On additive MDS codes with linear projections

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(Joint work with Simeon Ball)

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Central question

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What if we relax linearity to additivity? Are there long additive MDS codes over finite fields, which are not equivalent to linear codes?

Linear codes and their geometry

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Every column of G represents a point of $\mathrm{PG}(k-1, q)$. This gives us a (mutli)set of n points in $\mathrm{PG}(k-1, q)$.

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Vice versa, from a point set we can construct a code (by reversing the previous process). This can yield different point sets, which are an orbit under code equivalence.

The parameters of the code

	Linear code	Point set
n	length	size
k	dimension	(vector) dimension of the ambient projective space
d	minimum Hamming distance	minimum number of points outside any hyperplane

Additive codes and their geometry

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$$(\forall \mathbf{x}, \mathbf{y} \in C)(\mathbf{x} + \mathbf{y} \in C).$$

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In this talk we will consider codes over $\mathbb{F}_{q^h}$ which are linear over $\mathbb{F}_q$.
$q = q^h \rightarrow$ linear code
q prime $\rightarrow$ additive code

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Let G be a generator matrix for C over $\mathbb{F}_q$. This means that $G \in \mathbb{F}_{q^h}^{k \times n}$, and the rows of G are an $\mathbb{F}_q$-basis for C.

$$G = k \left[\begin{pmatrix} \boxed{\cdot \quad \cdot \qquad\qquad \cdot \quad \cdot} \\ \underbrace{\vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots}_{} \\ \boxed{\cdot \quad \cdot \qquad\qquad \cdot \quad \cdot} \end{pmatrix} \right.$$

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Take an $\mathbb{F}_q$-basis $\alpha_1, \dots, \alpha_h$ of $\mathbb{F}_{q^h}$ and write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_h)$. The j^{th} column of G is of the form $\boldsymbol{\alpha} G_j$ for some unique $G_j \in \mathbb{F}_q^{h \times k}$.

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$$\begin{pmatrix} g_{1j} \\ \vdots \\ g_{kj} \end{pmatrix} = \begin{pmatrix} \alpha_1 g_{1j}^{(1)} + \dots + \alpha_h g_{1j}^{(h)} \\ \vdots \\ \alpha_1 g_{kj}^{(1)} + \dots + \alpha_h g_{kj}^{(h)} \end{pmatrix} = \boldsymbol{\alpha} \begin{pmatrix} g_{1j}^{(1)} & \dots & g_{1j}^{(h)} \\ & \vdots & \\ g_{kj}^{(1)} & \dots & g_{kj}^{(h)} \end{pmatrix}$$

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Consider the subspaces $\mathrm{ColSp}(G_1), \dots, \mathrm{ColSp}(G_n)$ of $\mathrm{PG}(k-1, q)$.

Equivalence

Definition

Call two $\mathbb{F}_q$-linear codes C and D over $\mathbb{F}_{q^h}$ *$\mathbb{F}_q$-equivalent* if C can be transformed into D by

1. permuting the coordinate positions,
2. in each coordinate, apply an $\mathbb{F}_q$-linear bijection. This bijection can be different for different coordinates.

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There exist an equivalence between:

1. equivalence classes of $\mathbb{F}_q$-linear $(n, q^k, d)_{q^h}$ codes,
2. $\mathrm{PGL}(k, q)$-orbits of multisets of n subspaces in $\mathrm{PG}(k-1, q)$ of dimension at most $h-1$.

Parameters of the code

	$\mathbb{F}_q$-linear code over $\mathbb{F}_{q^h}$	Set of subspaces of dimension $< h$
n	length	size
k	$\mathbb{F}_q$-dimension	(vector) dimension of the ambient projective space
d	minimum Hamming distance	minimum number of subspaces not contained in a hyperplane

Recognising linear codes

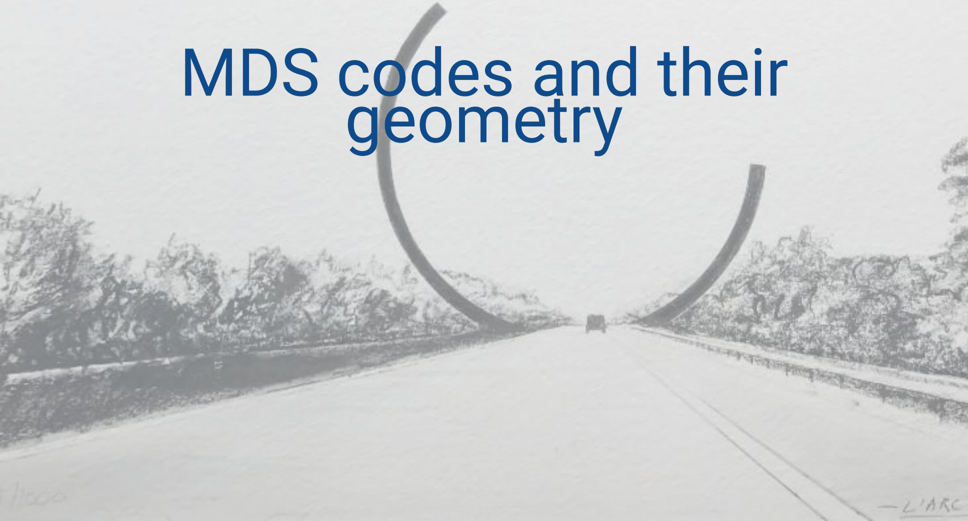
Theorem

An $\mathbb{F}_q$-linear $(n, q^k, d)_{q^h}$ code is $\mathbb{F}_q$-equivalent to a linear code

$$\Longleftrightarrow$$

its associated set of subspaces is a subset of a Desarguesian $(h-1)$-spread of $\mathrm{PG}(k-1, q)$.

MDS codes and their geometry



Linear MDS codes and arcs

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Proposition

A linear code is MDS

$$\iff$$

its associated point set is an arc, i.e. a set of points in $\mathrm{PG}(k-1, q)$ *of which any* k *span the space.*

Additive MDS codes and generalised arcs

Definition

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Proposition (Ball, Lavrauw, Gamboa; 2021)

An $\mathbb{F}_q$-linear $(n, q^{kh}, d)_{q^h}$ code is MDS

$$\iff$$

its associated set of subspaces is a generalised arc of $(h-1)$-spaces in $\mathrm{PG}(kh-1,q)$.

Question

Can we make long additive MDS codes over finite fields, which aren't equivalent to linear codes?

Can we make large generalised arcs which aren't contained in a Desarguesian spread?

Generalised arcs and translation generalised quadrangles

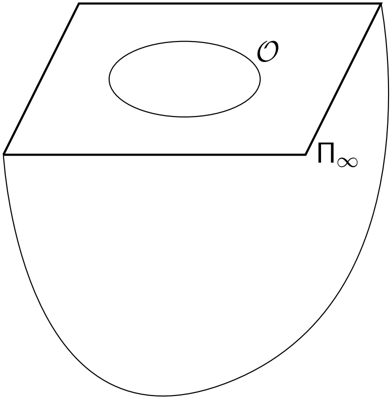
Generalised quadrangles

Definition

A *generalised quadrangle (GQ)* is a point- and block-regular incidence geometry such that given any point P and a line/block $\ell \not\ni P$, there is a unique point $Q \in \ell$ collinear to P.

The $\mathcal{T}_2(\mathcal{O})$ construction by Tits

In PG$(3, q)$, take a plane Π_∞ and an oval $\mathcal{O} \subset \Pi_\infty$. We can construct a GQ



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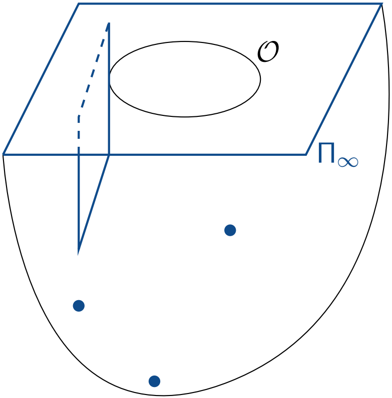

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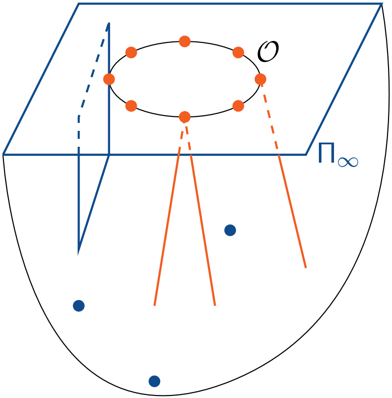

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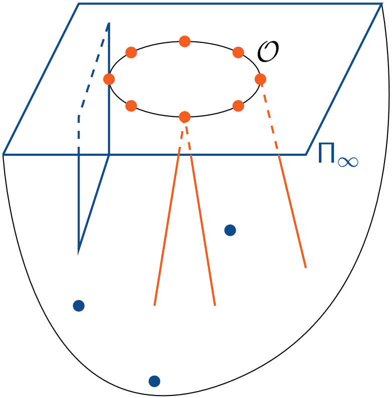
and the natural incidence inherited from PG$(3, q)$.



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Projections of a generalised arc

Definition

Let $\mathcal{A} = \{\pi_1, \ldots, \pi_n\}$ be a generalised arc of $(h-1)$-spaces in $\mathrm{PG}(kh-1, q)$. The *projection* of $\mathcal{A}$ from π_j is constructed as follows.

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If $\mathcal{A}$ is associated to the $\mathbb{F}_q$-linear MDS code over $\mathbb{F}_{q^h}$, then $\mathcal{A}'$ is associated to

$$\{(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_n) \,\|\, (c_1, \dots, c_{j-1}, 0, c_{j+1}, \dots, c_n) \in C\}.$$

Generalised arcs through projections

Let $\mathcal{A}$ be a generalised arc of n $(h-1)$-spaces in PG$(3h-1, q)$. Call a generalised arc *linear* if it is contained in a Desarguesian spread.

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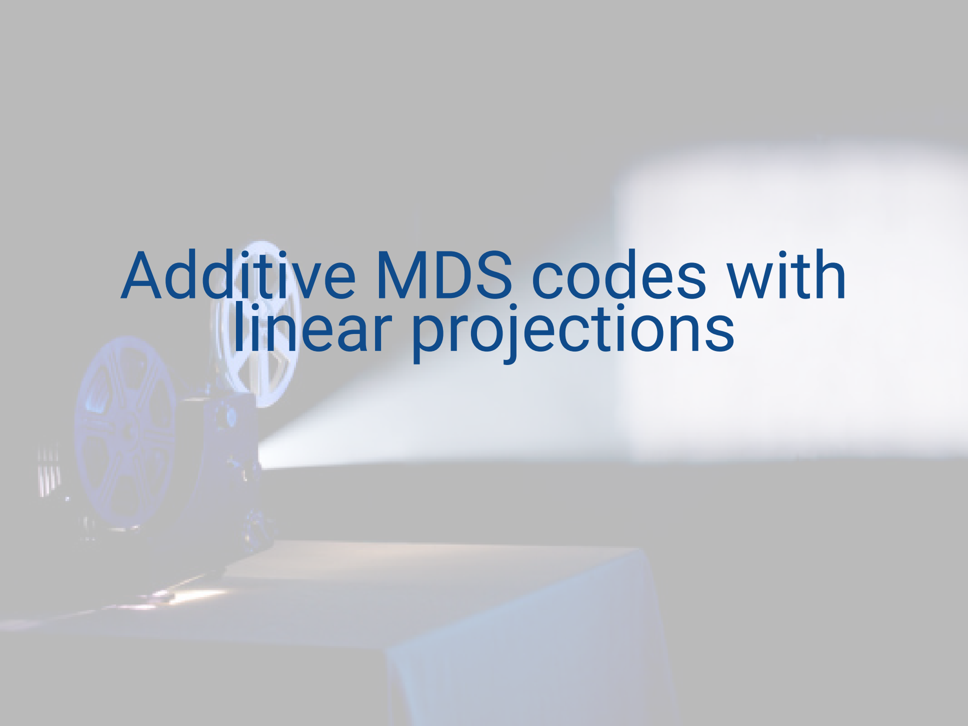
$\mathcal{A}$ is linear if

- (Penttila, Van de Voorde; 2013) q is odd, $n >$ size of the second largest complete arc in $\mathrm{PG}(2,q^h)$, $\mathcal{A}$ has at least 1 linear projection;
- (Rottey, Van de Voorde; 2015) (Thas; 2019) q is even, h is prime, $n = q^h + 1$, all projections of $\mathcal{A}$ are linear.

Additive MDS codes with linear projections



The projection of a code

Definition

Recall that the projection of a code C from the i^{th} coordinate equals

$$\{(\mathbf{c}_1, \mathbf{c}_2) \mid (\mathbf{c}_1, \underbrace{0}_{i^{\text{th}} \text{ coordinate}}, \mathbf{c}_2) \in C\}$$

The case $k > 3$

Theorem (A., Ball; 2022+)

Let C be an $\mathbb{F}_q$-linear $(n, q^{kh}, n-k+1)_{q^h}$ MDS code over $\mathbb{F}_{q^h}$. Suppose that

- $k > 3$,
- $n \geq q + k$,
- *there are two coordinates from which the projection of C is $\mathbb{F}_q$-equivalent to a linear code.*

Then C is $\mathbb{F}_q$-equivalent to an $\mathbb{F}_{q^s}$-linear code (for some $1 < s | h$).

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Corollary

If the above conditions hold and $n \geq q^e + k$, with $e = \max\{t < h \mid\mid t|h\}$, then C is equivalent to a linear code.

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Theorem (A., Ball; 2022+)

Suppose that C is an $\mathbb{F}_q$-linear $(n, q^{3h}, n-2)_{q^h}$ MDS code over $\mathbb{F}_{q^h}$, and suppose that

- $h \in \{2, 3\}$,
- $n \geq \max\{q^{h-1}, hq - 1\} + 4$,
- *There are 3 coordinates from which the projection of C is $\mathbb{F}_q$-equivalent to a linear code.*

Then C is $\mathbb{F}_q$-equivalent to a linear code.

Conclusion

We supported some evidence that if an additive MDS code over a finite field exists such that

- it is reasonably long,
- it is in a sense close to being (essentially) a linear code,

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Progress in this direction might help reduce the additive MDS conjecture to the linear MDS conjecture.


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