0, \nu_1, \nu_2$ chosen minimally with respect to the lexicographic order are called the Frobenius orders of $C$.

The curve $C$ is Frobenius classical if $(\nu_1, \nu_2) = (1, 2)$, Frobenius non–classical otherwise.

## Frobenius non–classical curves on surfaces

**Aim:** Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

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Let $C$ be a non–plane curve lying on a surface $S$. Assume that $C$ is Frobenius non–classical with $\nu_1 = 1$. Then $C$ is not a component of $C_\Phi^S$.

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## Frobenius classical components of $C_\Phi^S$

**Recap:** A component of $C_\Phi^S$ falls in one of the following cases:

- $\nu_1 > 1$: in this case, if it has $\delta \leq q$, it is plane;
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### Example of surface with highly reducible $C_\Phi^S$

Over $\mathbb{F}_5$, consider the surface $S$ defined by

$$\begin{aligned} f = \;& 2X_0X_1^2 + 2X_1^3 + 2X_0^2X_2 + 2X_0X_1X_2 + X_1^2X_2 + 2X_0X_2^2 + 3X_1X_2^2 \\ & +3X_2^3 + 4X_0^2X_3 + X_0X_1X_3 + X_1^2X_3 + 2X_1X_2X_3 + 2X_2^2X_3 \\ & +3X_0X_3^2 + 4X_1X_3^2 + X_2X_3^2. \end{aligned}$$

The curve $C_\Phi^S$ has degree 21 and is formed of 15 $\mathbb{F}_5$–lines and one non–plane sextic $(\delta = q + 1)$.

## Main result & Conclusion

**Theorem [BN21]**

Let $S$ be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let $C$ be a non–plane irreducible curve of degree $\delta \leq q$ lying on $S$. Suppose $C$ is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

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- A plane curve on a degree $d$ surface has $\delta \leq d \Rightarrow$ our bound holds for plane curves which have at least one point $P$ such that $\Phi(P) \notin T_P C$ by Stöhr–Voloch bound $(\delta(\delta+q-1)/2)$.

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- **Embedding entails arithmetic and geometric constraints on a variety:**
  For $\delta = 11$ and $d = 5$ over $\mathbb{F}_9$, $C$ has genus at most 17 and $\#C(\mathbb{F}_q) \leq 72$.
  In `ManyPoints`, maximal curves of genus 16 and 17 have 74 $\mathbb{F}_9$–points.
  These record curves cannot lie on a Frobenius classical surface in $\mathbb{P}^3$, unless being a component of $C^S_\Phi$.

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**Thank you for your attention!**

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Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of $\Phi$ restricted to the curve $C$,
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$.

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
*Remark:* $C_\Phi^S$ was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the $1^{st}$ projection.

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- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of $\Phi$ restricted to the curve $C$, (dim 1)
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$. (dim 4)

Then $C(\mathbb{F}_q) \overset{\Delta}{\hookrightarrow} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
*Remark:* $C_\Phi^S$ was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the $1^{st}$ projection.

$\Gamma_C$ and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
$\rightarrow$ bound the number of rational points on $C$ by a fraction of the intersection product $[\Gamma_C] \cdot [\mathcal{T}_S]$.

When $n \geq 4$, $[\Gamma_C] \cdot [\mathcal{T}_S] = 0$ while $\Gamma_C \cap \mathcal{T}_S \neq \varnothing$.

**Idea:** Fix this dimension incompatibility by blowing up $\mathcal{T}_S$ or $S \times S$.