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FACULTY OF MATHEMATICS AND PHYSICS
DEPARTMENT OF MATHEMATICS

Vito Vitrih

CORRECT INTERPOLATION PROBLEMS IN MULTIVARIATE POLYNOMIAL SPACES

Doctoral thesis

ADVISER: Prof. Dr. Jernej Kozak

COADVISER: Doc. Dr. Emil Žagar

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UNIVERZA V LJUBLJANI
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Vito Vitrih

KOREKTNI INTERPOLACIJSKI PROBLEMI V PROSTORIH POLINOMOV VEČ SPREMENLJIVK

Doktorska disertacija

MENTOR: Prof. Dr. Jernej Kozak

SOMENTOR: Doc. Dr. Emil Žagar

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Abstract

In the thesis, correct interpolation problems in multivariate polynomial spaces are considered. A general multivariate Lagrange interpolation problem, interpolation spaces, unisolvent sets of interpolation points and remainder formulas are outlined in the introduction, and main results about multivariate interpolation are presented. Sets of interpolation points, which imply correct interpolation problems in the space of polynomials in d variables of total degree $\leq n$, are considered. Among them, $(d+1)$-*pencil lattices* are particularly useful in practice, and are studied in detail. In Chapter 2, the barycentric representation of a $(d+1)$-pencil lattice on a simplex in $\mathbb{R}^d$ is derived. The representation provides shape parameters of a lattice having a clear geometric interpretation. Furthermore, the Lagrange polynomial interpolant is presented. In the next chapter, these results are extended to a global $(d+1)$-pencil lattice on a regular simply connected simplicial partition in $\mathbb{R}^d$. The global lattice provides at least continuous piecewise polynomial Lagrange interpolant over the partition, since lattice points coincide on common faces of adjacent simplices. The number of degrees of freedom of such a lattice is equal to the number of vertices of a simplicial partition. Non-simply connected simplicial partitions are also studied. It is shown, that this property can be used to increase the flexibility of a lattice on such a partition. Since in some applications slight changes in the topology of a partition may appear after the construction of a lattice, the problem, how to extend a lattice over a hole, is considered. In the last chapter, Newton-Cotes cubature rules over $(d+1)$-pencil lattices on a simplex are studied. These rules are combined with an adaptive algorithm and are applied on simplicial partitions. A subdivision step, that refines a $(d+1)$-pencil lattice on a simplex, is studied in detail. The additional freedom of $(d+1)$-pencil lattices may be used to decrease the number of function evaluations significantly.

Key-words: interpolation, polynomial, multivariate, lattice, barycentric coordinates, simplex, simplicial partition, integration, cubature rule, adaptiveness.

Math. Subj. Class. (2000): 41A05, 41A63, 65D05, 65D07.

Povzetek

V disertaciji so obravnavani korektni interpolacijski problemi v prostorih polinomov več spremenljivk. V uvodu so predstavljeni splošen večrazsežen Lagrangeev interpolacijski problem, interpolacijski prostori, unisolventne množice interpolacijskih točk in formule za napako. Podani so najpomembnejši rezultati s tega področja. Posebna pozornost je namenjena množicam interpolacijskih točk, ki porodijo korektnost interpolacijskih problemov v prostoru polinomov d spremenljivk, skupne stopnje $\leq n$. Podrobno so predstavljene *mreže* $d+1$ *šopov*, ki so še posebej uporabne v praksi. V drugem poglavju je izpeljana baricentrična predstavitev mreže $d+1$ šopov na simpleksu v $\mathbb{R}^d$. Predstavitev temelji na parametrih z jasno geometrijsko interpretacijo. Izpeljan je tudi Lagrangeev interpolacijski polinom. V naslednjem poglavju so ti rezultati posplošeni na globalno mrežo $d+1$ šopov na regularni enostavno povezani simplicialni particiji v $\mathbb{R}^d$. Ker se točke globalne mreže ujemajo na skupnih licih sosednjih simpleksov particije, nam to zagotavlja vsaj zveznost odsekoma polinomskega interpolanta nad particijo. Število prostostnih stopenj globalne mreže je enako številu vozlišč simplicialne particije. V nadaljevanju so obravnavane tudi simplicialne particije, ki niso enostavno povezane. Izkaže se, da lahko ta lastnost particije poveča fleksibilnost mreže na njej. Ker lahko v aplikacijah pride do nepredvidljivih naknadnih topoloških sprememb particije, je v disertaciji obravnavan tudi problem, kako razširiti globalno mrežo preko luknje particije. V zadnjem poglavju so izpeljana Newton-Cotesova kubaturna pravila nad mrežami $d+1$ šopov na simpleksu. Ta pravila so s pomočjo adaptivnega algoritma razširjena tudi na simplicialne particije. Pri tem ima ključno vlogo subdivizijski korak, ki zgosti mrežo $d+1$ šopov na simpleksu. Če je število izračunov funkcijskih vrednosti ključnega pomena, lahko dodatna svoboda mrež $d+1$ šopov pripomore k občutnemu zmanjšanju le-teh.

Ključne besede: interpolacija, polinom, večdimenzionalen, mreža, baricentrične koordinate, simpleks, simplicialna particija, integracija, kubaturno pravilo, adaptivnost.

Math. Subj. Class. (2000): 41A05, 41A63, 65D05, 65D07.

Contents

Chapter 1

Introduction

The approximation theory is one of the classical topics of numerical analysis. It is a basis for numerical algorithms in various fields of applied mathematics. Polynomial interpolation is particularly important, since it offers a closed form approximation function, which can be used in implementations. Polynomials are the most easily handled in practice, since they can be represented by finite information, evaluated in finite number of basic operations and easily integrated and differentiated. Thus, there is a wide field of applications for polynomial interpolation in several variables, such as surface reconstruction, cubature rules, finite elements, optimization...

A classical problem of approximation theory is the univariate *Lagrange interpolation problem*. For a given set of interpolation points (also called nodes, knots or parameters) $x_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, and given data y_i, $i = 0, 1, \ldots, n$, one has to find a polynomial p, such that

$$p(x_i) = y_i, \quad i = 0, 1, \ldots, n.$$

It is well-known, that the problem has a unique solution (is correct) for any data y_i iff the interpolation points x_i, $i = 0, 1, \ldots, n$, are pairwise distinct. If some of the points coalesce, we interpolate derivatives at these points. This problem is called the *Hermite interpolation problem*. There are several ways how to express an interpolant. While the Lagrange formula is appropriate only for Lagrange interpolation, the Newton formula can be used also for the Hermite interpolation.

Although the univariate interpolation theory is very well understood, this is not the case for the multivariate one. Here the problems arise even for such fundamental questions as the existence and the uniqueness of an interpolant. Let us first present some standard notation used in multivariate problems. The dimension of the space will be denoted by d, an arbitrary point in $\mathbb{R}^d$ by $\boldsymbol{x} := (x_1, x_2, \ldots, x_d)^T$, and the space of all d-variate polynomials with real coefficients by Π^d. Further, the subspace of polynomials of total degree at most n, will be denoted by Π_n^d. It is formed by polynomials

$$p(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}| \leq n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}},$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)^T$, $\alpha_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is a multiindex vector with the *length* $|\boldsymbol{\alpha}| := \sum_{i=1}^{d} \alpha_i$, and $c_{\boldsymbol{\alpha}} \in \mathbb{R}$. Moreover, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ denotes the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. It is easy to prove the following lemma.

LEMMA 1.1. *The dimension of the space* Π_n^d *is equal to* $\binom{n+d}{d}$.

Proof. Since there are $\binom{k+d-1}{k}$ ways in which k undistinguishable balls (exponents) can be distributed into d distinguishable boxes (variables), it follows

$$\dim \Pi_n^d = \sum_{k=0}^{n} \binom{k+d-1}{k} = \binom{n+d}{d}.$$

□

We will also need the following notation:

- $\boldsymbol{\beta} \leq \boldsymbol{\alpha} \ \Leftrightarrow \ \beta_i \leq \alpha_i, \quad i = 1, 2, \ldots, d,$
- $\boldsymbol{\alpha}! := \alpha_1! \, \alpha_2! \cdots \alpha_d!,$
- $(\boldsymbol{x} + \boldsymbol{y})^{\boldsymbol{\alpha}} := \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}, \quad \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \dfrac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}! \, (\boldsymbol{\alpha} - \boldsymbol{\beta})!},$
- $D^{\boldsymbol{\alpha}} := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d}, \quad D_i^{\alpha_i} := \dfrac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}.$

The multivariate Lagrange interpolation problem can now be stated similarly as the univariate one.

DEFINITION 1.2. *For a given N-dimensional subspace $\mathcal{P} \subset \Pi^d$, a given set of distinct interpolation points $\boldsymbol{x}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$, and given data $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, a polynomial $p \in \mathcal{P}$, for which*

$$p(\boldsymbol{x}_i) = y_i, \quad i = 1, 2, \ldots, N,$$

is called a Lagrange interpolating polynomial for the given interpolation space, points and data.

Usually the data y_i, $i = 1, 2, \ldots, N$, are sampled from some function $f : \mathbb{R}^d \to \mathbb{R}$, and the definition can be reformulated to

DEFINITION 1.3. *For a given N-dimensional subspace $\mathcal{P} \subset \Pi^d$, a given set of distinct interpolation points $\boldsymbol{x}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$, and a given function $f : \mathbb{R}^d \to \mathbb{R}$, a polynomial $p \in \mathcal{P}$, for which*

$$p(\boldsymbol{x}_i) = f(\boldsymbol{x}_i), \quad i = 1, 2, \ldots, N,$$

is called a Lagrange interpolating polynomial for the given interpolation space, points and function f.

The phrase multivariate polynomial interpolation has been first used in 1860 and 1865 by W. Borchardt and L. Kronecker. It was also mentioned in the work of the Prussian Academy of Sciences, the Encyklopädie der Mathematischen Wissenschaften, where one type of the multivariate interpolation, namely (tensor) products of sine and cosine functions in two variables, has been presented. The French counterpart, the Encyclopédie de Sciences Mathematiques, also contains a section on interpolation. They have the following opinion: "It is clear that the interpolation of functions of several variables does not demand any new principles because in the above exposition the fact that the variable was unique has not played frequently any role." In spite of this negative assessment, the multivariate polynomial interpolation has received increasing further attention. More about the history in the field of the multivariate polynomial interpolation can be found in [29].

In the univariate interpolation, the space of polynomials Π_n^1 is an example of a Haar space.

DEFINITION 1.4. *An N-dimensional linear subspace V of all continuous functions is called a Haar space of order N, if for any set of N pairwise distinct interpolation points $\boldsymbol{x}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$, and data $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, there exists a unique function $f \in V$, such that*

$$f(\boldsymbol{x}_i) = y_i, \quad i = 1, 2, \ldots, N.$$

Unfortunately, there are no Haar spaces of order greater than one for the d-dimensional case, where $d > 1$. This is one of the most significant differences between the univariate and the multivariate interpolation.

While in the univariate case $n+1$ points will always be interpolated by polynomials from the space Π_n^1, it is not clear, which interpolation subspace to choose in the multivariate case for a given set of interpolation points. Namely, the dimensions of the standard polynomial interpolation spaces belong only to some subset of $\mathbb{N}$. For example, for Π_n^d this subset is

$$\left\{ d+1, \binom{d+2}{2}, \binom{d+3}{3}, \ldots \right\} \subseteq \mathbb{N}.$$

Hence it is not possible that the interpolation problem will be correct in such a space for an arbitrary given set of interpolation points. In other words, the first fact is, that the number of interpolation points has to match the dimension of the polynomial interpolation space. But even if this is true, the interpolation problem does not always have a solution or the solution is not necessarily unique. Consider the following examples:

- Take three interpolation points in the plane. If they are not collinear, we can interpolate at those points by a unique polynomial from the space Π_1^2. Suppose now that they are collinear. Then the given function f, which we interpolate, has to be linear over the line containing interpolation points, otherwise no interpolating polynomial exists. But on the other hand, if f is linear over this line, there are infinitely many different interpolants.

- Take now 6 planar interpolation points on a unit circle $x^2 + y^2 - 1 = 0$. Suppose that $p \in \Pi_2^2$ is an interpolating polynomial for given data at those points. Then $p(x, y) + x^2 + y^2 - 1$ is an another interpolating polynomial for the same data (Figure 1.1).

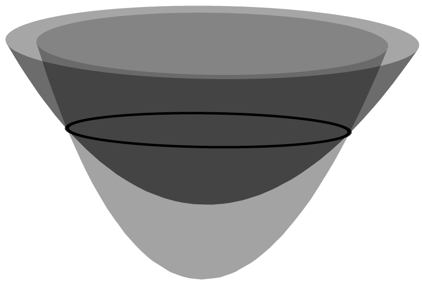


Figure 1.1: Two different interpolants from Π_2^2 for the data on a unit circle.

In general we have the following definition.

DEFINITION 1.5. *Let $\mathcal{P} \subset \Pi^d$ be an N-dimensional interpolation subspace. The Lagrange interpolation problem on a set of N interpolation points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N \in \mathbb{R}^d$ is called correct in $\mathcal{P}$, if for any data $y_1, y_2, \ldots, y_N \in \mathbb{R}$, there exists a unique polynomial $p \in \mathcal{P}$, such that $p(\boldsymbol{x}_i) = y_i,\ i = 1, 2, \ldots, N$.*

Some authors rather use terms *poised* or *unisolvent* instead of correct. The term *unisolvent set* for the set of interpolation points is also very common. It is straightforward to prove the following two theorems (see [31], e.g.).

THEOREM 1.6. *The Lagrange interpolation problem with respect to interpolation points X is correct in a space $\mathcal{P} \subset \Pi^d$ iff the interpolation points X do not lie on any algebraic hypersurface, with the polynomial, which represents the hypersurface in the implicit form, being in $\mathcal{P}$.*

THEOREM 1.7. *The Lagrange interpolation problem with respect to interpolation points X is correct in a space $\mathcal{P} \subset \Pi^d$ iff the Vandermonde matrix of the linear system*

$$\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}_i^{\boldsymbol{\alpha}} = f(\boldsymbol{x}_i), \quad \boldsymbol{x}_i \in X,$$

is nonsingular.

Since the main emphasis of the thesis will be given to the multivariate Lagrange interpolation, we have only considered this case until now. But it seems this is the right point to say something about the multivariate Hermite interpolation, too. When some of the interpolation points coalesce in the univariate case, the interpolating polynomials converge to the Hermite interpolating polynomial which interpolates function values and derivatives. In general, this does not hold in more variables, since here things become even more complicated than in the multivariate Lagrange case. There exist interpolation problems which are unsolvable only for some special selections of interpolation points (as in the Lagrange case) and there are interpolation problems which are generically unsolvable. The simplest example of the latter is the interpolation of a function f and its gradient at two distinct points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $\mathbb{R}^2$. This is the limit case of the Lagrange interpolation problem at six points

$$\boldsymbol{x}_j, \quad \boldsymbol{x}_j + h\,\boldsymbol{e}_1, \quad \boldsymbol{x}_j + h\,\boldsymbol{e}_2, \quad j = 1, 2,$$

where the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are of the form $\boldsymbol{e}_i = (\delta_{i,j})_{j=1}^2$, and $\delta_{i,j}$ is the Kronecker's delta. This Lagrange problem is correct with respect to Π_2^2 for all $h \neq 0$ and almost all choices of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. But, the original Hermite interpolation problem is never correct in Π_2^2, for any choice of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. An interpolation problem is called *singular* for a given space if the problem is not correct for any set of interpolation points. Note that the Lagrange interpolation problems are never singular. For more details on multivariate Hermite interpolation see [45], e.g.

We have seen that studying the multivariate Lagrange polynomial interpolation leads to new questions and problems, which are not encountered in the univariate situation. There are actually two important points of view. In the first approach, the interpolation points are given in advance, for example they come from some measurements of physical quantities, and an interpolation space, which gives rise to the correct interpolation problem is searched for. In the second approach, the problem how to construct the interpolation points that admit the correct interpolation problem for the given interpolation space (in particular for Π_n^d) is studied.

From interpolation points to an interpolation space

Let us first consider the situation, where a given set of interpolation points $X := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ does not allow a correct interpolation in Π_n^d for any n. This can be due to the inappropriate number of points, or because the points lie on an algebraic hypersurface of sufficiently low degree. Therefore, an another type of interpolation subspace in Π^d is searched for. However, it can be proved that there always exist several subspaces in Π_N^d which admit a unique interpolation, so one has to impose further restrictions to the interpolation space. Let us introduce the following class of interpolation spaces.

DEFINITION 1.8. *Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\} \subset \mathbb{R}^d$ be a set of interpolation points. A polynomial space $\mathcal{P} \subset \Pi_N^d$ is called a minimal degree interpolation space with respect to X if*

- *$\mathcal{P}$ admits a correct interpolation, i.e., for any function $f : X \to \mathbb{R}$ there exists a unique polynomial $p \in \mathcal{P}$, such that $p(\boldsymbol{x}_i) = f(\boldsymbol{x}_i)$, $i = 1, 2, \ldots, N$.*

- *$\mathcal{P}$ is of a minimal degree n, i.e., $\mathcal{P} \subset \Pi_n^d$ and there exists no subspace in Π_{n-1}^d, which admits correct interpolation.*

- *$\mathcal{P}$ is degree reducing, i.e., if $q \in \Pi^d$ is any polynomial, then the degree of its interpolant from $\mathcal{P}$ w.r.t. interpolation points X is not larger than the degree of q.*

For a given set X of interpolation points there usually exist several minimal degree interpolation spaces. There is only one exception: the minimal degree interpolation space $\mathcal{P}$ is unique if and only if $\mathcal{P} = \Pi_n^d$ for some $n \in \mathbb{N}$. Except for this case, we can come from one minimal degree interpolation space to another by knowing the so-called Newton basis of the interpolation space.

DEFINITION 1.9. *Let $X \subset \mathbb{R}^d$ be a set of interpolation points. If there exist two subsets $I_n, I'_n \subset \{\boldsymbol{\alpha},\ |\boldsymbol{\alpha}| \leq n\}$, an indexation $X = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in I_n\}$ and polynomials $\mathcal{N}_{\boldsymbol{\alpha}}$, such that*

- $\mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\beta}}) = \begin{cases} 1, & \boldsymbol{\alpha} = \boldsymbol{\beta} \\ 0, & \text{otherwise} \end{cases}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in I_n,\ |\boldsymbol{\beta}| \leq |\boldsymbol{\alpha}|,$

- $\mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\beta}}) = 0, \quad \boldsymbol{\alpha} \in I'_n,\ \boldsymbol{\beta} \in I_n,$

- $\{\mathcal{N}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in I_n \cup I'_n\}$ *is a basis of* Π_n^d,

then the set of polynomials $\{\mathcal{N}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in I_n\}$ is called a Newton basis for X.

Moreover, the polynomials $\mathcal{N}_{\boldsymbol{\alpha}}$ are called the *Newton fundamental polynomials*. The following theorem indicates that the minimal degree interpolation spaces and Newton bases are deeply connected ([42]).

THEOREM 1.10. *A polynomial space $\mathcal{P} \subset \Pi_n^d$ is a minimal degree interpolation space with respect to X if and only if it is spanned by a Newton basis for X.*

Let $\mathcal{P} \subset \Pi_n^d$ be a minimal degree interpolation space with respect to interpolation points X and let $\mathcal{N}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in I_n$, be a Newton basis for $\mathcal{P}$. Then the set of polynomials

$$\{\mathcal{N}_{\boldsymbol{\alpha}} + q_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in I_n,\ q_{\boldsymbol{\alpha}} \in \Pi_{|\boldsymbol{\alpha}|}^d,\ q_{\boldsymbol{\alpha}}(X) = 0\}$$

is an another Newton basis with respect to X and any Newton basis can be obtained in this way. Hence, the Newton basis and the minimal degree interpolation space $\mathcal{P}$ are unique if and only if $\Pi_n^d \cap \{q,\ q(X) = 0\} = \{0\}$.

Let us now consider two examples of minimal degree interpolation spaces. Suppose that

$$\begin{aligned} X_1 &= \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2)\}, \\ X_2 &= \{(0,0),(0,1),(1,0),(0,2),(1,1),(0,3),(1,2),(1,3)\} \end{aligned}$$

are two sets of interpolation points. It is trivial to see that the minimal interpolation space for the points X_2 can not be a subset in Π_3^2. Namely, if we would extend X_2 with any two points to X_2', interpolation points from X_2' would not imply correct interpolation in Π_3^2, since they would lie on an algebraic curve of degree 3 (product of three lines). Let

$$I_3 = \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2)\}, \quad I_3' = \{(2,1),(3,0)\},$$

and

$$\begin{aligned} I_4 &= \{(0,0),(0,1),(1,0),(0,2),(1,1),(0,3),(1,2),(1,3)\}, \\ I_4' &= \{(2,0),(2,1),(3,0),(2,2),(3,1),(4,0),(0,4)\}. \end{aligned}$$

Then

$$X_1 = \{x_{\boldsymbol{\alpha}} = \boldsymbol{\alpha},\ \ \boldsymbol{\alpha} \in I_3\} \quad \text{and} \quad X_2 = \{x_{\boldsymbol{\alpha}} = \boldsymbol{\alpha},\ \ \boldsymbol{\alpha} \in I_4\}.$$

Moreover, the Newton polynomials are equal to

$$\begin{aligned} &\mathcal{N}_{(0,0)}(u,v) = 1,\ \ \mathcal{N}_{(0,1)}(u,v) = v,\ \ \mathcal{N}_{(1,0)}(u,v) = u,\ \ \mathcal{N}_{(0,2)}(u,v) = \frac{1}{2}\,v(v-1), \\ &\mathcal{N}_{(1,1)}(u,v) = uv,\ \ \mathcal{N}_{(2,0)}(u,v) = \frac{1}{2}\,u(u-1),\ \ \mathcal{N}_{(0,3)}(u,v) = \frac{1}{6}\,v(v-1)(v-2), \\ &\mathcal{N}_{(1,2)}(u,v) = \frac{1}{2}\,uv(v-1),\ \ \mathcal{N}_{(1,3)}(u,v) = \frac{1}{6}\,uv(v-1)(v-2), \end{aligned}$$

thus

$$\begin{aligned} \mathcal{P}(X_1) &= \mathrm{Lin}\,\{\mathcal{N}_{\boldsymbol{\alpha}},\ \ \boldsymbol{\alpha} \in I_3\} = \mathrm{Lin}\,\{1,u,v,u^2,uv,v^2,uv^2,v^3\}, \\ \mathcal{P}(X_2) &= \mathrm{Lin}\,\{\mathcal{N}_{\boldsymbol{\alpha}},\ \ \boldsymbol{\alpha} \in I_4\} = \mathrm{Lin}\,\{1,u,v,uv,v^2,uv^2,v^3,uv^3\}. \end{aligned}$$

For more information on this topic see [2], [30], [42] and [43].

From an interpolation space to interpolation points

Since it is hard to verify the algebraic characterization of the correctness, given in Theorem 1.6, for example in the floating point arithmetic, many researchers (e.g., C. de Boor, J. M. Carnicer, K. C. Chung, M. Gasca, J. Maeztu, G. M. Phillips, T. Sauer, T. H. Yao) put their effort into finding appropriate sets of interpolation points, which will, for a given interpolation space (Π_n^d, e.g.), imply the correctness of the interpolation problem in advance.

We will now restrict the discussion to the most important interpolation space in practice, namely to Π_n^d. Let us present some best-known and most often used techniques for choosing sets of interpolation points $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$, such that the interpolation problem with respect to these points will be correct in Π_n^d. Clearly, this requires

$$N = \dim \Pi_n^d = \binom{n+d}{d}.$$

The first and the most natural approach how to choose such interpolation points are *principal lattices* on simplices in $\mathbb{R}^d$ (for $d=2$ see Figure 1.2, left), where the points are intersections of $d+1$ pencils of $n+1$ *parallel* hyperplanes. Each point is an intersection of $d+1$ hyperplanes, one from each pencil. Clearly, the number of points obtained this way is $\binom{n+d}{d}$. The barycentric coordinates of lattice points w.r.t. vertices of a simplex are

$$\left\{\frac{1}{n}\boldsymbol{\alpha},\ \ \boldsymbol{\alpha}\in\mathbb{N}_0^{d+1},\ |\boldsymbol{\alpha}|=n\right\}.$$

In such a form, these lattices were first introduced in [40]. This paper apparently motivated the construction in the paper [16], where K. C. Chung and T. H. Yao introduced a very important property, called geometric characterization condition (see Figure 1.2).

DEFINITION 1.11. *A set of interpolation points $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_N\}$ satisfies the geometric characterization (GC) condition, if for each point $\boldsymbol{x}_i\in X$ there exist hyperplanes $H_{i,j}$, $j=1,2,\ldots,n$, such that $\boldsymbol{x}_i$ is not on any of these hyperplanes, and all points of $X\backslash\{\boldsymbol{x}_i\}$ lie on at least one of them. More precisely,*

$$\boldsymbol{x}_\ell\in\bigcup_{j=1}^{n}H_{i,j}\ \Leftrightarrow\ i\neq\ell,\quad i,\ell=1,2,\ldots,N.$$

Sometimes we will rather write GC_n instead of just GC, in order to emphasize the number of hyperplanes, associated with a particular point. Moreover, let $\Gamma_X:=\{H_{i,j},\ i=1,2,\ldots,N,\ j=1,2,\ldots,n\}$.

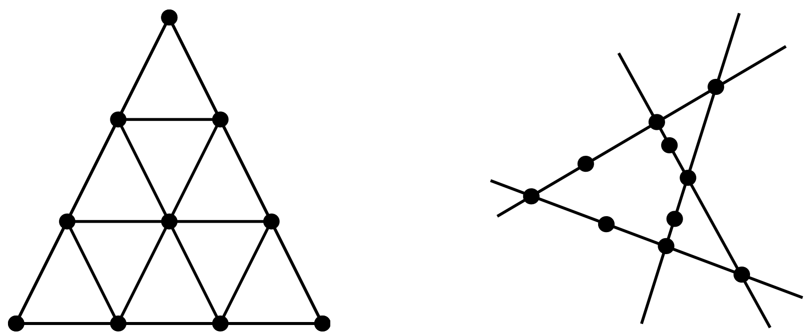


Figure 1.2: An example of a principal lattice (left) and an another lattice satisfying the GC condition (right).

In the planar case, these sets have some interesting properties (see [8], e.g.).

PROPOSITION 1.12. *Let X, $|X|=\binom{n+2}{2}$, be a set of interpolation points, which satisfies the GC_n condition. Then*

(a) $|\Gamma_X|\geq n+2$ and each line from Γ_X contains at least two points from X;

(b) no line contains more than $n+1$ points from X;

(c) two lines, containing $n+1$ points from X, meet at a point from X;

(d) three lines, containing $n+1$ points from X, are not concurrent;

(e) there are at most $n+2$ lines containing $n+1$ points from X.

The next theorem shows, why the GC_n condition is useful.

THEOREM 1.13. *Let the set of interpolation points $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N\}$, $N = \binom{n+d}{d}$, satisfy the GC_n condition. Then X admits a correct interpolation in the space Π_n^d.*

Proof. Let $h_{i,j}(\cdot) = 0$ be the equation of the hyperplane $H_{i,j}$. Then

$$p = \sum_{i=1}^{N} f(\boldsymbol{x}_i) \prod_{j=1}^{n} \frac{h_{i,j}}{h_{i,j}(\boldsymbol{x}_i)}$$

is the explicit solution of the Lagrange interpolation problem (for an arbitrary function f) w.r.t. X in Π_n^d. Since $N = \binom{n+d}{d}$, the theory of systems of linear equations yields the uniqueness of the interpolant. □

REMARK 1.14. *The Lagrange fundamental polynomials*

$$\mathcal{L}_{i,n} = \prod_{j=1}^{n} \frac{h_{i,j}}{h_{i,j}(\boldsymbol{x}_i)},$$

where $h_{i,j}(\cdot) = 0$ is the equation of the hyperplane $H_{i,j} \in \Gamma_X$, are the products of linear polynomials. Although this is always true in the univariate situation, it only holds for the sets satisfying GC condition in the multivariate case. However, it is a very important property for the implementation.

It is not difficult to see, that principal lattices satisfy the GC condition, so they assure a correct interpolation in Π_n^d in advance. Let us now introduce additional two interesting classes of GC sets (see [14], [15], [27], e.g.).

DEFINITION 1.15. *A natural lattice of order n in $\mathbb{R}^d$ (for $d = 2$ see Figure 1.3, left) is a set of $\binom{n+d}{d}$ points*

$$X = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\ \ \boldsymbol{\alpha} \in \mathbb{N}^d,\ \alpha_i \in \{i, i+1, \dots, i+n\},\ \alpha_i < \alpha_{i+1}\},$$

for which there exist pairwise distinct hyperplanes $(H_i)_{i=1}^{n+d}$, such that each $\boldsymbol{x}_{\boldsymbol{\alpha}} \in X$ is obtained as

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \bigcap_{i=1}^{d} H_{\alpha_i}.$$

DEFINITION 1.16. *A generalized principal lattice of order n in $\mathbb{R}^d$ (for $d = 2$ see Figure 1.3, right) is a set of $\binom{n+d}{d}$ points*

$$X = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\ \ \boldsymbol{\alpha} \in \mathbb{N}_0^{d+1},\ |\boldsymbol{\alpha}| = n\},$$

for which there exist $d+1$ pencils of $n+1$ hyperplanes $(H_{i,r})_{r=0}^{n}$, $i = 0, 1, \dots, d$, such that each $\boldsymbol{x}_{\boldsymbol{\alpha}} \in X$ is obtained as

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \bigcap_{i=0}^{d} H_{i,\alpha_i}.$$

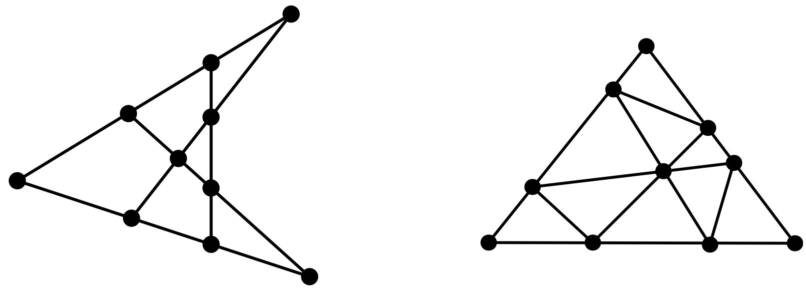


Figure 1.3: Examples of a natural lattice (left) and a generalized principal lattice (right).

In the plane, we have the following proposition.

PROPOSITION 1.17. *Let $X \subset \mathbb{R}^2$, $|X| = \binom{n+2}{2}$, be a set of points, which satisfies the GC_n condition.*

(a) If X is a generalized principal lattice of order n, then there exist exactly three lines containing $n+1$ points from X.

(b) If there are exactly three lines containing $n+1$ points from X and if $n \le 7$, then X is a generalized principal lattice of order n.

A special and a very important example of generalized principal lattices are so-called $(d+1)$-*pencil lattices* (for $d = 2$ see Figure 1.4, left), where the hyperplanes of each pencil intersect in a *center*, which is a plane of codimension two. These lattices will be described in detail in the next chapter.

Another sets of interpolation points, which assure correct interpolation in advance, are the so-called *decreasing hyperplanes (DH) sets*. In the planar case, the points of a DH (or DH$_n$) set X are lying on $n+1$ lines $L_0, L_1, \ldots, L_n$, such that

$$|L_i \cap X \backslash (L_0 \cup L_1 \cup \cdots \cup L_{i-1})| = n + 1 - i, \quad i = 0, 1, \ldots, n.$$

There exists a well-known conjecture connecting DH and GC sets in the planar case. It was stated in [28].

CONJECTURE 1.18. *If a set $X \subset \mathbb{R}^2$, $|X| = \binom{n+2}{2}$, satisfies the GC_n condition, then it is a DH_n set.*

On the other hand, it is trivial to find an example of a DH set, which does not satisfy the GC condition (Figure 1.4, right). Conjecture 1.18 can be rewritten to an equivalent conjecture.

CONJECTURE 1.19. *Let $X \subset \mathbb{R}^2$, $|X| = \binom{n+2}{2}$, be a set satisfying the GC_n condition. Then there exists a line containing $n+1$ points from X.*

If the conjecture holds, then there are at least three such lines ([12]). There are a lot of papers concerning this conjecture (e.g., [5], [7], [8], [9], [10], [11], [12], [13], [35]), but it still remains unconfirmed for $n > 4$.

Recently, an another family of unisolvent sets, called *Padua points*, was introduced. For more details on these sets of interpolation points see [3],[4] and [6].

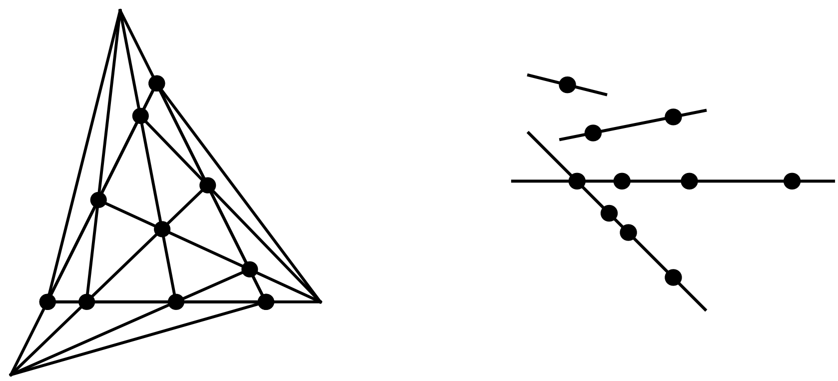


Figure 1.4: A special case of a generalized principal lattice, $(d+1)$-pencil lattice, and an example of a DH lattice, which does not satisfy the GC condition.

Remainder formula

Here we will describe an approach to obtain the remainder formula for the multivariate interpolation. Recall first the well-known univariate Newton approach. The main idea is to solve the interpolation problem by beginning with a very simple subproblem and then successively add more and more points to the interpolation problem and increase the degree of the polynomial at the same time. Using the principle of divided differences $[x_0, x_1, \ldots, x_n]\, f$, the Newton form of the univariate interpolant and the remainder formula can be written as

$$p(x) = \sum_{j=0}^{n} [x_0, x_1, \ldots, x_j]\, f \prod_{i=0}^{j-1} (x - x_i), \quad f(x) - p(x) = [x_0, x_1, \ldots, x_n, x]\, f \prod_{i=0}^{n} (x - x_i).$$

If we try to extend the idea of the Newton interpolation to the multivariate case, we have at least two options: we may either add one point at each step, or increase the degree of the interpolation polynomial by one at each step, which corresponds to adding not one but $\binom{k+d-1}{d-1}$ points at a time. This latter strategy is called *blockwise Newton interpolation* and has been introduced in [44]. Suppose that distinct points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N \in \mathbb{R}^d$, $N = \binom{n+d}{d}$, are given, which admit unique polynomial interpolation of total degree at most n. Since the interpolation points can be re-indexed as

$$X = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\ |\boldsymbol{\alpha}| \le n\},$$

in such a way that all interpolation problems based on the nested subsets

$$X_k = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\ |\boldsymbol{\alpha}| \le k\}$$

are correct in Π_k^d, $k = 0, 1, \ldots, n$, there exist Newton fundamental polynomials

$$\mathcal{N}_{\boldsymbol{\alpha}} \in \Pi_{|\boldsymbol{\alpha}|}^d, \quad |\boldsymbol{\alpha}| \le n,$$

such that

$$\mathcal{N}_{\boldsymbol{\alpha}}(x_{\boldsymbol{\beta}}) = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \quad |\boldsymbol{\beta}| \le |\boldsymbol{\alpha}| \le n.$$

With these at hand, we are able to construct *finite differences*

$$\lambda_{k+1}[X_k, \boldsymbol{x}]\, f, \quad k = -1, 0, \ldots, n,$$

as

$$\lambda_0[\boldsymbol{x}]\, f = f(\boldsymbol{x}), \quad \lambda_{k+1}[X_k, \boldsymbol{x}]\, f = \lambda_k[X_{k-1}, \boldsymbol{x}]\, f - \sum_{|\boldsymbol{\alpha}|=k} \lambda_k[X_{k-1}, \boldsymbol{x}_{\boldsymbol{\alpha}}]\, f \cdot \mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{x}).$$

THEOREM 1.20. *Let the Lagrange interpolation problem with respect to X be correct in Π_n^d. Then the interpolant and the remainder formula can be written as*

$$p(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|\le n} \lambda_{|\boldsymbol{\alpha}|}[X_{|\boldsymbol{\alpha}|-1}, \boldsymbol{x}_{\boldsymbol{\alpha}}]\, f \cdot \mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{x}), \qquad f(\boldsymbol{x}) - p(\boldsymbol{x}) = \lambda_{n+1}[X_n, \boldsymbol{x}]\, f.$$

Computationally, we first generate the Newton fundamental polynomials by a Gram-Schmidt orthogonalization process and then compute the finite differences by a triangular scheme, similar to the one for univariate divided differences.

As an example let us consider the principal lattice

$$X = \{(0,0), (0,1), (1,0), (0,2), (1,1), (2,0)\}$$

of order 2 in $\mathbb{R}^2$. Then

$$X_0 = \{(0,0)\},\ \ X_1 = \{(0,1), (1,0)\},\ \ X_2 = \{(0,2), (1,1), (2,0)\},$$

and the Newton fundamental polynomials become

$$\begin{aligned}
&\mathcal{N}_{(0,0)}(u,v) = 1,\ \ \mathcal{N}_{(0,1)}(u,v) = v,\ \ \mathcal{N}_{(1,0)}(u,v) = u,\\
&\mathcal{N}_{(0,2)}(u,v) = \frac{1}{2}\, v(v-1),\ \ \mathcal{N}_{(1,1)}(u,v) = uv,\ \ \mathcal{N}_{(2,0)}(u,v) = \frac{1}{2}\, u(u-1).
\end{aligned}$$

We can now compute finite differences

$$\begin{aligned}
\lambda_0[(u,v)]\, f &= f(u,v),\\
\lambda_1[X_0,(u,v)]\, f &= f(u,v) - f(0,0),\\
\lambda_2[X_1,(u,v)]\, f &= (u+v-1)f(0,0) - vf(0,1) - uf(1,0) + f(u,v),\\
\lambda_3[X_2,(u,v)]\, f &= \frac{1}{2}\Big((u+v-1)(2-u-v)f(0,0) + v^2(2f(0,1) - f(0,2))\\
&\quad + v\big(2(u-2)f(0,1) + f(0,2) + 2u(f(1,0) - f(1,1))\big)\\
&\quad + u\big(2(u-2)f(1,0) + (1-u)f(2,0)\big)\Big) + f(u,v).
\end{aligned}$$

The density plot of the error, given by Theorem 1.20, for the function

$$f(u,v) = \sqrt{1 - \left(\frac{u}{4}\right)^2 - \left(\frac{v}{4}\right)^2} \tag{1.1}$$

is presented in Figure 1.5.

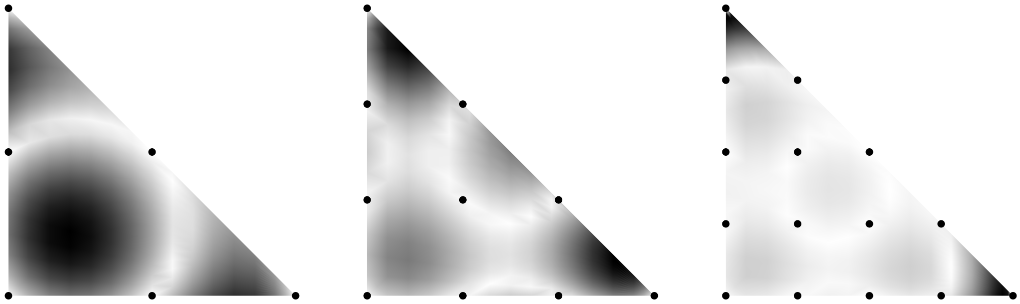


Figure 1.5: Density plots of the interpolation error for the function (1.1) on planar principal lattices of order $2, 3$ and 4, respectively. Darker the colour is, larger the error is.

A remainder formula can now be obtained in a closed form by finding a representation for the finite difference $\lambda_{n+1}[\cdot]\, f$ in terms of certain derivatives of f (see [44]). According to this, we have to introduce some new notation. Let

$$\Xi_n := \{\, \underline{\boldsymbol{\mu}} = (\boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n)^T,\ \ \boldsymbol{\mu}_j \in \mathbb{N}_0^d,\ |\boldsymbol{\mu}_j| = j,\, j = 0, 1, \ldots, n\} \tag{1.2}$$

be an index set. Elements of Ξ_n are called paths. For any path $\underline{\boldsymbol{\mu}} \in \Xi_n$, let us define the quantities

$$X_{\underline{\boldsymbol{\mu}}} := \{\boldsymbol{x}_{\boldsymbol{\mu}_0}, \boldsymbol{x}_{\boldsymbol{\mu}_1} \ldots, \boldsymbol{x}_{\boldsymbol{\mu}_n}\},$$

$$\pi_{\underline{\boldsymbol{\mu}}} := \prod_{j=0}^{n-1} \mathcal{N}_{\boldsymbol{\mu}_j}(\boldsymbol{x}_{\boldsymbol{\mu}_{j+1}}),$$

$$D^n_{\underline{\boldsymbol{\mu}}} := D_{\boldsymbol{x}_{\boldsymbol{\mu}_n} - \boldsymbol{x}_{\boldsymbol{\mu}_{n-1}}} D_{\boldsymbol{x}_{\boldsymbol{\mu}_{n-1}} - \boldsymbol{x}_{\boldsymbol{\mu}_{n-2}}} \cdots D_{\boldsymbol{x}_{\boldsymbol{\mu}_1} - \boldsymbol{x}_{\boldsymbol{\mu}_0}}.$$

Further, let us introduce the simplex spline integral

$$\int_{[\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n]} f := \int_{\Delta^n_{n+1}} f(u_0\, \boldsymbol{y}_0 + u_1\, \boldsymbol{y}_1 + \ldots + u_n\, \boldsymbol{y}_n)\, \mathrm{d}\boldsymbol{u},$$

where

$$\Delta^n_{n+1} := \left\{ \boldsymbol{u} = (u_0, u_1, \ldots, u_n)^T,\ \ u_j \geq 0,\ j = 0, 1, \ldots, n,\ \sum_{j=0}^{n} u_j = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

is an n-simplex in $\mathbb{R}^{n+1}$. Now we can state the following result (see [44]).

THEOREM 1.21. *Let $\Omega \subset \mathbb{R}^d$ be a convex set and let $X \subset \Omega$. Further, let $f \in \mathcal{C}^{n+1}(\Omega)$. Then, for any $\boldsymbol{x} \in \Omega$*

$$f(\boldsymbol{x}) - p(\boldsymbol{x}) = \sum_{\underline{\boldsymbol{\mu}} \in \Xi_n} \mathcal{N}_{\boldsymbol{\mu}_n}(\boldsymbol{x})\, \pi_{\underline{\boldsymbol{\mu}}} \int_{[X_{\underline{\boldsymbol{\mu}}},\, \boldsymbol{x}]} D_{\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{\mu}_n}} D^n_{\underline{\boldsymbol{\mu}}} f.$$

More details on this approach can be found in [31] or [44], e.g.

Some other approaches that can be used to obtain the remainder formula for the multivariate polynomial interpolant are given in [1] and [17], e.g.

In the thesis, an approach how to find appropriate sets of interpolation points for the interpolation polynomial space Π_n^d is studied. The special case of generalized principal lattices, $(d+1)$-pencil lattice, is considered, since this type of lattices is useful in many practical applications, such as interpolation of multivariate functions, numerical methods for multidimensional integrals, finite element methods for solving partial differential equations... In the following chapter, $(d+1)$-pencil lattice on a simplex in $\mathbb{R}^d$ is studied in detail. First, the closed form formula for a 3-pencil lattice on a triangle in the plane is obtained, which is further generalized to the barycentric representation of a $(d+1)$-pencil lattice on a simplex in $\mathbb{R}^d$. In contrast to [38], this representation provides shape parameters of a lattice with a clear geometric interpretation. To conclude the chapter, Lagrange polynomial interpolant over a $(d+1)$-pencil lattice on a simplex and its closed form formula are derived. Chapter 3 extends these results from a $(d+1)$-pencil lattice on a simplex to a global $(d+1)$-pencil lattice on a simplicial partition. The extension is based on the barycentric representation of a $(d+1)$-pencil lattice, given in Chapter 2. It is shown how to construct a global $(d+1)$-pencil lattice on a given regular simply connected simplicial partition with V vertices, such that the lattice points agree on common faces of adjacent simplices. Such a lattice provides at least a continuous piecewise polynomial Lagrange interpolant over the given simplicial partition. It is proved that such a global lattice in the plane and in the space has exactly V degrees of freedom, that can be used as shape parameters. Further, the conjecture, which states that the same holds for any d-dimensional space, is confirmed. The chapter is concluded by observing more general simplicial partitions, which are not simply connected (have holes). Since such simplicial partitions often appear in practice, they have to be considered too. It is shown, how the fact, that a partition is not simply connected, can be used to increase the flexibility of a lattice. On the other hand, a local modification algorithm is proposed also to deal with slight changes in the topology of a partition that may appear after a lattice has already been constructed. In other words, the problem how to extend a lattice over a hole is considered. In Chapter 4, Newton-Cotes cubature rules over $(d+1)$-pencil lattices are studied. Closed form cubature rules as well as error terms are determined. Further, the basic cubature rules are combined with an adaptive algorithm and carried over to simplicial partitions. The key point of the algorithm is a subdivision step that refines a $(d+1)$-pencil lattice on a simplex. Moreover, it is proved, that the additional freedom provided by $(d+1)$-pencil lattices may be used to decrease the number of function evaluations significantly.

The results of the thesis are presented in the papers: [32], [33], [34], [36], [47] and [48]. The first five have already been published, and the last is submitted.

Chapter 2

$(d+1)$-pencil lattice

In this chapter, a $(d+1)$-*pencil lattice* of *order* n on a simplex in $\mathbb{R}^d$, as a special case of generalized principal lattices, will be studied. The lattice consists of $\binom{n+d}{d}$ points on a simplex in $\mathbb{R}^d$, which are generated by particular $d+1$ pencils of $n+1$ hyperplanes. Since $(d+1)$-pencil lattice satisfies the GC condition, the Lagrange interpolating polynomial over the lattice is uniquely determined.

2.1. Definitions

A *d-simplex* (or shortly simplex, when the dimension is known) in $\mathbb{R}^d$ is a convex hull of $d+1$ distinct points $\boldsymbol{T}_i \in \mathbb{R}^d$, $i = 0, 1, \ldots, d$. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron. A single point may be considered as a 0-simplex, and a line segment may be viewed as an 1-simplex. The convex hull of any nonempty subset of points $\boldsymbol{T}_i$, $i = 0, 1, \ldots, d$, is called a *face* of a simplex. Faces are simplices in lower dimensions. The 0-faces are called the vertices, the 1-faces are called the edges, and the $(d-1)$-faces are called the *facets* (see [21], e.g.). In general, the number of k-faces is $\binom{d+1}{k+1}$. Moreover, a k-simplex may be constructed from a $(k-1)$-simplex by connecting a new vertex with all original vertices.

Since for our purposes the ordering of the vertices of a simplex will be important, the notation

$$\triangle := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle,$$

which defines a simplex with a prescribed order of the vertices $\boldsymbol{T}_i$, will be used. The *standard simplex* in $\mathbb{R}^d$ on vertices

$$\boldsymbol{T}_i = (\delta_{i,j})_{j=1}^{d}, \quad i = 0, 1, \ldots, d,$$

where

$$\delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.1}$$

is the Kronecker's delta, will be denoted by

$$\triangle^d := \triangle^d_d \subset \mathbb{R}^d.$$

Furthermore, $\triangle^d_{d+1} \subset \mathbb{R}^{d+1}$ will denote a d-simplex

$$\triangle^d_{d+1} := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle, \quad \boldsymbol{T}_i = (\delta_{i,j})_{j=0}^d, \quad i = 0, 1, \ldots, d. \tag{2.2}$$

A $(d+1)$-*pencil lattice* of *order* n on a simplex $\triangle$ consists of $\binom{n+d}{d}$ points, generated by particular $d+1$ pencils of $n+1$ hyperplanes, such that each lattice point is an intersection of $d+1$ hyperplanes, one from each pencil. Furthermore, each pencil intersects at a center

$$\boldsymbol{C}_i \subset \mathbb{R}^d, \ i = 0, 1, \ldots, d,$$

which is a plane of codimension two. Let us first consider the planar case. A 3-pencil lattice on a triangle $\triangle$ is a set of $\binom{n+2}{2}$ points, which are determined by 3 pencils of $n+1$ lines. Each pencil now intersects at a center, that is a point in the plane, and each lattice point is obtained as an intersection of precisely three lines, one from each pencil (see Figure 2.1).

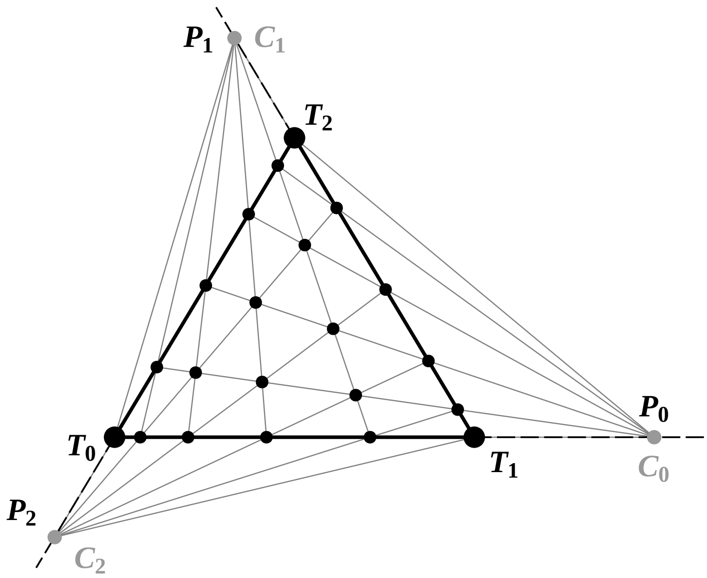


Figure 2.1: A 3-pencil lattice of order 5 on a triangle $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$ in $\mathbb{R}^2$.

Let us now consider higher dimensional cases. In order to determine the positions of centers, a more precise definition of a lattice is needed. The lattice is actually based upon affinely independent control points

$$\boldsymbol{P}_i \in \mathbb{R}^d, \quad i = 0, 1, \ldots, d,$$

where $\boldsymbol{P}_i$ lies on the line through the simplex vertices $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$, outside of the segment $\boldsymbol{T}_i\boldsymbol{T}_{i+1}$ (Figure 2.2). Note, that affine independence of control points $\boldsymbol{P}_i$, $i = 0, 1, \ldots, d$, is equivalent to linear independence of vectors $\boldsymbol{P}_i - \boldsymbol{P}_0$, $i = 1, 2, \ldots, d$. Each center $\boldsymbol{C}_i$ is then uniquely determined by a sequence of $d-1$ consecutive control points

$$\boldsymbol{P}_i, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_{i+d-2}, \tag{2.3}$$

where

$$\{\boldsymbol{P}_{i+1}, \boldsymbol{P}_{i+2}, \ldots, \boldsymbol{P}_{i+d-2}\} \subseteq \boldsymbol{C}_i \cap \boldsymbol{C}_{i+1}.$$

If $d = 2$, the centers $\boldsymbol{C}_i$ are simply the control points $\boldsymbol{P}_i$ (Figure 2.1), while for $d > 2$ more control points are needed to determine the centers (Figure 2.2).

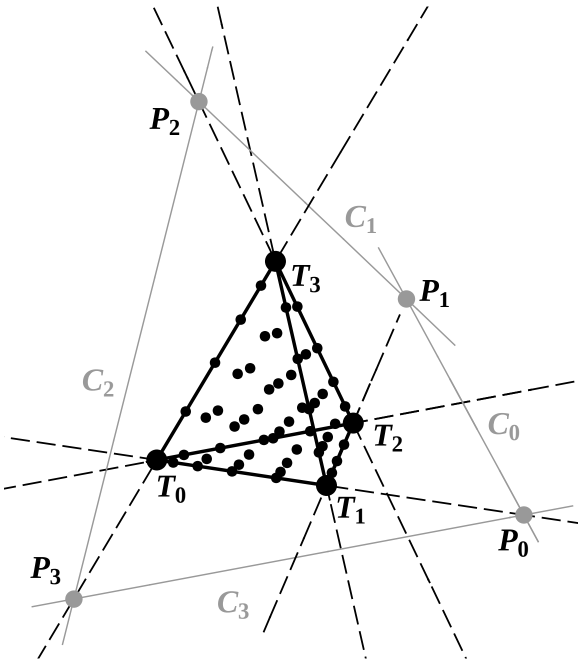


Figure 2.2: A 4-pencil lattice on a tetrahedron $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ in $\mathbb{R}^3$, lattice control points $\boldsymbol{P}_i$ and centers $\boldsymbol{C}_i$.

Thus with the given control points, the lattice on a simplex is determined. Quite clearly, the construction of the lattice assures

$$\boldsymbol{C}_i \cap \triangle = \emptyset, \quad i = 0, 1, \ldots, d,$$

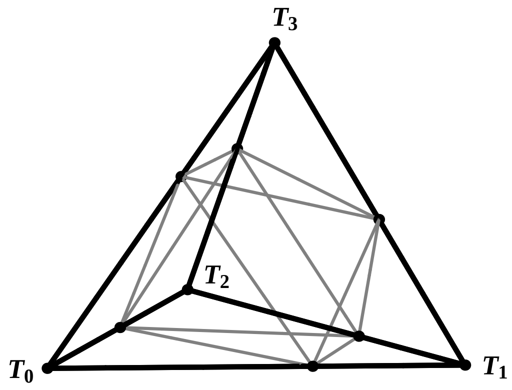
and also that each $\boldsymbol{C}_i$ is lying in a supporting hyperplane of a facet $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1}, \ldots, \boldsymbol{T}_{i+d-1} \rangle$ of $\triangle$ (Figure 2.2 and Figure 2.3).

Here and throughout the dissertation, indices of simplex vertices, control points, lattice points, centers, lattice parameters, etc., are assumed to be taken modulo $d+1$. Wherever necessary, an emphasis on this assumption will be given explicitly by a function

$$m(i) := i \bmod (d+1).$$

With d prescribed, indices considered belong to

$$\mathbb{Z}_{d+1} := \{0, 1, \ldots, d\} = m(\mathbb{Z}).$$



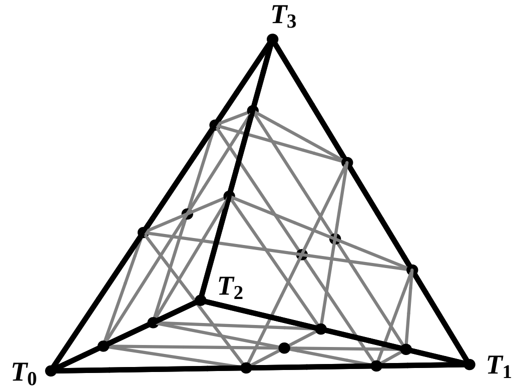


Figure 2.3: Two 4-pencil lattices of order $n = 2, 3$ on a simplex $\triangle$ and the intersections of hyperplanes through the centers of the lattice with facets of $\triangle$.

Let us consider the set of particular multiindex vectors

$$\mathcal{I}_n^d := \left\{ \boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T \in \mathbb{N}_0^{d+1}, \quad |\boldsymbol{\gamma}| = \sum_{i=0}^{d} \gamma_i = n \right\}. \tag{2.4}$$

Since

$$\left|\mathcal{I}_n^d\right| = \binom{n+d}{d},$$

we will be able to represent all lattice points with multiindices in $\mathcal{I}_n^d$.

In the next chapter, two special mappings will be of a particular importance and will significantly simplify further discussion. The first one is a natural bijective imbedding

$$u : \mathbb{Z}_{d+1}^{r+1} \to \mathbb{N}_0^{r+1},$$

defined as

$$u\left((i_j)_{j=0}^r \right) := \left(i_j + (d+1) \sum_{k=0}^{j-1} \chi\left(i_k - i_{k+1}\right) \right)_{j=0}^r, \tag{2.5}$$

where

$$\chi(s) := \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is the usual Heaviside step function. A graphical interpretation of this map (see Figure 2.4) explains also the second map

$$w : \mathbb{Z}_{d+1}^{r+1} \to \mathbb{N},$$

defined as

$$w\left((i_j)_{j=0}^r \right) := \sum_{k=0}^{r-1} \chi\left(i_k - i_{k+1}\right) \, + \, \chi\left(i_r - i_0\right). \tag{2.6}$$

The image of this map will be called a *winding number* of an index vector $(i_j)_{j=0}^r$.

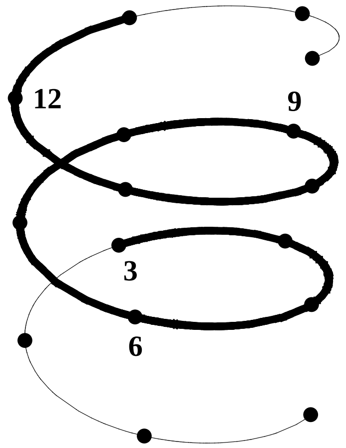


Figure 2.4: Let $d = 4, \boldsymbol{i} = (3, 1, 4, 2)^T$ and $r = 3$. Then $u(\boldsymbol{i}) = (3, 6, 9, 12)^T$ and $w(\boldsymbol{i}) = 2$.

In order to shorten the notation, for $j \in \mathbb{N}_0$ also the symbol

$$[j]_\alpha := \sum_{i=0}^{j-1} \alpha^i = \begin{cases} j, & \alpha = 1, \\ \dfrac{1-\alpha^j}{1-\alpha}, & \text{otherwise}, \end{cases} \tag{2.7}$$

will be used.

2.2. Three-pencil lattice on a triangle

The most trivial but probably the most important case in the practice, the planar case, will be considered first. If one is looking for a three-pencil lattice on a triangle $\triangle$, the answer will undoubtedly depend on the coordinates of the vertices of $\triangle$. But a general approach should work for any given triangle. So it is natural to switch to barycentric coordinates with respect to the vertices $\boldsymbol{T}_0, \boldsymbol{T}_1$ and $\boldsymbol{T}_2$ of a triangle $\triangle$, and apply a simple transformation of coordinates for each particular case separately.

Let us recall the definition of barycentric coordinates.

DEFINITION 2.1. *Let Ω be a convex polygon in the plane with vertices $\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots \boldsymbol{T}_n$, $n \geq 2$. Functions $\nu^i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, n+1$, are called barycentric coordinates if they satisfy, for all $\boldsymbol{x} \in \Omega$, the following three properties*

$$\nu^i(\boldsymbol{x}) \geq 0, \;\; i = 1, 2, \ldots, n+1, \qquad \sum_{i=1}^{n+1} \nu^i(\boldsymbol{x}) = 1, \qquad \sum_{i=0}^{n} \nu^{i+1}(\boldsymbol{x}) \, \boldsymbol{T}_i = \boldsymbol{x}.$$

Note that for $\boldsymbol{x} \notin \Omega$ some of the coordinates ν^i are negative. This definition generalizes the well-known triangular barycentric coordinates. For more details about generalized barycentric coordinates see [22], [23], [24] and [25], e.g. As a special case, when $n = 2$, a polygon Ω is a triangle $\triangle = \langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle$, and the second and third property alone determine the three coordinates uniquely, namely

$$\nu^1(\boldsymbol{x}) = \frac{\mathrm{vol}(\langle\, \boldsymbol{x}, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle)}{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle)}, \quad \nu^2(\boldsymbol{x}) = \frac{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{x}, \boldsymbol{T}_2 \,\rangle)}{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle)}, \quad \nu^3(\boldsymbol{x}) = \frac{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{x} \,\rangle)}{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle)},$$

where

$$\mathrm{vol}(\langle\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \,\rangle)$$

denotes the signed volume (area) of the triangle $\langle\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \,\rangle$. For example, with

$$\boldsymbol{x} = (x, y)^T \quad \text{and} \quad \boldsymbol{T}_i = (x_i, y_i)^T, \quad i = 1, 2,$$

we have

$$\mathrm{vol}(\langle\, \boldsymbol{x}, \boldsymbol{T}_1, \boldsymbol{T}_2 \,\rangle) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & x_1 & x_2 \\ y & y_1 & y_2 \end{vmatrix}.$$

The functions $\nu^i : \triangle \to \mathbb{R}$, $i = 1, 2, 3$, are thus nonnegative linear functions and possess the property

$$\nu^i(\boldsymbol{T}_j) = \delta_{i-1,j}.$$

The notion of barycentric coordinates can be generalized also to higher dimensions. In particular, if $\Omega = \triangle$ is a simplex in $\mathbb{R}^d$, then

$$\nu^i(\boldsymbol{x}) = \frac{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{i-2}, \boldsymbol{x}, \boldsymbol{T}_i, \ldots, \boldsymbol{T}_d \,\rangle)}{\mathrm{vol}(\langle\, \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \,\rangle)}, \quad i = 1, 2, \ldots, d+1, \tag{2.8}$$

where vol is a signed volume in $\mathbb{R}^d$. For more details on generalized barycentric coordinates in higher dimensions see [26], e.g.

The lattice points on a triangle $\triangle$ will be indexed by multiindices in $\mathcal{I}_n^2$. In barycentric coordinates w.r.t. vertices of $\triangle$, they can be written as

$$\boldsymbol{B}_{n-k-j,k,j}, \quad k, j \geq 0, \quad k + j \leq n, \tag{2.9}$$

with known triangle vertices

$$\boldsymbol{B}_{n,0,0} = (1, 0, 0)^T, \quad \boldsymbol{B}_{0,n,0} = (0, 1, 0)^T, \quad \boldsymbol{B}_{0,0,n} = (0, 0, 1)^T.$$

Let us write the centers in the barycentric form as

$$\boldsymbol{C}_0 = \begin{pmatrix} \dfrac{1}{1-\xi_0} \\ -\dfrac{\xi_0}{1-\xi_0} \\ 0 \end{pmatrix}, \quad \boldsymbol{C}_1 = \begin{pmatrix} 0 \\ \dfrac{1}{1-\xi_1} \\ -\dfrac{\xi_1}{1-\xi_1} \end{pmatrix}, \quad \boldsymbol{C}_2 = \begin{pmatrix} -\dfrac{\xi_2}{1-\xi_2} \\ 0 \\ \dfrac{1}{1-\xi_2} \end{pmatrix}, \tag{2.10}$$

where $\xi_i > 0$, $i = 0, 1, 2$, are free parameters (Figure 2.5). Note that a special form of barycentric coordinates is used in order to cover also the cases of parallel lines ($\xi_i = 1$).

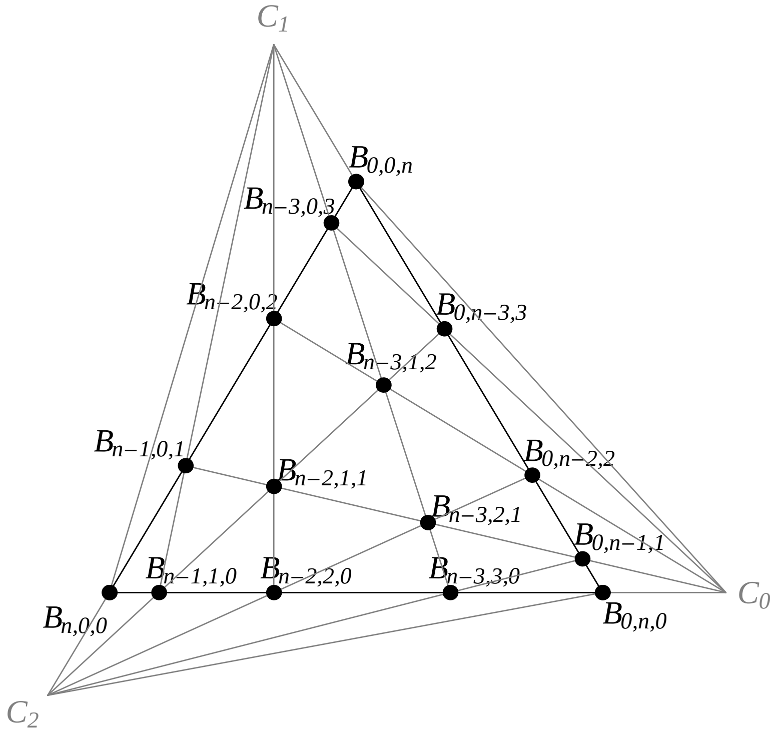


Figure 2.5: A three-pencil lattice of order n.

The range $0 < \xi_i < 1$ covers positions from the ideal line (line at infinity) to the vertex $\boldsymbol{T}_i$, and $1 < \xi_i < \infty$ the half-line from $\boldsymbol{T}_{i+1}$ to the ideal line (see Figure 2.6).

Recall that centers $\boldsymbol{C}_i$ coincide with control points $\boldsymbol{P}_i$ in the planar case and we will use centers rather than control points in this section. Of a particular importance will be a constant $\alpha > 0$, defined as

$$\alpha := \alpha\left(\xi_0, \xi_1, \xi_2\right) := \sqrt[n]{\xi_0\,\xi_1\,\xi_2}. \tag{2.11}$$

If triangle vertices are not given in advance, a three-pencil lattice can be determined by 3 centers, two lines that define one vertex of a triangle, and an additional line that completely determines the geometric construction (see Figure 2.7). One can assume that this construction starts at $\boldsymbol{B}_{n,0,0}$, with $\boldsymbol{B}_{n-1,0,1}\boldsymbol{C}_0$ as the chosen additional line (see Figure 2.5). The first step to determine the points $\boldsymbol{B}_{n-k-j,k,j}$ is given in the following lemma.

LEMMA 2.2. *Let $\boldsymbol{B}_{n-k-j,k,j}$, $k, j \geq 0$, $k + j \leq n$, be the barycentric coordinates of lattice points of a three-pencil lattice, generated by centers $\boldsymbol{C}_i$ given in* (2.10)*. Then*

$$\boldsymbol{B}_{n-k,k,0} = \begin{pmatrix} \tau_k \\ 1-\tau_k \\ 0 \end{pmatrix}, \quad k = 0, 1, \ldots, n, \tag{2.12}$$

where

$$\tau_k := \tau_k\left(\xi_0\right) := \frac{\alpha^n - \alpha^k}{\alpha^n - \alpha^k + \left(\alpha^k - 1\right)\xi_0}, \quad k = 0, 1, \ldots, n, \tag{2.13}$$

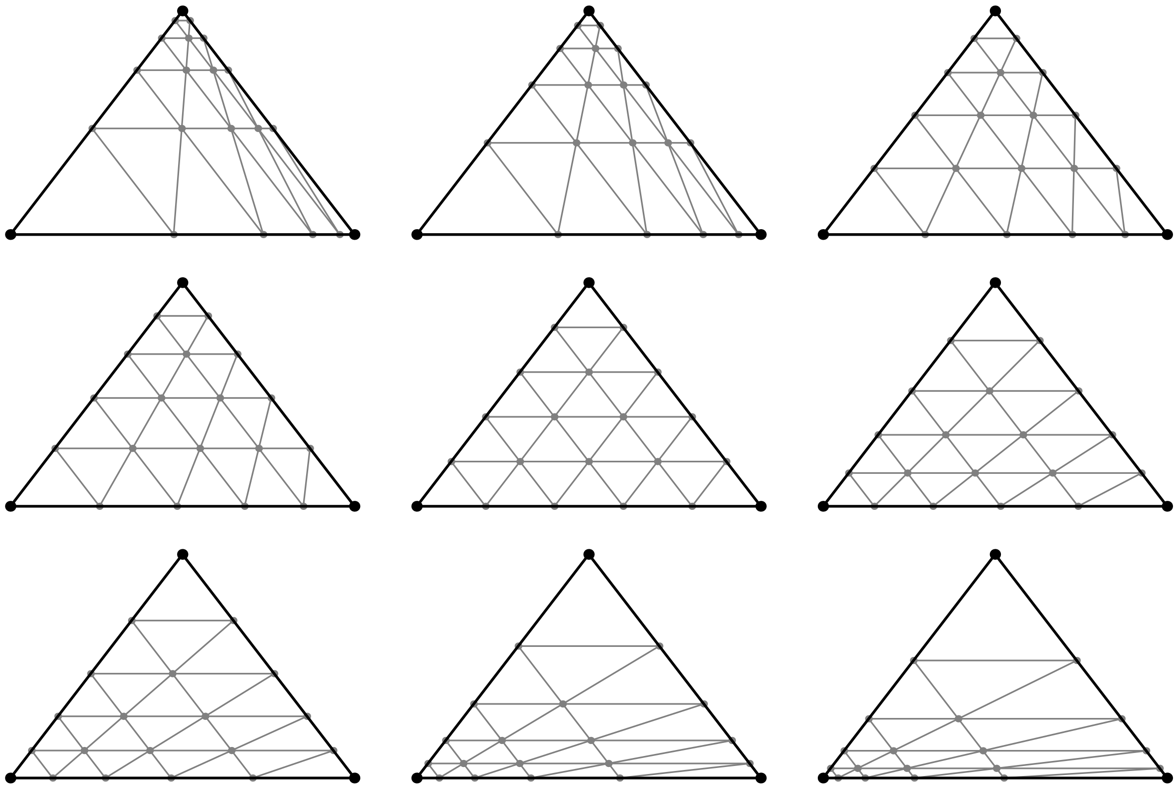


Figure 2.6: Different three-pencil lattices with parameters $\xi_0 = 1$, $\xi_1 = 1$ and $\xi_2 = \frac{1}{20}, \frac{1}{10}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 10, 20$, respectively.

and α is defined in (2.11)*.*

Proof. Suppose first that $\alpha \neq 1$. Let us choose $\omega \in (0,1)$, so that the point

$$\boldsymbol{U}_1 := \begin{pmatrix} 1-\omega \\ 0 \\ \omega \end{pmatrix}$$

is on the edge $\boldsymbol{B}_{n,0,0}\boldsymbol{B}_{0,0,n}$, and let $\boldsymbol{\ell}$ denote the line connecting $\boldsymbol{U}_1$ and $\boldsymbol{C}_0$. To start with, let us assume

$$\boldsymbol{L}_0 := \boldsymbol{B}_{n,0,0},$$

and consider the following geometric construction for $k = 1, 2, \ldots, n$ (Figure 2.8):

- *Forward step:* determine a point $\boldsymbol{U}_k$ as the intersection of the lines $\boldsymbol{C}_2\boldsymbol{L}_{k-1}$ and $\boldsymbol{\ell}$.
- *Backward step:* determine a point $\boldsymbol{L}_k$ as the intersection of the edge $\boldsymbol{B}_{n,0,0}\boldsymbol{B}_{0,n,0}$ and the line $\boldsymbol{C}_1\boldsymbol{U}_k$.

This "zig-zag" procedure produces points

$$\boldsymbol{L}_0, \boldsymbol{U}_1, \boldsymbol{L}_1, \ldots, \boldsymbol{U}_n, \boldsymbol{L}_n, \tag{2.14}$$

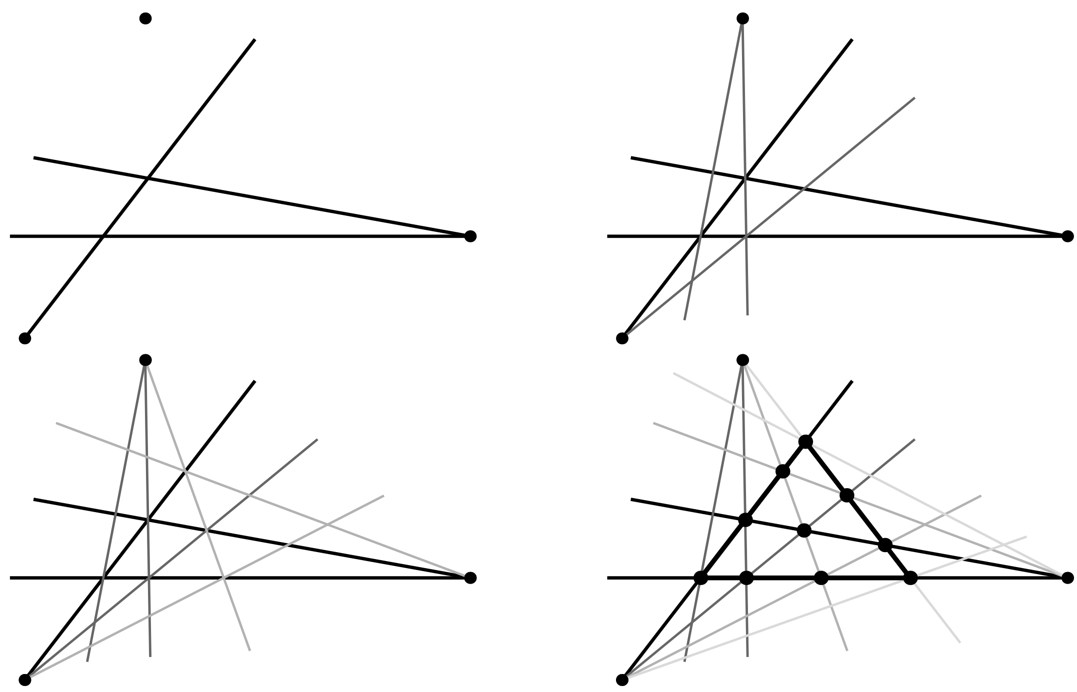


Figure 2.7: A geometric construction of a three-pencil lattice, determined by 3 centers, two lines that define one vertex of a triangle, and an additional line.

that are clearly part of a three-pencil lattice for some triangle, since each is determined as an intersection of lines from all the centers. We proceed to find a unique $\omega \in (0,1)$ such that the points (2.14) are the lattice points for the given $\triangle$, i.e., the equation

$$\boldsymbol{L}_n = \boldsymbol{B}_{0,n,0} \tag{2.15}$$

is satisfied. This is the key point of our algebraic construction. Since the points $\boldsymbol{L}_k$ lie on the edge $\boldsymbol{B}_{n,0,0}\boldsymbol{B}_{0,n,0}$, their barycentric coordinates are

$$\boldsymbol{L}_k = \begin{pmatrix} \phi_k(\omega) \\ 1-\phi_k(\omega) \\ 0 \end{pmatrix}, \quad k = 0, 1, \ldots, n,$$

with

$$\phi_0(\omega) := 1.$$

The forward step determines the intersection point $\boldsymbol{U}_k$. Since we are dealing with barycentric coordinates, it can be written as

$$\boldsymbol{U}_k = \mu_k \boldsymbol{C}_2 + (1-\mu_k)\boldsymbol{L}_{k-1} = \rho_k \boldsymbol{U}_1 + (1-\rho_k)\boldsymbol{C}_0, \tag{2.16}$$

and an elimination from the right-hand side equation yields

$$\begin{aligned} \mu_k &= \frac{\omega\,(1-\xi_2)\,((\xi_0-1)\,\phi_{k-1}(\omega)+1)}{\omega\,(1-\xi_2)\,(\xi_0-1)\,(\phi_{k-1}(\omega)-1)+\xi_0}, \\ \rho_k &= \frac{1}{\omega\,(1-\xi_2)}\,\mu_k. \end{aligned} \tag{2.17}$$

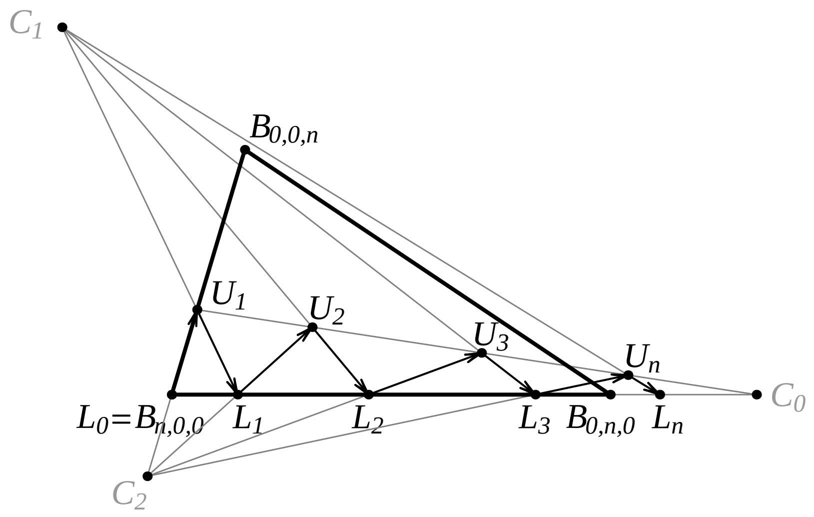


Figure 2.8: The “zig-zag” construction with $\xi_0 > 1$.

Similarly, the backward step determines $\boldsymbol{L}_k$ as

$$\boldsymbol{L}_k = \gamma_k \boldsymbol{C}_1 + (1-\gamma_k)\boldsymbol{U}_k, \tag{2.18}$$

with $\boldsymbol{U}_k$ given by (2.16), and ρ_k by (2.17). However, the third component of $\boldsymbol{L}_k$ is 0, which implies

$$\gamma_k = \frac{(\xi_1-1)\,\mu_k}{(\xi_1-1)\,\mu_k + \xi_1\,(\xi_2-1)}.$$

But then the first component of (2.18) reveals the recurrence relation for $\phi_k(\omega)$, namely

$$\phi_k(\omega) = \frac{a\,\phi_{k-1}(\omega)+b}{c\,\phi_{k-1}(\omega)+d}, \quad \phi_0(\omega) := 1, \tag{2.19}$$

where the coefficient matrix is obtained as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \xi_1\,(\omega\xi_2-(\omega-1)\xi_0) & -\omega\xi_1\xi_2 \\ \omega\,(\xi_0-1)\,(1-\xi_1\xi_2) & \omega+\xi_1\,(\omega\xi_2\,(\xi_0-1)-(\omega-1)\xi_0) \end{pmatrix}.$$

The difference equation (2.19) admits a closed form solution (cf. [39, p. 146])

$$\phi_k(\omega) = \frac{\psi(\omega)^k - \xi_0\xi_1\xi_2}{(1-\xi_0)\,\psi(\omega)^k + \xi_0\,(1-\xi_1\xi_2)}, \tag{2.20}$$

where

$$\psi(\omega) := \frac{(1-\omega+\omega\xi_2)\,\xi_0\xi_1}{\omega+(1-\omega)\,\xi_0\xi_1}. \tag{2.21}$$

The numerator and the denominator in (2.20) have clearly no common root $\psi(\omega)^k$, and the equation (2.15) simplifies to

$$\psi(\omega)^n - \xi_0\xi_1\xi_2 = \psi(\omega)^n - \alpha^n = 0.$$

But this is a well-known equation with solutions proportional to the roots of unity,

$$\psi(\omega) = \alpha \exp\left(\frac{2\pi i}{n} k\right), \quad k = 1, 2, \ldots, n,$$

and with precisely one positive real root

$$\psi(\omega) = \alpha \neq 1.$$

From (2.21), it is now straightforward to derive

$$\omega = \psi^{-1}(\alpha) = \frac{1}{1 + \dfrac{1}{\xi_0 \xi_1} \dfrac{\alpha^n - \alpha}{\alpha - 1}}.$$

Obviously, $0 < \omega < 1$, even if $\alpha \to 1$, since then

$$\frac{\alpha^n - \alpha}{\alpha - 1} = \sum_{j=0}^{n-2} \alpha^{j+1} \to n - 1, \quad \omega \to \frac{1}{1 + (n-1)\,\xi_2}.$$

Finally, the claim (2.13) is confirmed by simplifying

$$\tau_k = \phi_k\left(\psi^{-1}(\alpha)\right).$$

Note that the expression (2.13) makes sense as $\alpha \to 1$ too, namely

$$\frac{\alpha^k - 1}{\alpha^n - \alpha^k} = \frac{\sum\limits_{j=0}^{k-1} \alpha^j}{\sum\limits_{j=0}^{n-1-k} \alpha^{n-1-j}} \to \frac{k}{n-k}, \quad k = 0, 1, \ldots, n-1,$$

and

$$\tau_k \to \frac{n-k}{n - k + k\,\xi_0}, \quad k = 0, 1, \ldots, n.$$

This concludes the proof of the lemma. □

In order to continue, we will need another lemma.

LEMMA 2.3. *Pappus' hexagon theorem: Let* $\{\ell_i\}_{i=1}^3$ *and* $\{\ell_i{}'\}_{i=1}^3$ *be two sets of concurrent lines. Then the lines, defined by the pairs of points*

$$\ell_1 \cap \ell_2{}',\ \ell_2 \cap \ell_1{}'; \qquad \ell_1 \cap \ell_3{}',\ \ell_3 \cap \ell_1{}'; \qquad \ell_2 \cap \ell_3{}',\ \ell_3 \cap \ell_2{}',$$

are concurrent (see Figure 2.9, left)

Proof. See [20, Axiom 14.15]), e.g. □

The following theorem reveals the whole lattice.

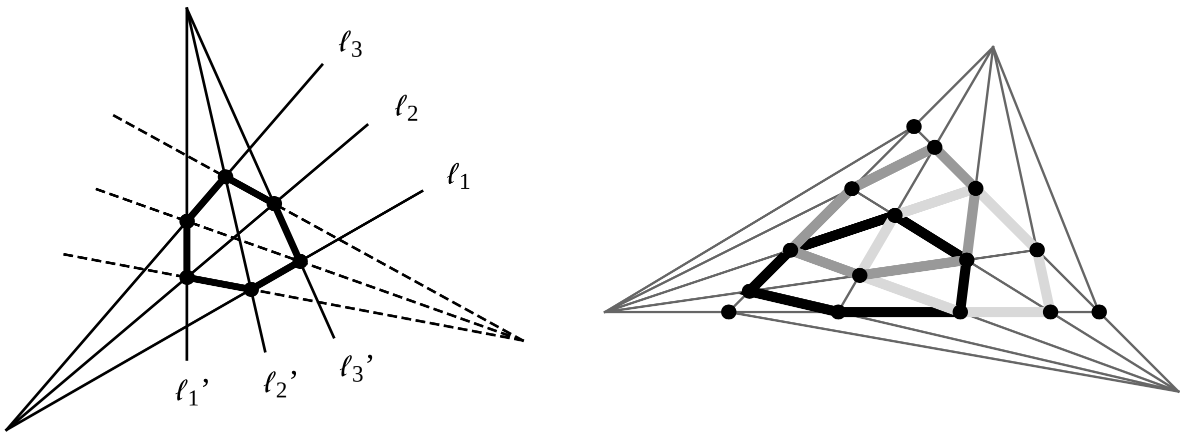


Figure 2.9: An example which illustrates the Pappus' hexagon theorem (left), and an example of an application of this theorem on a three-pencil lattice (right).

THEOREM 2.4. *Suppose that the centers $\boldsymbol{C}_i$ of a three-pencil lattice are prescribed by ξ_i as in* (2.10)*, and that the corresponding α is determined by* (2.11)*. Let*

$$v_i := \alpha^i, \quad w_i := \sum_{j=0}^{i-1} v_j, \qquad i = 0, 1, \ldots, n.$$

The points of a three-pencil lattice of order n are given as

$$\boldsymbol{B}_{n-k-j,k,j} = \begin{pmatrix} \dfrac{v_{k+j} w_{n-k-j}}{v_{k+j} w_{n-k-j} + (v_j w_k + w_j \xi_1)\,\xi_0} \\ \dfrac{v_{n-k} w_k}{v_{n-k} w_k + (v_{n-k-j} w_j + w_{n-k-j} \xi_2)\,\xi_1} \\ \dfrac{v_{n-j} w_j}{v_{n-j} w_j + (v_k w_{n-k-j} + w_k \xi_0)\,\xi_2} \end{pmatrix}. \tag{2.22}$$

Proof. Let us prove (2.22) constructively. Consider Figure 2.5. The point $\boldsymbol{B}_{0,n-j,j}$ is clearly determined by the line $\boldsymbol{C}_2\boldsymbol{B}_{j,n-j,0}$, and it is enough to compute the second component only. With the help of Lemma 2.2 and (2.13), it turns out that

$$\frac{(\alpha^n - \alpha^j)\,\xi_0\xi_2}{\alpha^n(\alpha^j - 1) + (\alpha^n - \alpha^j)\,\xi_0\xi_2} = \frac{\alpha^n - \alpha^j}{\alpha^n - \alpha^j + (\alpha^j - 1)\,\xi_1} = \tau_j(\xi_1),$$

since $\xi_0\xi_1\xi_2 = \alpha^n$. Similarly, the third component of the point $\boldsymbol{B}_{n-k-j,0,k+j}$, determined by the line $\boldsymbol{C}_1\boldsymbol{B}_{n-k-j,k+j,0}$, is

$$\frac{\left(\alpha^{k+j} - 1\right)\xi_0\xi_1}{\alpha^n - \alpha^{k+j} + (\alpha^{k+j} - 1)\,\xi_0\xi_1} = \frac{\alpha^n - \alpha^{n-k-j}}{\alpha^n - \alpha^{n-k-j} + (\alpha^{n-k-j} - 1)\,\xi_2} = \tau_{n-k-j}(\xi_2).$$

Lines $\boldsymbol{C}_2\boldsymbol{B}_{n-k,k,0}$ and $\boldsymbol{C}_1\boldsymbol{B}_{n-k-j,0,k+j}$ are not parallel, so they meet at some point which is the lattice point $\boldsymbol{B}_{n-k-j,k,j}$,

$$\boldsymbol{B}_{n-k-j,k,j} = \begin{pmatrix} \dfrac{\alpha^n - \alpha^{k+j}}{\alpha^n - \alpha^{k+j} + \left(\alpha^{k+j} - \alpha^j + \left(\alpha^j - 1\right)\xi_1\right)\xi_0} \\ \dfrac{\alpha^n - \alpha^{n-k}}{\alpha^n - \alpha^{n-k} + \left(\alpha^{n-k} - \alpha^{n-k-j} + \left(\alpha^{n-k-j} - 1\right)\xi_2\right)\xi_1} \\ \dfrac{\alpha^n - \alpha^{n-j}}{\alpha^n - \alpha^{n-j} + \left(\alpha^{n-j} - \alpha^k + \left(\alpha^k - 1\right)\xi_0\right)\xi_2} \end{pmatrix}, \tag{2.23}$$

iff it lies on the line $\boldsymbol{C}_0\boldsymbol{B}_{0,n-j,j}$ too. In order to show this, one may base the argument on Lemma 2.3 and Figure 2.9 (right). The substitution stated in the theorem and (2.23) conclude the proof. □

The points $\boldsymbol{B}_{n-k-j,k,j}$ can be computed efficiently and stably, avoiding any cancellations. Indeed, one is able to obtain $v_i, w_i,\ i = 0, 1, \ldots, n$, in

$$2n + \mathcal{O}(1)$$

floating point operations only.

Let us conclude the section with a numerical example and compute the barycentric coordinates of three-pencil lattice points with $n = 3$ by using formula (2.22). Let $\xi_0 = 1/2$, $\xi_1 = 1/6$ and $\xi_2 = 3/2$. Then $\alpha = 1/2$ and

$$v_i = \frac{1}{2^i}, \quad w_i = 2 - \frac{1}{2^{i-1}}, \qquad i = 0, 1, 2, 3.$$

The barycentric coordinates are then equal to

$$\begin{aligned}
&\boldsymbol{B}_{3,0,0} = (1,0,0)^T, && \boldsymbol{B}_{2,0,1} = \left(\frac{9}{10}, 0, \frac{1}{10}\right)^T, && \boldsymbol{B}_{1,0,2} = \left(\frac{2}{3}, 0, \frac{1}{3}\right)^T,\\
&\boldsymbol{B}_{0,0,3} = (0,0,1)^T, && \boldsymbol{B}_{2,1,0} = \left(\frac{3}{5}, \frac{2}{5}, 0\right)^T, && \boldsymbol{B}_{1,1,1} = \left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right)^T,\\
&\boldsymbol{B}_{0,1,2} = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T, && \boldsymbol{B}_{1,2,0} = \left(\frac{1}{4}, \frac{3}{4}, 0\right)^T, &&\\
&\boldsymbol{B}_{0,2,1} = \left(0, \frac{9}{11}, \frac{2}{11}\right)^T, && \boldsymbol{B}_{0,3,0} = (0,1,0)^T. &&
\end{aligned}$$

2.3. $(d+1)$-pencil lattice on a simplex

In this section, the barycentric representation of a three-pencil lattice on a triangle will be generalized to a $(d+1)$-pencil lattice on a simplex (see Figure 2.10 for $d = 3$).

Although a similar geometric construction as in the previous section can be applied, the generalization will be based on the approach introduced in [38]. This approach heavily depends on homogenous coordinates, and a nice illustrative explanation can be found in [41], where the cases perhaps most often met in practice, i.e., $d = 2$ and $d = 3$,

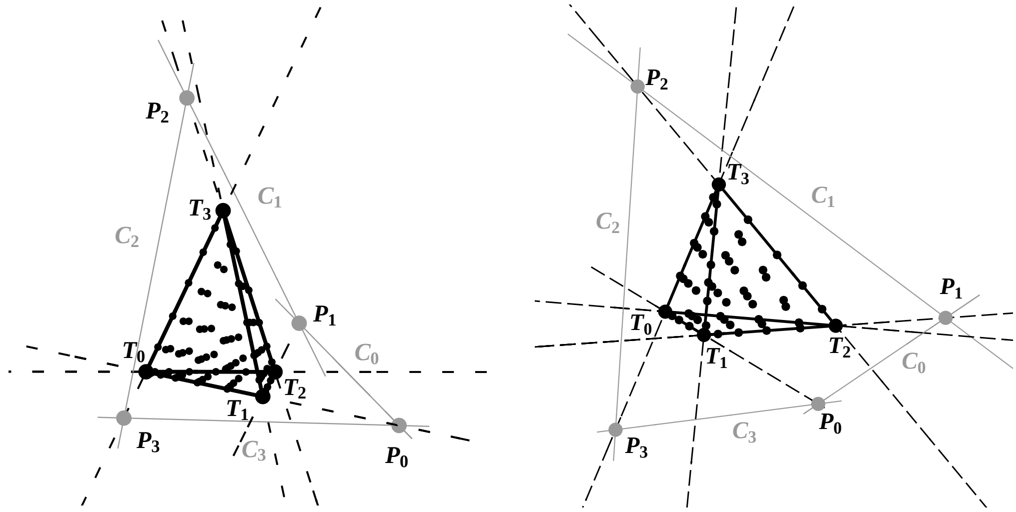


Figure 2.10: Two different four-pencil lattices on a tetrahedron in $\mathbb{R}^3$.

are outlined. Here our goal is an explicit representation in barycentric coordinates, since this enables a natural extension from a simplex to a simplicial partition, that will be presented in the next chapter.

A $(d+1)$-pencil lattice of order n on the standard simplex $\triangle^d \subseteq \mathbb{R}^d$, as introduced in [38], is given by free parameters

$$\alpha > 0 \quad \text{and} \quad \boldsymbol{\beta} := (\beta_0, \beta_1, \ldots, \beta_d)^T, \ \beta_i > 0, \ i = 0, 1, \ldots, d.$$

Control points $\boldsymbol{P}_i = \boldsymbol{P}_i(\alpha, \boldsymbol{\beta})$ of the lattice on $\triangle^d$ are determined as

$$\begin{aligned}
\boldsymbol{P}_0 &= \left(\frac{\beta_1}{\beta_1 - \beta_0}, \underbrace{0, 0, \ldots, 0}_{d-1} \right)^T, \\
\boldsymbol{P}_i &= \left(\underbrace{0, 0, \ldots, 0}_{i-1}, \frac{\beta_i}{\beta_i - \beta_{i+1}}, \frac{\beta_{i+1}}{\beta_{i+1} - \beta_i}, \underbrace{0, 0, \ldots, 0}_{d-1-i} \right)^T, \quad i = 1, 2, \ldots, d-1, \\
\boldsymbol{P}_d &= \left(\underbrace{0, 0, \ldots, 0}_{d-1}, \frac{\beta_d}{\beta_d - \alpha^n \beta_0} \right)^T.
\end{aligned} \tag{2.24}$$

If $\beta_{i+1} = \beta_i$ for some $0 \le i \le d$ (and $\alpha = 1$ if $i = d$), then the control point $\boldsymbol{P}_i$ is at the ideal line. Recall (2.7). Lattice points are then given as

$$\left(\boldsymbol{Q}_{\boldsymbol{\gamma}}\right)_{\substack{\boldsymbol{\gamma} \in \mathbb{N}_0^d \\ |\boldsymbol{\gamma}| \le n}} := \left(\boldsymbol{Q}_{\boldsymbol{\gamma}}(\alpha, \boldsymbol{\beta})\right)_{\substack{\boldsymbol{\gamma} \in \mathbb{N}_0^d \\ |\boldsymbol{\gamma}| \le n}},$$

where

$$\boldsymbol{Q}_{\boldsymbol{\gamma}}(\alpha, \boldsymbol{\beta}) = \frac{1}{D} \left(\beta_1 \alpha^{|\boldsymbol{\gamma}| - \gamma_1} [\gamma_1]_\alpha, \beta_2 \alpha^{|\boldsymbol{\gamma}| - \gamma_1 - \gamma_2} [\gamma_2]_\alpha, \beta_3 \alpha^{|\boldsymbol{\gamma}| - \gamma_1 - \gamma_2 - \gamma_3} [\gamma_3]_\alpha, \ldots, \beta_d \alpha^0 [\gamma_d]_\alpha \right)^T, \tag{2.25}$$

and

$$D = \beta_0 \alpha^{|\boldsymbol{\gamma}|} \left[n - |\boldsymbol{\gamma}|\right]_\alpha + \beta_1 \alpha^{|\boldsymbol{\gamma}|-\gamma_1} \left[\gamma_1\right]_\alpha + \beta_2 \alpha^{|\boldsymbol{\gamma}|-\gamma_1-\gamma_2} \left[\gamma_2\right]_\alpha + \cdots + \beta_d \alpha^0 \left[\gamma_d\right]_\alpha .$$

Since the points $\boldsymbol{P}_i$, $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$ are collinear, the barycentric coordinates of $\boldsymbol{P}_i$ w.r.t. $\triangle$, which will here be denoted by $\boldsymbol{P}_{i,\triangle}$, are particularly simple,

$$\begin{aligned} \boldsymbol{P}_{i,\triangle} &= \left(\underbrace{0,0,\ldots,0}_{i}, \frac{1}{1-\xi_i}, -\frac{\xi_i}{1-\xi_i}, \underbrace{0,0,\ldots,0}_{d-1-i}\right)^T, \ i = 0,1,\ldots,d-1, \\ \boldsymbol{P}_{d,\triangle} &= \left(-\frac{\xi_d}{1-\xi_d}, \underbrace{0,0,\ldots,0}_{d-1}, \frac{1}{1-\xi_d}\right)^T, \end{aligned} \tag{2.26}$$

where

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$$

are new free parameters of the lattice. Quite clearly, $\xi_i > 0$, since $\boldsymbol{P}_i$ is not on the line segment $\boldsymbol{T}_i\boldsymbol{T}_{i+1}$. Again the range $0 < \xi_i < 1$ covers positions from the ideal line to the vertex $\boldsymbol{T}_i$, and $1 < \xi_i < \infty$ the half-line from $\boldsymbol{T}_{i+1}$ to the ideal line (see Figure 2.6, e.g.). Recall that a special form of barycentric coordinates is used in order to cover also the cases of parallel hyperplanes ($\xi_i = 1$). If all of the control points that determine the center $\boldsymbol{C}_i$ are on the ideal line, so is $\boldsymbol{C}_i$, and the corresponding hyperplanes are parallel. We are now able to give the relations between parameters $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$.

THEOREM 2.5. *Let $\triangle \subset \mathbb{R}^d$ be a d-simplex, and let the barycentric representation $\boldsymbol{P}_{i,\triangle}, i = 0,1,\ldots,d$, of control points $\boldsymbol{P}_i$ of a $(d+1)$-pencil lattice on $\triangle$ be prescribed by $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ as in (2.26). Then the lattice is determined by parameters α and $\boldsymbol{\beta}$ that satisfy*

$$\alpha = \sqrt[n]{\prod_{i=0}^{d} \xi_i}, \qquad \frac{\beta_i}{\beta_0} = \prod_{j=0}^{i-1} \xi_j, \quad i = 1,2,\ldots,d. \tag{2.27}$$

Proof. An affine map $\mathcal{A}$ carries $\triangle$ to the standard simplex

$$\triangle^d = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle, \quad \boldsymbol{T}_i = (\delta_{i,j})_{j=1}^d ,$$

where the lattice is given by (2.25) with the control points (2.24). The i-th barycentric coordinate of a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^T$ w.r.t. $\triangle^d$ is obtained by (2.8) and can be written as

$$\frac{\operatorname{vol}(\langle \overbrace{\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{i-2}}^{i-1}, \boldsymbol{x}, \boldsymbol{T}_i, \ldots, \boldsymbol{T}_d \rangle)}{\operatorname{vol}(\triangle^d)} = \begin{cases} 1 - \sum_{j=1}^{d} x_j, & i = 1, \\ x_{i-1}, & i = 2,3,\ldots,d+1. \end{cases} \tag{2.28}$$

Thus it is straightforward to compute the barycentric coordinates of (2.24) w.r.t. $\triangle^d$. The inverse map $\mathcal{A}^{-1}$ brings control points (2.24) as well as the lattice from $\triangle^d$ back to

$\triangle$. But barycentric coordinates are affinely invariant, so the barycentric coordinates of transformed control points w.r.t. $\triangle$ do not change and are given by (2.26). Therefore

$$\begin{aligned}
\frac{\beta_1}{\beta_1-\beta_0} &= -\frac{\xi_0}{1-\xi_0}, \\
\frac{\beta_i}{\beta_i-\beta_{i+1}} &= \frac{1}{1-\xi_i}, \quad i=1,2,\ldots,d-1, \\
\frac{\beta_d}{\beta_d-\alpha^n\beta_0} &= \frac{1}{1-\xi_d}.
\end{aligned}$$

This describes the system of $d+1$ equations for $d+1$ unknowns

$$\alpha, \quad \frac{\beta_i}{\beta_0}, \ i=1,2,\ldots,d.$$

Since the solution is given by (2.27), the proof is completed. □

Note that, in contrast to parameters $\boldsymbol{\beta}$, the introduced parameters $\boldsymbol{\xi}$ have a clear geometric interpretation, and can be used as shape parameters of the lattice.

Recall (2.4). The following theorem is a direct consequence of Theorem 2.5.

THEOREM 2.6. *The barycentric coordinates of a $(d+1)$-pencil lattice of order n on a simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ w.r.t. $\triangle$ are determined by $d+1$ positive parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ as*

$$(\boldsymbol{B}_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_n^d} := (\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}))_{\boldsymbol{\gamma}\in\mathcal{I}_n^d},$$

where

$$\boldsymbol{B}_{\boldsymbol{\gamma}} = \frac{1}{D_{\boldsymbol{\gamma},\boldsymbol{\xi}}} \left(\alpha^{n-\gamma_0}\left[\gamma_0\right]_\alpha, \xi_0\,\alpha^{n-\gamma_0-\gamma_1}\left[\gamma_1\right]_\alpha, \xi_0\xi_1\,\alpha^{n-\sum_{i=0}^2\gamma_i}\left[\gamma_2\right]_\alpha, \ldots, \xi_0\cdots\xi_{d-1}\,\alpha^0\left[\gamma_d\right]_\alpha\right)^T, \tag{2.29}$$

with

$$D_{\boldsymbol{\gamma},\boldsymbol{\xi}} = \alpha^{n-\gamma_0}\left[\gamma_0\right]_\alpha + \xi_0\,\alpha^{n-\gamma_0-\gamma_1}\left[\gamma_1\right]_\alpha + \ldots + \xi_0\xi_1\cdots\xi_{d-1}\,\alpha^0\left[\gamma_d\right]_\alpha, \quad \alpha^n = \prod_{i=0}^d \xi_i.$$

Proof. By applying (2.27) and (2.28) to (2.25), one obtains

$$\boldsymbol{B}_{\boldsymbol{\gamma}'} = \frac{1}{\boldsymbol{1}^T\boldsymbol{x}}\boldsymbol{x}, \quad \boldsymbol{x} = \left(\alpha^{|\boldsymbol{\gamma}'|}\left[n-|\boldsymbol{\gamma}'|\right]_\alpha, \xi_0\,\alpha^{|\boldsymbol{\gamma}'|-\gamma_1}\left[\gamma_1\right]_\alpha, \ldots, \xi_0\cdots\xi_{d-1}\alpha^0\left[\gamma_d\right]_\alpha\right)^T,$$

where $\boldsymbol{1} = (1,1,\ldots,1)^T$ and $\boldsymbol{\gamma}' \in \mathbb{N}_0^d$, $|\boldsymbol{\gamma}'| \le n$. To make the formula more symmetric, we can, without loss of generality, replace the index vector $\boldsymbol{\gamma}' = (\gamma_1, \ldots, \gamma_d)^T$ by the index vector

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T, \quad \gamma_0 := n - |\boldsymbol{\gamma}'|,$$

and (2.29) follows. □

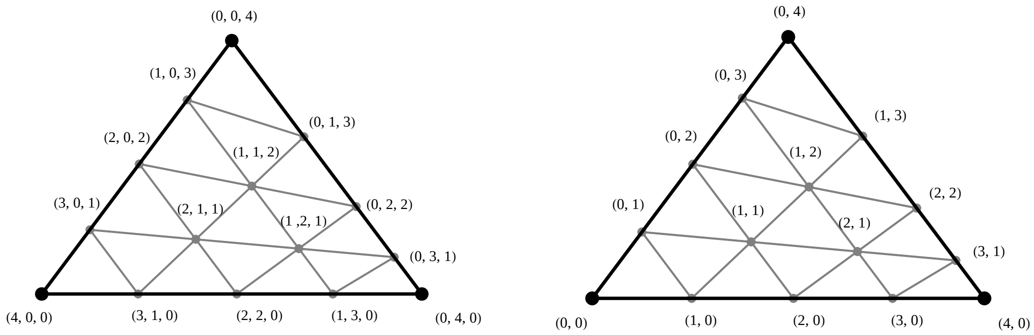


Figure 2.11: Indices of lattice points: $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T$, $|\boldsymbol{\gamma}| = n$, (left), and $\boldsymbol{\gamma}' = (\gamma_1, \gamma_2, \ldots, \gamma_d)^T$, $|\boldsymbol{\gamma}'| \leq n$, (right).

Note that ξ_d appears in (2.29) implicitly, since $\alpha^n = \prod_{i=0}^{d} \xi_i$.

Moreover, the indices $\boldsymbol{\gamma}$ in (2.29) are determined by hyperplanes $H_{i,j}$ such that

$$\boldsymbol{B}_{\boldsymbol{\gamma}} := \bigcap_{i=0}^{d} H_{i,\gamma_i} ,$$

where $H_{i,j}$ is the $(j+1)$-th hyperplane passing through the center $\boldsymbol{C}_{i+1}$. Since $|\boldsymbol{\gamma}| = n$, one can drop any fixed component of the index, and the lattice points will still be uniquely denoted. So, with $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T$ and $\boldsymbol{\gamma}' = (\gamma_1, \gamma_2, \ldots, \gamma_d)^T$,

$$\{\boldsymbol{B}_{\boldsymbol{\gamma}},\ \boldsymbol{\gamma} \in \mathbb{N}_0^{d+1}, |\boldsymbol{\gamma}| = n\} \quad \text{and} \quad \{\boldsymbol{B}_{\boldsymbol{\gamma}'},\ \boldsymbol{\gamma}' \in \mathbb{N}_0^{d}, |\boldsymbol{\gamma}'| \leq n\} \tag{2.30}$$

refer to the same set of points (see Figure 2.11).

2.4. Lagrange polynomial interpolant

In this section, the Lagrange interpolating polynomial over $(d+1)$-pencil lattice will be considered. One of the main advantages of lattices satisfying the GC condition is the fact that they provide an explicit construction of Lagrange basis polynomials as products of linear factors. Therefore some simplifications can be done in order to decrease the amount of work needed. Details on how this can be done in the barycentric coordinates are summarized in the following theorem.

Let $(v)_k$ denote the k-th component of a vector v.

THEOREM 2.7. *Let a $(d+1)$-pencil lattice of order n on a simplex $\triangle$ be given in the barycentric form by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ as in Theorem 2.6 and let data*

$$f_{\boldsymbol{\gamma}} \in \mathbb{R}, \quad \boldsymbol{\gamma} \in \mathcal{I}_n^d = \left\{\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T \in \mathbb{N}_0^{d+1},\ |\boldsymbol{\gamma}| = n\right\},$$

be prescribed. The polynomial $p_n \in \Pi_n^d$ that interpolates the data $(f_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ at the points $(\boldsymbol{B}_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ is given as

$$p_n(\boldsymbol{x}) := p_n(\boldsymbol{x}; \boldsymbol{\xi}) = \sum_{\boldsymbol{\gamma} \in \mathcal{I}_n^d} f_{\boldsymbol{\gamma}}\, \mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}; \boldsymbol{\xi}), \quad \boldsymbol{x} \in \mathbb{R}^{d+1}, \sum_{i=0}^{d} x_i = 1. \tag{2.31}$$

The Lagrange basis polynomial $\mathcal{L}_{\boldsymbol{\gamma}}$ is a product of hyperplanes, i.e.,

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}) = \prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1} h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}), \tag{2.32}$$

where

$$h_{i,j,\boldsymbol{\gamma}}\left(\boldsymbol{x};\boldsymbol{\xi}\right) := \frac{c_{i,\boldsymbol{\gamma}}}{1-\dfrac{[n-\gamma_i]_\alpha}{[n-j]_\alpha}}\left(x_i + \left(1-\frac{[n]_\alpha}{[n-j]_\alpha}\right) q_i\left(\boldsymbol{x};\boldsymbol{\xi}\right)\right), \tag{2.33}$$

and

$$q_i\left(\boldsymbol{x};\boldsymbol{\xi}\right) := \sum_{t=i+1}^{i+d} \frac{1}{\prod\limits_{k=i}^{t-1}\xi_k} x_t, \quad c_{i,\boldsymbol{\gamma}} := \left(1-\frac{[n-\gamma_i]_\alpha}{[n]_\alpha}\right)\frac{1}{\left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{i+1}}.$$

Proof. Let $\boldsymbol{\gamma}\in\mathcal{I}_n^d$ be a given index vector. Let us construct the Lagrange basis polynomial $\mathcal{L}_{\boldsymbol{\gamma}}$ that satisfies

$$\mathcal{L}_{\boldsymbol{\gamma}}\left(\boldsymbol{B}_{\boldsymbol{\gamma}'};\boldsymbol{\xi}\right) = \begin{cases} 1, & \boldsymbol{\gamma}'=\boldsymbol{\gamma},\\ 0, & \boldsymbol{\gamma}'\neq\boldsymbol{\gamma}.\end{cases}$$

Based upon the GC approach, this polynomial can be found as a product of hyperplanes $H_{i,j}$ with the equations

$$h_{i,j,\boldsymbol{\gamma}} = 0, \quad j=0,1,\ldots,\gamma_i-1, \quad i=0,1,\ldots,d,$$

where $H_{i,j}$ contains lattice points (see (2.30))

$$\boldsymbol{B}_{\boldsymbol{\gamma}'}, \quad \boldsymbol{\gamma}'\in\mathcal{I}_n^d,\ \gamma_i'=j.$$

Quite clearly, the total degree of such a polynomial is bounded above by

$$\sum_{i=0}^{d}\sum_{j=0}^{\gamma_i-1} 1 = n.$$

But, for fixed i and $j, 0\leq j\leq \gamma_i-1$, a hyperplane $H_{i,j}$ is determined by the center $\boldsymbol{C}_{i+1}$ and the point $\boldsymbol{U}$ at the edge $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1}\rangle$ with the barycentric coordinates w.r.t. $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1}\rangle$ equal to (see (2.29))

$$\left(\frac{[n]_\alpha-[n-j]_\alpha}{[n]_\alpha-[n-j]_\alpha+[n-j]_\alpha\,\xi_i}, \frac{[n-j]_\alpha\,\xi_i}{[n]_\alpha-[n-j]_\alpha+[n-j]_\alpha\,\xi_i}\right)^T.$$

The equation $h_{i,j,\boldsymbol{\gamma}}=0$ is by (2.3) given as

$$\det\left(\boldsymbol{x},\boldsymbol{U},\boldsymbol{P}_{i+1},\boldsymbol{P}_{i+2},\ldots,\boldsymbol{P}_{i+d-1}\right)=0. \tag{2.34}$$

Let us recall the barycentric representation (2.26) of $\boldsymbol{P}_i$. A multiplication of the matrix in (2.34) by a nonsingular diagonal matrix

$$\operatorname{diag}\left(1,[n]_\alpha-[n-j]_\alpha+[n-j]_\alpha\,\xi_i, 1-\xi_{i+1}, 1-\xi_{i+2},\ldots,1-\xi_{i+d-1}\right),$$

and a circular shift of columns simplifies the equation (2.34) to

$$\det\begin{pmatrix} x_i & x_{i+1} & \dots & \dots & x_{i+d} \\ [n]_\alpha - [n-j]_\alpha & [n-j]_\alpha\, \xi_i & & & \\ & 1 & -\xi_{i+1} & & \\ & & 1 & -\xi_{i+2} & \\ & & & \ddots & \ddots \\ & & & 1 & -\xi_{i+d-1} \end{pmatrix} = 0.$$

Recall that indices are taken modulo $d+1$ here. A straightforward evaluation of the determinant yields

$$[n-j]_\alpha \left(\prod_{k=i}^{i+d-1} \xi_k\right) x_i - \left([n]_\alpha - [n-j]_\alpha\right) \sum_{t=i+1}^{i+d} x_t \left(\prod_{k=t}^{i+d-1} \xi_k\right) = 0,$$

and further

$$[n-j]_\alpha\, x_i - \left([n]_\alpha - [n-j]_\alpha\right) \sum_{t=i+1}^{i+d} \frac{1}{\prod\limits_{k=i}^{t-1} \xi_k} x_t = [n-j]_\alpha\, x_i - \left([n]_\alpha - [n-j]_\alpha\right) q_i(\boldsymbol{x};\boldsymbol{\xi}) = 0.$$

If $j > 0$, this gives also a relation

$$q_i(\boldsymbol{x};\boldsymbol{\xi}) = \frac{[n-j]_\alpha}{[n]_\alpha - [n-j]_\alpha} x_i \tag{2.35}$$

for a particular $\boldsymbol{x}$ that satisfies (2.34). Note that $0 \le j < \gamma_i \le n$. Now the equation of the hyperplane $h_{i,j,\boldsymbol{\gamma}}$ can be written as

$$h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}) = \frac{[n-j]_\alpha\, x_i - \left([n]_\alpha - [n-j]_\alpha\right) q_i(\boldsymbol{x};\boldsymbol{\xi})}{[n-j]_\alpha \left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{i+1} - \left([n]_\alpha - [n-j]_\alpha\right) q_i(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi})}. \tag{2.36}$$

Consider now an index vector $\boldsymbol{\gamma}' \neq \boldsymbol{\gamma}$. Since $|\boldsymbol{\gamma}'| = |\boldsymbol{\gamma}| = n$, there must exist an index i, $0 \le i \le d$, such that $\gamma_i' < \gamma_i$. So γ_i' appears as one of the indices j in the product (2.32). Thus

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{B}_{\boldsymbol{\gamma}'};\boldsymbol{\xi}) = 0.$$

But from (2.36) one deduces

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi}) = 1.$$

Furthermore, if we can prove that

$$q_i(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi}) = \frac{[n-\gamma_i]_\alpha}{[n]_\alpha - [n-\gamma_i]_\alpha} \left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{i+1}, \tag{2.37}$$

then (2.36) can be simplified to the assertion (2.33). Note that the lattice point with barycentric coordinates $\boldsymbol{B}_{\boldsymbol{\gamma}}$ does not satisfy (2.34), thus (2.35) can not be used for it.

By simplifying the left-hand side of the equation (2.37), one obtains

$$\begin{aligned} q_i\left(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi}\right) &= \sum_{t=i+1}^{d}\left(\prod_{k=0}^{i-1}\xi_k\ \alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha\right)+\sum_{t=0}^{i-1}\left(\prod_{k=0}^{t-1}\xi_k\prod_{k=i}^{t+d}\xi_k^{-1}\ \alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha\right)\\ &= \prod_{k=0}^{i-1}\xi_k\left(\sum_{t=i+1}^{d}\alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha+\frac{1}{\alpha^n}\sum_{t=0}^{i-1}\alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha\right). \end{aligned}$$

On the other hand, the right-hand side of (2.37) can be written as

$$\begin{aligned} \frac{\left[n-\gamma_i\right]_\alpha}{\left[n\right]_\alpha-\left[n-\gamma_i\right]_\alpha}\left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{i+1} &= \prod_{k=0}^{i-1}\xi_k\ \frac{\left[n-\gamma_i\right]_\alpha\left[\gamma_i\right]_\alpha}{\left[n\right]_\alpha-\left[n-\gamma_i\right]_\alpha}\ \alpha^{n-\gamma_0-\ldots-\gamma_i}\\ &= \prod_{k=0}^{i-1}\xi_k\ \alpha^{n-\gamma_0-\ldots-\gamma_i}\left[\gamma_i\right]_\alpha\left(-1+\frac{\left[n\right]_\alpha}{\alpha^{n-\gamma_i}\left[\gamma_i\right]_\alpha}\right). \end{aligned}$$

Thus we have to prove that

$$\frac{1}{\alpha^n}\sum_{t=0}^{i-1}\alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha+\sum_{t=i}^{d}\alpha^{n-\gamma_0-\ldots-\gamma_t}\left[\gamma_t\right]_\alpha=\alpha^{-\gamma_0-\ldots-\gamma_{i-1}}\left(1+\alpha+\ldots+\alpha^{n-1}\right). \tag{2.38}$$

Since the left-hand side of (2.38) can be rewritten to

$$\begin{aligned} &\alpha^{-\gamma_0}\left(1+\alpha+\ldots+\alpha^{\gamma_0-1}\right)+\ldots+\alpha^{-\gamma_0-\ldots-\gamma_{i-1}}\left(1+\alpha+\ldots+\alpha^{\gamma_{i-1}-1}\right)\\ &+\alpha^{-\gamma_0-\ldots-\gamma_i+n}\left(1+\alpha+\ldots+\alpha^{\gamma_i-1}\right)+\ldots+\alpha^0(1+\alpha+\ldots+\alpha^{\gamma_d-1})\\ &=\alpha^{-\gamma_0-\ldots-\gamma_{i-1}}+\ldots+\frac{1}{\alpha}+1+\alpha+\ldots+\alpha^{n-\gamma_0-\ldots-\gamma_{i-1}-1}, \end{aligned}$$

the proof is completed. □

Note that $h_{i,j,\boldsymbol{\gamma}}$ in (2.33) depends only on the $(i+1)$-th component of the corresponding point $\boldsymbol{B}_{\boldsymbol{\gamma}}$. This is not obvious from the classical representation of Lagrange basis polynomial, and is vital for an efficient computation.

Let us now give some remarks on how to organize the computations. Let $\triangle$ be a simplex in $\mathbb{R}^d$ with vertices $\boldsymbol{V}_i$, $i=0,1,\ldots,d$, given in Cartesian coordinates. The Cartesian coordinates of lattice points are

$$\boldsymbol{X}_{\boldsymbol{\gamma}}=\sum_{j=0}^{d}\left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{j+1}\boldsymbol{V}_j. \tag{2.39}$$

Let us use formula (2.39) on a trivial example. Let $d=2$, $n=2$ and

$$\boldsymbol{V}_0=(0,0)^T,\quad \boldsymbol{V}_1=(2,0)^T,\quad \boldsymbol{V}_2=(1,1)^T. \tag{2.40}$$

Then

$$\boldsymbol{B}_{(2,0,0)}=(1,0,0)^T,\qquad \boldsymbol{B}_{(1,0,1)}=\left(\frac{\alpha}{\alpha+\xi_0\xi_1},0,\frac{\xi_0\xi_1}{\alpha+\xi_0\xi_1}\right)^T,\qquad \boldsymbol{B}_{(0,0,2)}=(0,0,1)^T,$$

$$\boldsymbol{B}_{(1,1,0)}=\left(\frac{\alpha}{\alpha+\xi_0},\frac{\xi_0}{\alpha+\xi_0},0\right)^T,\ \boldsymbol{B}_{(0,1,1)}=\left(0,\frac{\alpha}{\alpha+\xi_1},\frac{\xi_1}{\alpha+\xi_1}\right)^T,\ \boldsymbol{B}_{(0,2,0)}=(0,1,0)^T,$$

and

$$\boldsymbol{X}_{(2,0,0)} = (0,0)^T, \qquad \boldsymbol{X}_{(1,0,1)} = \left(\frac{\xi_0\xi_1}{\alpha+\xi_0\xi_1}, \frac{\xi_0\xi_1}{\alpha+\xi_0\xi_1}\right)^T, \quad \boldsymbol{X}_{(0,0,2)} = (1,1)^T,$$
$$\boldsymbol{X}_{(1,1,0)} = \left(\frac{2\xi_0}{\alpha+\xi_0}, 0\right)^T, \quad \boldsymbol{X}_{(0,1,1)} = \left(\frac{2\alpha+\xi_1}{\alpha+\xi_1}, \frac{\xi_1}{\alpha+\xi_1}\right)^T, \qquad \boldsymbol{X}_{(0,2,0)} = (2,0)^T.$$

If one is looking for an explicit representation of the Lagrange interpolating polynomial

$$p_n(\boldsymbol{u};\boldsymbol{\xi}), \quad \boldsymbol{u} = (u_1, u_2, \ldots, u_d)^T \in \mathbb{R}^d,$$

over the given $\triangle$, the symbolic system

$$\sum_{j=0}^{d} (\boldsymbol{V}_j)_i \, x_j = u_i, \quad i = 1,2,\ldots,d, \quad \sum_{j=0}^{d} x_j = 1,$$

has to be solved. This leads to the solution of the form

$$x_j = g_j(\boldsymbol{u}), \quad j = 0,1,\ldots,d. \tag{2.41}$$

For $d = 2$ and vertices (2.40), e.g., it follows

$$x_0 = 1 - \frac{1}{2}(u_1+u_2), \quad x_1 = \frac{1}{2}(u_1-u_2), \quad x_2 = u_2.$$

After inserting (2.41) into (2.31), one obtains the interpolating polynomial $p_n(\boldsymbol{u};\boldsymbol{\xi})$ over the lattice on $\triangle$.

Let $\boldsymbol{U}$ be an arbitrary point in $\triangle$ now. We would like to evaluate the interpolating polynomial p_n at the point $\boldsymbol{U}$ efficiently (see Figure 2.12).

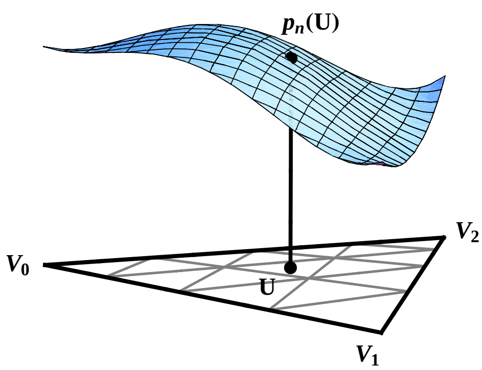


Figure 2.12: An evaluation of a polynomial at the point $\boldsymbol{U}$.

The previous observation gives one of the possible ways, but one can use a more efficient method by computing the barycentric coordinates $\widetilde{\boldsymbol{U}}$ of the point $\boldsymbol{U}$ w.r.t. $\triangle$. By inserting $\widetilde{\boldsymbol{U}}$ into (2.31), the desired value $p_n(\boldsymbol{U})$ is obtained.

To conclude the section, let us consider the Lagrange interpolating polynomial

$$p_3(\boldsymbol{x};\boldsymbol{\xi}), \quad \boldsymbol{x} \in \mathbb{R}^3, \; \sum_{i=0}^{2} x_i = 1,$$

over a 3-pencil lattice of order 3, given on a triangle $\triangle := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$ in $\mathbb{R}^2$, where $f_{\boldsymbol{\gamma}} \in \mathbb{R}$, $\boldsymbol{\gamma} \in \mathcal{I}_3^2$, are the prescribed data at lattice points.

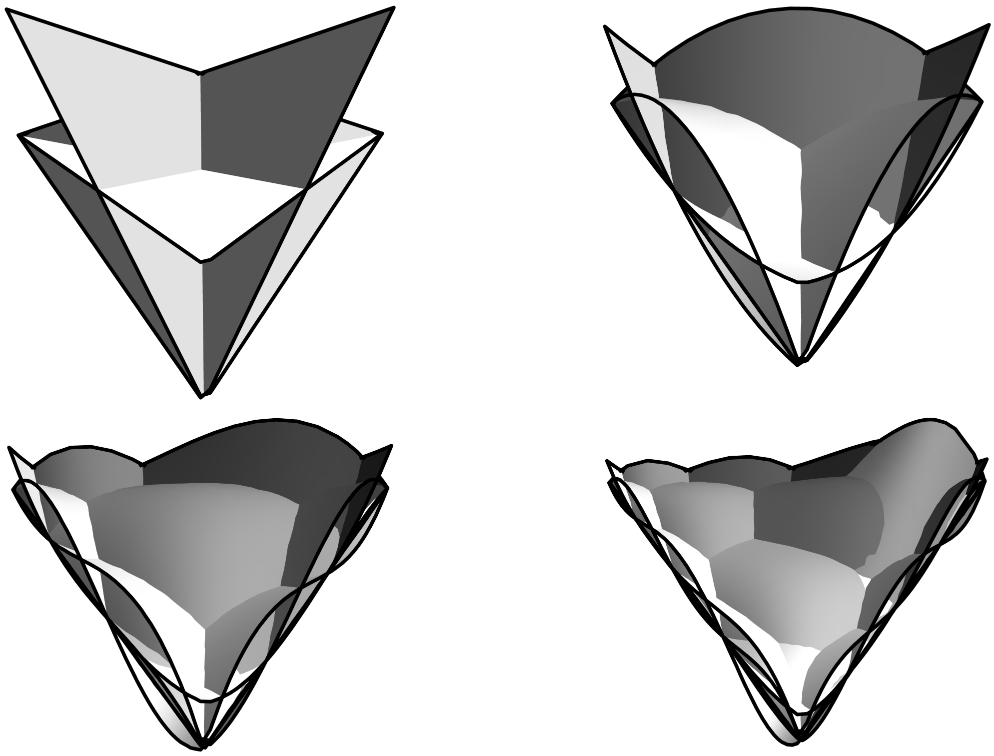


Figure 2.13: Lagrange basis polynomials $\mathcal{L}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma} \in \mathcal{I}_n^2$, over a three-pencil lattice of order n with parameters $\xi_0 = 2, \xi_1 = 1/2$ and $\xi_2 = 2/3$, for $n = 1, 2, 3, 4$.

By Theorem 2.7, the interpolating polynomial is equal to

$$p_3(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{\boldsymbol{\gamma} \in \mathcal{I}_3^2} f_{\boldsymbol{\gamma}} \mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}),$$

where $\mathcal{L}_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \mathcal{I}_3^2$, is determined by one of the following cases

$$\begin{array}{ll} \prod_{j=0}^{2} h_{i,j}, & \exists\, i,\; \gamma_i = 3, \\ \left(\frac{\left(\alpha^2+\alpha+\prod_{k=i}^{j-1} \xi_k\right)^3}{\alpha^2(1+\alpha+\alpha^2)\left(\prod_{k=i}^{j-1} \xi_k\right)} \right) \left(h_{j,0} \prod_{k=0}^{1} h_{i,k}\right), & \exists\, i < j,\; \gamma_i = 2,\; \gamma_j = 1, \end{array}$$

$$
\begin{aligned}
&\left(\tfrac{\left(\alpha^2+(\alpha+1)\prod_{k=j}^{i-1}\xi_k\right)^3}{\alpha^2(1+\alpha+\alpha^2)\left(\prod_{k=j}^{i-1}\xi_k\right)^2}\right)\left(h_{j,0}\prod_{k=0}^{1}h_{i,k}\right), \quad && \exists\, i>j,\ \gamma_i=2,\ \gamma_j=1,\\
&\left(\tfrac{\left(\alpha^2+\xi_0(\alpha+\xi_1)\right)^3}{\alpha^3\xi_0^2\xi_1}\right)\left(\prod_{i=0}^{2}h_{i,0}\right), && \boldsymbol{\gamma}=(1,1,1)^T,
\end{aligned}
$$

with

$$h_{i,j}(\boldsymbol{x};\boldsymbol{\xi}) = x_i + \frac{[n-j]_\alpha - [n]_\alpha}{[n-j]_\alpha\,\xi_i}\,x_{i+1} + \frac{[n-j]_\alpha - [n]_\alpha}{[n-j]_\alpha\,\xi_i\,\xi_{i+1}}\,x_{i+2}.$$

The polynomials $\mathcal{L}_{\boldsymbol{\gamma}}$ over a three-pencil lattice with parameters $\xi_0 = 2$, $\xi_1 = \frac{1}{2}$ and $\xi_2 = \frac{2}{3}$ on a triangle $\triangle = \langle\, (0,0), (2,0), (1,1)\,\rangle$ are shown in Figure 2.13.

Chapter 3

Lattices on simplicial partitions

In this chapter, an extension of a $(d+1)$-pencil lattice from a single simplex to a regular simplicial partition will be considered. Simplicial partitions are the most natural subdivisions of bounded complex domains. Lattices on regular simplicial partitions, where the lattice points coincide on adjacent faces, are particularly important, since they provide at least continuous piecewise polynomial interpolants. A simplicial partition in $\mathbb{R}^d$ is called *regular* if every pair of adjacent simplices has an r-face in common, $r \in \{0, 1, \ldots, d-1\}$ (see Figure 3.1 for $d = 2$). Furthermore, all simplicial partitions considered in the dissertation are assumed to be finite. The barycentric representation of a $(d + 1)$-pencil lattice, derived in the previous chapter, turns out to be a natural tool for the extension, since it keeps clear track of geometric properties.

3.1. Lattices on triangulations

In this section, a three-pencil lattice on a triangle will be first extended to adjacent triangle and further to a regular triangulation (see Figure 3.1). This will serve as a basis for the similar study in higher dimensions. Since our aim is to obtain a global three-pencil lattice on a regular triangulation, which will provide continuous piecewise polynomial interpolants, every two adjacent triangles have to share all lattice points on the common edge. This implies some relations between the center positions which are revealed in the following theorem. Recall again, that centers coincide with control points in the planar case.

THEOREM 3.1. *Let* $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$ *and* $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2 \rangle$ *be given triangles, and let the corresponding three-pencil lattices be determined by parameters* $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2)^T$ *and* $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \xi'_2)^T$*, respectively. Barycentric coordinates of lattice points at the edges* $\langle \boldsymbol{T}_0, \boldsymbol{T}_1 \rangle$ *and* $\langle \boldsymbol{T}'_0, \boldsymbol{T}'_1 \rangle$ *agree iff*

$$\xi_0 \xi'_1 \xi'_2 = \xi'_0 \xi_1 \xi_2, \tag{3.1}$$

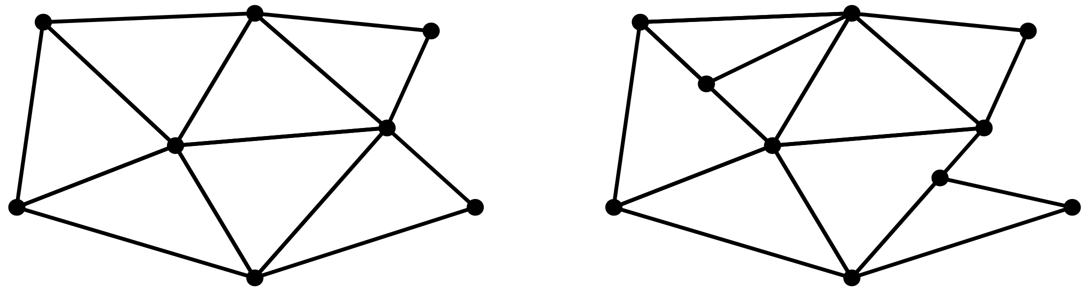


Figure 3.1: Example of a regular triangulation (left) and an irregular one (right).

in the case $n = 2$*, and*

$$\xi_1\xi_2 = \xi_1'\xi_2',\ \xi_0 = \xi_0',\ (\alpha' = \alpha), \quad \text{or} \quad \xi_0\xi_1'\xi_2' = 1,\ \xi_0'\xi_1\xi_2 = 1,\ (\alpha'\alpha = 1)\,, \tag{3.2}$$

for $n \geq 3$.

In the proof of the theorem we will use a well-known Descartes' rule of signs, given in the following lemma.

LEMMA 3.2. *The number of positive real roots of a real polynomial is bounded by the number of changes of signs in the sequence of its coefficients. Moreover, the number of positive real roots, counted with multiplicity, is of the same parity as the number of changes of signs.*

Proof. See [49], e.g. □

We are now able to prove Theorem 3.1.

Proof. Recall that the barycentric coordinates of lattice points can be written as (2.9). Therefore, one has to verify

$$\boldsymbol{B}_{n-k,k,0} \;=\; \boldsymbol{B}'_{n-k,k,0}, \quad k = 1, 2, \ldots, n-1, \tag{3.3}$$

only. But then (2.13) simplifies (3.3) to

$$\frac{\alpha^k - 1}{\alpha^n - \alpha^k}\,\xi_0 \;=\; \frac{\alpha'^{\,k} - 1}{\alpha'^{\,n} - \alpha'^{\,k}}\,\xi_0', \quad k = 1, 2, \ldots, n-1, \tag{3.4}$$

where

$$\alpha^n = \xi_0\xi_1\xi_2, \quad \alpha'^{\,n} = \xi_0'\xi_1'\xi_2'.$$

Consider the case $n = 2$ first. Then (3.4) simplifies to

$$\frac{\alpha - 1}{\alpha^2 - \alpha}\,\xi_0 = \frac{\alpha' - 1}{\alpha'^{\,2} - \alpha'}\,\xi_0'$$

and further to (3.1). Let now $n \geq 3$. Dividing equations in (3.4) for $k = 1$ and $k = 2$ leads to $f(\alpha) = f(\alpha')$, where

$$f(\alpha) = \frac{\alpha\,(\alpha^{n-2} - 1)}{(\alpha + 1)\,(\alpha^{n-1} - 1)} = \frac{\sum_{j=1}^{n-2} \alpha^j}{\sum_{j=1}^{n-1} (\alpha^j + \alpha^{j-1})}.$$

Since

$$f\left(\frac{1}{\alpha}\right) = \frac{\frac{1}{\alpha}\left(\frac{1-\alpha^{n-2}}{\alpha^{n-2}}\right)}{\left(\frac{1+\alpha}{\alpha}\right)\left(\frac{1-\alpha^{n-1}}{\alpha^{n-1}}\right)} = f(\alpha),$$

it follows $\alpha' = \alpha$ or $\alpha' = 1/\alpha$. It remains to prove that there are no other positive solutions. Since f is nonnegative, it is enough to prove that $f(\alpha) = c$, $c \geq 0$, has at most two positive solutions (see Figure 3.2). Since $\alpha > 0$, the equation $c - f(\alpha) = 0$ is

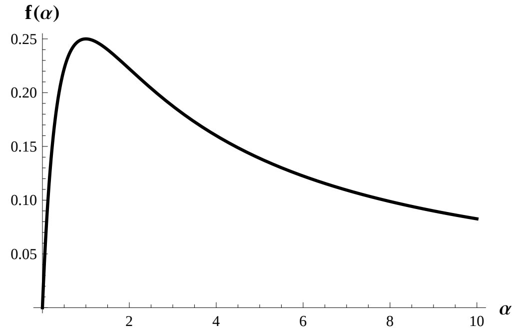


Figure 3.2: Function f for $n = 3$.

equivalent to

$$\left(\sum_{j=1}^{n-1}\left(\alpha^j + \alpha^{j-1}\right)\right)(c - f(\alpha)) = c\sum_{j=1}^{n-1}\left(\alpha^j + \alpha^{j-1}\right) - \sum_{j=1}^{n-2}\alpha^j$$
$$= c + \sum_{j=1}^{n-2}(2\,c-1)\;\alpha^j + c\,\alpha^{n-1} = 0.$$

In the sequence of coefficients

$$c, 2c-1, 2c-1, \ldots, 2c-1, c$$

there are at most two changes of signs. By Lemma 3.2 there are at most two roots in $[0, \infty)$. The relation (3.4) for $k = 1$ now gives the desired results stated in the theorem. It can also be easily verified that all solutions satisfy (3.4) for any suitable k as well. $\square$

Let us simplify the notation in this section and denote the triangle vertices $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, by $0, 1, 2$, respectively.

Consider the following example. Let $n \geq 3$ and

$$\xi_0 = 5/4,\;\; \xi_1 = 2,\;\; \xi_2 = 2/3.$$

The relations between ξ_1' and ξ_2', outlined in Theorem 3.1, for the transformations $\alpha \to \alpha$ and $\alpha \to 1/\alpha$, are

$$\xi_2' = \frac{4}{3\xi_1'}, \quad \xi_2' = \frac{4}{5\xi_1'},$$

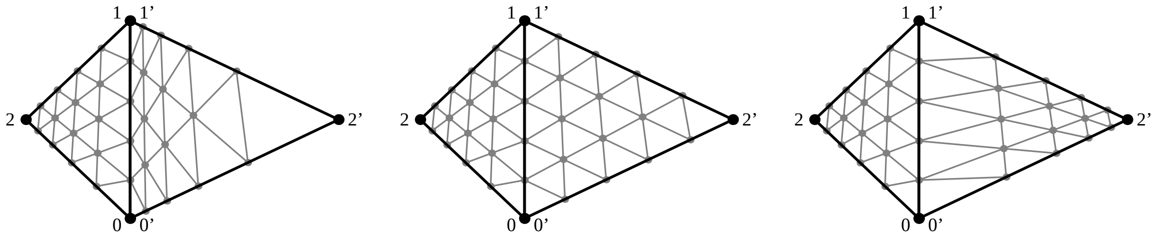


Figure 3.3: The lattices obtained for parameters $\xi_1' = \frac{1}{3}, 1, 3$ and the transformation $\alpha \to \alpha$.

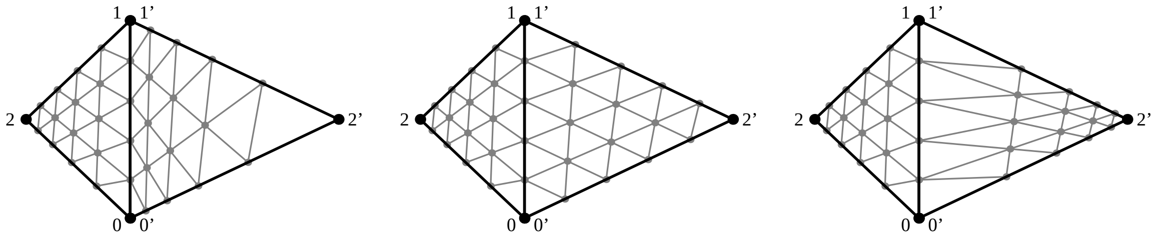


Figure 3.4: The lattices obtained for parameters $\xi_1' = \frac{1}{3}, 1, 3$ and the transformation $\alpha \to 1/\alpha$.

respectively. In Figure 3.3 and Figure 3.4, the lattices that correspond to points $\xi_1' = \frac{1}{3}, 1, 3$ are presented for both cases.

Theorem 3.1 gives relations which assure that the lattice points agree on the common edge for a particular labeling of triangle vertices. But, in order to construct a lattice on the whole triangulation, similar results for every possible pair of edges would be needed. Instead, only Theorem 3.1 together with rotations and mirror maps on the labels of the triangle vertices can be used. It is well-known that these transformations form the symmetric group S_3. Reflections of the triangle around one of the angle bisectors give the permutations (0 1), (0 2), (1 2), and the rotations are given by (0 1 2), (0 2 1), (0)(1)(2). The question is how the group transformations transform the center parameters ξ_i. By (2.10), it is easy to verify that the rotation (0 1 2) yields

$$\xi_0 \to \xi_1, \quad \xi_1 \to \xi_2, \quad \xi_2 \to \xi_0, \quad \alpha \to \alpha, \tag{3.5}$$

and the mirror map (0 1) gives

$$\xi_0 \to \xi_0^{-1}, \quad \xi_1 \to \xi_2^{-1}, \quad \xi_2 \to \xi_1^{-1}, \quad \alpha \to \alpha^{-1}. \tag{3.6}$$

Since S_3 is generated by (0 1 2) and (0 1), all the other transformations of centers can be obtained by compositions of (3.5) and (3.6):

$$\begin{aligned}
(0)(1)(2): &\quad \xi_0 \to \xi_0, &\quad \xi_1 \to \xi_1, &\quad \xi_2 \to \xi_2, &\quad \alpha \to \alpha,\\
(0\ 2\ 1): &\quad \xi_0 \to \xi_2, &\quad \xi_1 \to \xi_0, &\quad \xi_2 \to \xi_1, &\quad \alpha \to \alpha,\\
(0\ 2): &\quad \xi_0 \to \xi_1^{-1}, &\quad \xi_1 \to \xi_0^{-1}, &\quad \xi_2 \to \xi_2^{-1}, &\quad \alpha \to \alpha^{-1},\\
(1\ 2): &\quad \xi_0 \to \xi_2^{-1}, &\quad \xi_1 \to \xi_1^{-1}, &\quad \xi_2 \to \xi_0^{-1}, &\quad \alpha \to \alpha^{-1}.
\end{aligned}$$

A lattice on a given regular triangulation can now be constructed in the following way. Choose an arbitrary triangle first and use Theorem 2.4 to obtain the lattice. Then repeat the following steps until the whole triangulation is covered:

- add a triangle at a time in such a way that the obtained subtriangulation is simply connected,
- use transformations from the group S_3 and Theorem 3.1 to construct the lattice on the new triangle.

Obviously, new triangles can be added in various ways. Here they will be added so that the stars, also called cells (see Figure 3.5), at the boundary of the current subtriangulation will be completed in the positive direction around the star's inner vertex. Note that a *star of degree m* in $\mathbb{R}^2$ ([37]) is a triangulation with exactly one inner vertex (of the degree m).

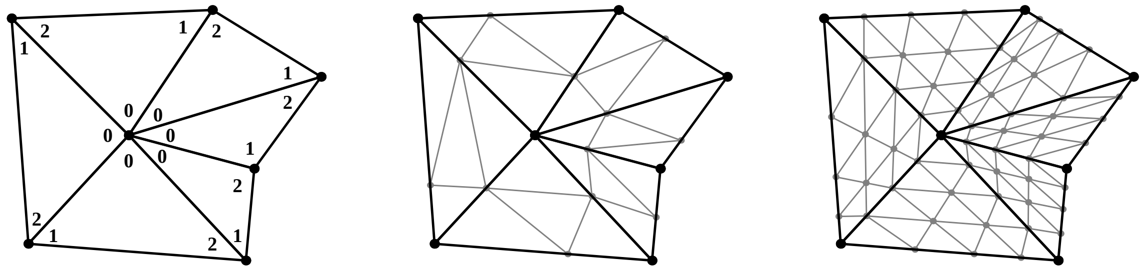


Figure 3.5: A star of degree 6 in $\mathbb{R}^2$ with global three-pencil lattices of orders 2 and 4.

Theorem 3.1 points out that each triangle added to a regular simply connected triangulation brings up an additional free center position unless the lattice points have already been prescribed on two edges. This happens when a star around an inner vertex is completed. In this case there are two additional equations to be fulfilled. The first chosen triangle brings 3 degrees of freedom, every other triangle adds one, and every star diminishes the degree by one as will be shown later on. Therefore one can conclude that a global three-pencil lattice on a regular simply connected triangulation with V vertices has V degrees of freedom. A more detailed analysis is given in the proof of the following theorem.

THEOREM 3.3. *Let $n > 2$. A global three-pencil lattice on a regular simply connected triangulation $\mathcal{T}$ with V vertices can be constructed by using Theorem 3.1 and transformations from the group S_3 (see Figure 3.7). There are V degrees of freedom.*

Proof. By Theorem 3.1, the result obviously holds for two triangles. Consider now a star of degree m. Let the starting triangle be chosen arbitrarily and let the rest of triangles be numbered consecutively in the positive direction around the star's inner vertex. Suppose that on the i-th triangle a lattice is given by parameters

$$\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)}, \quad i = 1, 2, \ldots, m,$$

where each $\xi_j^{(i)}$ defines a center $\boldsymbol{C}_j^{(i)}$ as in (2.10). Similarly, let each triangle be labeled in the positive direction starting with the inner point of the star (see Figure 3.6). The connections between parameters $\xi_j^{(i)}$ must be found so that the lattice points on common edges will coincide.

Let us choose the parameters for the first triangle as $\xi_j^{(1)} := \xi_j$, $j = 0, 1, 2$, and consider the i-th and $(i+1)$-th triangle (see Figure 3.6). In order to use Theorem 3.1, vertices of the common edge of the triangles considered must be labeled by 0 and 1. Therefore the transformation $(1\ 2) = (0\ 1)(0\ 1\ 2)$ for which

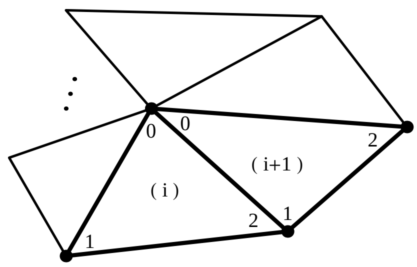


Figure 3.6: Labeling of the i-th and $(i+1)$-th triangle before the transformation.

$$\xi_0^{(i)} \to \frac{1}{\xi_2^{(i)}} =: \tilde{\xi}_0, \quad \xi_1^{(i)} \to \frac{1}{\xi_1^{(i)}} =: \tilde{\xi}_1, \quad \xi_2^{(i)} \to \frac{1}{\xi_0^{(i)}} =: \tilde{\xi}_2,$$

must first be used on the labels of the vertices of the i-th triangle. Now, Theorem 3.1 gives two possibilities, $\alpha \to \frac{1}{\alpha}$ and $\alpha \to \alpha$. In the first case the required equations are fulfilled iff

$$\xi_0^{(i+1)} = \frac{1}{\tilde{\xi}_1 \tilde{\xi}_2} = \xi_0^{(i)} \xi_1^{(i)}, \quad \xi_1^{(i+1)} = \sigma_i, \quad \xi_2^{(i+1)} = \frac{1}{\sigma_i \tilde{\xi}_0} = \frac{\xi_2^{(i)}}{\sigma_i}, \tag{3.7}$$

and in the second case iff

$$\xi_0^{(i+1)} = \tilde{\xi}_0 = \frac{1}{\xi_2^{(i)}}, \quad \xi_1^{(i+1)} = \sigma_i \tilde{\xi}_1 = \frac{\sigma_i}{\xi_1^{(i)}}, \quad \xi_2^{(i+1)} = \frac{\tilde{\xi}_2}{\sigma_i} = \frac{1}{\sigma_i \xi_0^{(i)}}, \tag{3.8}$$

where a new free parameter σ_i follows from Theorem 3.1. In the case (3.7), induction shows that

$$\xi_0^{(i)} = \xi_0 \xi_1 \prod_{j=1}^{i-2} \sigma_j, \quad \xi_1^{(i)} = \sigma_{i-1}, \quad \xi_2^{(i)} = \xi_2 \prod_{j=1}^{i-1} \sigma_j^{-1}, \quad i = 2, 3, \ldots, m. \tag{3.9}$$

Since the lattice points on the edge between the first and the last triangle must also agree, the final step gives the restriction

$$\prod_{i=1}^{m-1} \sigma_i \xi_1 = 1$$

on the choice of parameters σ_i. Therefore, it is clear that in this case the lattice on the star is determined by $m+1$ free parameters. In the second case, expressions (3.8) imply a distinction between odd and even m. For even numbered triangles, i.e., $i = 2k$, one obtains

$$\xi_0^{(2k)} = \frac{1}{\xi_2} \prod_{j=1}^{k-1} \sigma_{2j}, \quad \xi_1^{(2k)} = \frac{1}{\xi_1} \prod_{j=1}^{k-1} \sigma_{2j}^{-1} \prod_{j=1}^{k} \sigma_{2j-1}, \quad \xi_2^{(2k)} = \frac{1}{\xi_0} \prod_{j=1}^{k} \sigma_{2j-1}^{-1},$$

and for odd numbered triangles, $i = 2k+1$,

$$\xi_0^{(2k+1)} = \xi_0 \prod_{j=1}^{k} \sigma_{2j-1}, \quad \xi_1^{(2k+1)} = \xi_1 \prod_{j=1}^{k} \sigma_{2j}\, \sigma_{2j-1}^{-1}, \quad \xi_2^{(2k+1)} = \xi_2 \prod_{j=1}^{k} \sigma_{2j}^{-1},$$

for $i = 1, 2, \ldots, m$. If the degree of the star is even, $m = 2k$, the lattice points on the common edge between the first and the last triangle agree if

$$\prod_{i=1}^{k} \sigma_{2i-1} = 1,$$

which gives $m+1$ degrees of freedom. For odd degrees, $m = 2k-1$, this is true if

$$\prod_{i=1}^{k-1} \sigma_{2i} = \xi_0 \xi_2 \quad \text{and} \quad \xi_0 \xi_1 \xi_2 = 1,$$

therefore the number of degrees of freedom drops for one, because the equation $\alpha = 1$ must be fulfilled.

Since methods $\alpha \to \alpha$ and $\alpha \to \frac{1}{\alpha}$ can also be combined, the conclusions obtained above yield the only restriction: the method $\alpha \to \alpha$ in Theorem 3.1 must be used even number of times for this particular labeling of triangle vertices. But triangle vertices can be labeled arbitrarily. Each particular labeling determines the number of group transformations that give $\alpha \to \frac{1}{\alpha}$ so that Theorem 3.1 can be used as explained. If this number is odd (even), the method $\alpha \to \frac{1}{\alpha}$ in Theorem 3.1 must be used odd (even) number of times. That assures that the number of degrees of freedom is $m+1$.

Suppose now that the lattice has already been constructed on a simply connected subtriangulation $\mathcal{T}'$ of the triangulation $\mathcal{T}$. In the next step of the algorithm pick a vertex $\boldsymbol{P}$ at the boundary of $\mathcal{T}'$ and continue with the construction of the lattice on the star $\mathcal{S}$ around $\boldsymbol{P}$ in the positive direction. The lattice on a subtriangulation $\mathcal{S}'$ of $\mathcal{S}$ has already been computed. Let the triangle in $\mathcal{S}'$ that is adjacent to the starting triangle in $\mathcal{S} \setminus \mathcal{S}'$ be denoted by $\triangle^F$, and the triangle in $\mathcal{S}'$ adjacent to the last triangle in $\mathcal{S} \setminus \mathcal{S}'$ by $\triangle^L$. Let each triangle in $\mathcal{S} \setminus \mathcal{S}'$ be oriented in the positive direction with the point $\boldsymbol{P}$ corresponding to $\boldsymbol{T}_0$. The same must be done for triangles $\triangle^F$ and $\triangle^L$ by using transformations from S_3. The problem of determining the lattice parameters for triangles in $\mathcal{S} \setminus \mathcal{S}'$ is now the same as for the star. As shown before, each new triangle brings an additional degree of freedom, and the last triangle reduces the degree by one. Therefore the number of degrees of freedom increases by the number of points added to a triangulation. This concludes the proof of the theorem. $\square$

Since the case $n = 2$ is special in Theorem 3.1, the number of degrees of freedom in this case can be larger than the number of vertices of the triangulation. More precisely, the first chosen triangle brings 3 degrees of freedom, every other triangle adds two, and every star diminishes the degree by one. Therefore, for a regular triangulation with E edges there are E degrees of freedom.

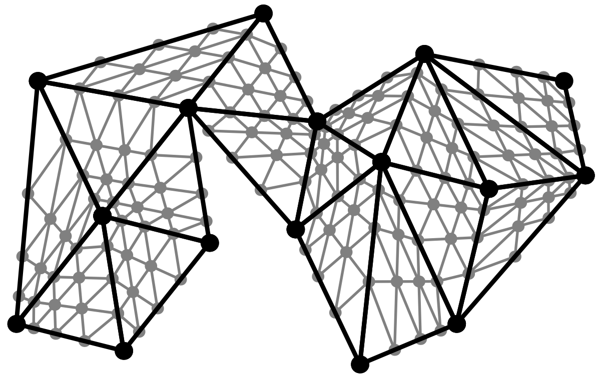


Figure 3.7: Lattice on a regular simply connected triangulation.

Theorem 3.3 gives the impetus to state the following conjecture.

CONJECTURE 3.4. *Let $\mathcal{T}$ be a regular simply connected simplicial partition with V vertices in $\mathbb{R}^d$. Then there exists a global $(d+1)$-pencil lattice on $\mathcal{T}$. Moreover, there are V degrees of freedom to construct it.*

Let us now consider a numerical example of a global three-pencil lattice on a star of degree 6 in $\mathbb{R}^2$ (Figure 3.5) and let the vertices of triangles be labeled as in Figure 3.5, left. Let first $n > 2$. Then by Theorem 3.3 the whole global lattice is determined by 7 parameters. Let us choose them as

$$\xi_0^{(1)} = \frac{3}{5}, \quad \xi_1^{(1)} = 1, \quad \xi_2^{(1)} = \frac{2}{3}, \quad \xi_1^{(2)} = \frac{4}{5}, \quad \xi_1^{(3)} = \frac{3}{4}, \quad \xi_1^{(4)} = \frac{4}{3}, \quad \xi_1^{(5)} = \frac{5}{6}.$$

Then the particular local lattices are by (3.9) determined with parameters

$$\begin{aligned}
\boldsymbol{\xi}^{(1)} &= \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}\right)^T = \left(\frac{3}{5}, 1, \frac{2}{3}\right)^T, \\
\boldsymbol{\xi}^{(2)} &= \left(\xi_0^{(1)}\xi_1^{(1)}, \xi_1^{(2)}, \frac{\xi_2^{(1)}}{\xi_1^{(2)}}\right)^T = \left(\frac{3}{5}, \frac{4}{5}, \frac{5}{6}\right)^T, \\
\boldsymbol{\xi}^{(3)} &= \left(\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(2)}, \xi_1^{(3)}, \frac{\xi_2^{(1)}}{\xi_1^{(2)}\xi_1^{(3)}}\right)^T = \left(\frac{12}{25}, \frac{3}{4}, \frac{10}{9}\right)^T,
\end{aligned}$$

$$\boldsymbol{\xi}^{(4)} = \left(\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(2)}\xi_1^{(3)}, \xi_1^{(4)}, \frac{\xi_2^{(1)}}{\xi_1^{(2)}\xi_1^{(3)}\xi_1^{(4)}} \right)^T = \left(\frac{9}{25}, \frac{4}{3}, \frac{5}{6} \right)^T,$$

$$\boldsymbol{\xi}^{(5)} = \left(\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(2)}\xi_1^{(3)}\xi_1^{(4)}, \xi_1^{(5)}, \frac{\xi_2^{(1)}}{\xi_1^{(2)}\xi_1^{(3)}\xi_1^{(4)}\xi_1^{(5)}} \right)^T = \left(\frac{12}{25}, \frac{5}{6}, 1 \right)^T,$$

$$\boldsymbol{\xi}^{(6)} = \left(\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(2)}\xi_1^{(3)}\xi_1^{(4)}\xi_1^{(5)}, \frac{1}{\xi_1^{(1)}\xi_1^{(2)}\xi_1^{(3)}\xi_1^{(4)}\xi_1^{(5)}}, \xi_1^{(1)}\xi_2^{(1)} \right)^T = \left(\frac{2}{5}, \frac{3}{2}, \frac{2}{3} \right)^T.$$

The whole global three-pencil lattice of order 4 is shown in Figure 3.5, right. Let now $n = 2$. Then the global lattice is determined by 12 parameters. Let us choose them as

$$\begin{aligned} &\xi_0^{(1)} = \frac{3}{5}, \quad \xi_1^{(1)} = 1, \quad \xi_2^{(1)} = \frac{2}{3}, \quad \xi_0^{(2)} = \frac{4}{5}, \quad \xi_1^{(2)} = \frac{3}{4}, \quad \xi_0^{(3)} = \frac{4}{3}, \\ &\xi_1^{(3)} = \frac{5}{6}, \quad \xi_0^{(4)} = \frac{2}{3}, \quad \xi_1^{(4)} = \frac{3}{2}, \quad \xi_0^{(5)} = 1, \quad \xi_1^{(5)} = 2, \quad \xi_0^{(6)} = \frac{1}{3}. \end{aligned}$$

Local lattices are now determined by

$$\boldsymbol{\xi}^{(j)} = \left(\xi_0^{(j)}, \xi_1^{(j)}, \frac{\xi_0^{(j)}\xi_2^{(1)}}{\xi_0^{(1)} \prod_{i=1}^{j-1} \left(\xi_1^{(i)}\xi_1^{(i+1)} \right)} \right)^T, \quad j = 1, 2 \ldots, 5,$$

$$\boldsymbol{\xi}^{(6)} = \left(\xi_0^{(6)}, \frac{1}{\prod_{i=1}^{5} \xi_1^{(i)}}, \frac{\xi_0^{(6)}\xi_2^{(1)}}{\xi_0^{(1)} \prod_{i=2}^{5} \xi_1^{(i)}} \right)^T.$$

The global three-pencil lattice of order 2 is shown in Figure 3.5, center.

To conclude the section, let us combine the results obtained here with the results in Section 2.4, where Lagrange interpolating polynomials have been studied. We are able to construct continuous piecewise polynomial interpolants over a triangulation (Figure 3.8).

3.2. Lattices on tetrahedral partitions

In the previous section, Conjecture 3.4 has been proved for the planar case. In this section, our aim is to prove it for the case $d = 3$. Recall (Theorem 2.6) that the barycentric coordinates of a four-pencil lattice on a tetrahedron $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ w.r.t. $\triangle$ are determined by four free parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3)^T$, $\xi_i > 0$, as

$$\left(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) \right)_{\boldsymbol{\gamma} \in I_n^3},$$

$$\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = \frac{1}{D_{\boldsymbol{\gamma},\boldsymbol{\xi}}} \left(\alpha^{n-\gamma_0} [\gamma_0]_\alpha, \xi_0 \alpha^{n-\gamma_0-\gamma_1} [\gamma_1]_\alpha, \xi_0\xi_1 \alpha^{\gamma_3} [\gamma_2]_\alpha, \xi_0\xi_1\xi_2 [\gamma_3]_\alpha \right)^T, \tag{3.10}$$

where

$$D_{\boldsymbol{\gamma},\boldsymbol{\xi}} = \alpha^{n-\gamma_0} [\gamma_0]_\alpha + \xi_0 \alpha^{n-\gamma_0-\gamma_1} [\gamma_1]_\alpha + \xi_0\xi_1 \alpha^{\gamma_3} [\gamma_2]_\alpha + \xi_0\xi_1\xi_2 [\gamma_3]_\alpha \quad \text{and} \quad \alpha^n = \prod_{i=0}^{3} \xi_i.$$

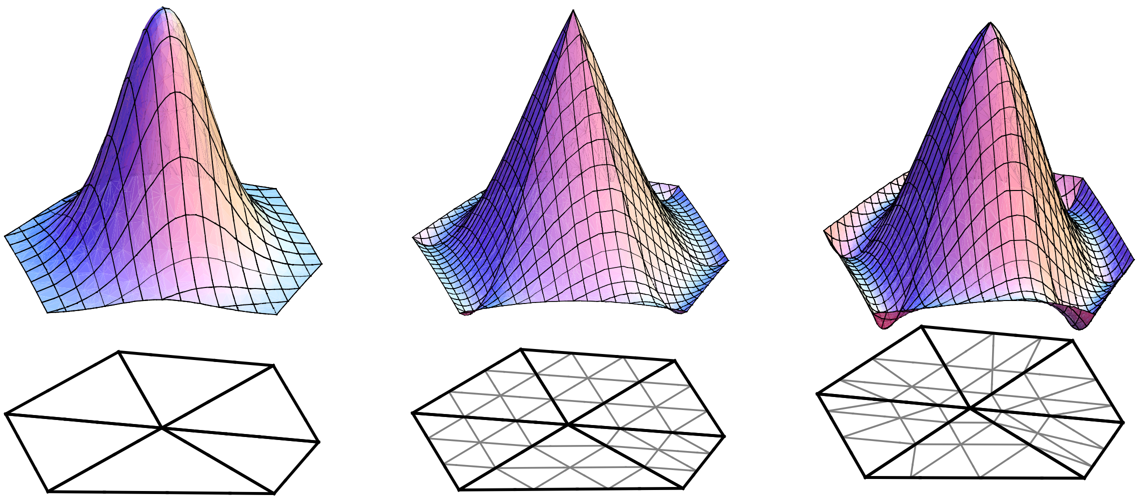


Figure 3.8: A surface over a star with two different continuous piecewise polynomial interpolants.

The results which follow will be a basis for the extension of a four-pencil lattice from a tetrahedron to a regular tetrahedral partition. As in the planar case, we will first answer the question when two lattices on tetrahedrons $\triangle$ and $\triangle'$ match on a common face. Suppose first, that a common face is an edge (1-face). The following theorem is an extension of Theorem 3.1.

THEOREM 3.5. *Let $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ and $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ be given tetrahedrons and let $(\boldsymbol{B}_\gamma(\boldsymbol{\xi}))_{\gamma \in I_n^3}$ and $\left(\boldsymbol{B}_\gamma(\boldsymbol{\xi}')\right)_{\gamma \in I_n^3}$ be the barycentric coordinates w.r.t. $\triangle$ and $\triangle'$ of four-pencil lattices of order n, $n \geq 3$, on $\triangle$ and $\triangle'$, respectively. The lattices coincide on the common edge*

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle, \qquad 0 \leq i_0 < i_1 \leq 3, \quad 0 \leq i'_0 < i'_1 \leq 3,$$

iff

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j = \prod_{j=i'_0}^{i'_1-1} \xi'_j \ \ \textit{and} \ \ \alpha' = \alpha \right) \ \textit{or} \ \left(\prod_{j=i_0}^{i_1-1} \xi_j = \alpha^n \prod_{j=i'_0}^{i'_1-1} \xi'_j \ \ \textit{and} \ \ \alpha'\alpha = 1 \right), \qquad (3.11)$$

where $\alpha^n = \prod_{j=0}^{3} \xi_j$ and $\alpha'^n = \prod_{j=0}^{3} \xi'_j$.

Proof. By (3.10), the barycentric coordinates w.r.t. $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle$, $0 \leq i_0 < i_1 \leq 3$, of the lattice on $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle$ are

$$\left(\frac{\alpha^n - \alpha^k}{\alpha^n - \alpha^k + (\alpha^k - 1) \prod_{j=i_0}^{i_1-1} \xi_j}, \frac{(\alpha^k - 1) \prod_{j=i_0}^{i_1-1} \xi_j}{\alpha^n - \alpha^k + (\alpha^k - 1) \prod_{j=i_0}^{i_1-1} \xi_j} \right)^T, \ k = 0, 1, \ldots, n.$$

Clearly, this corresponds to (2.12) and (2.13) with

$$\tau_k = \tau_k \left(\prod_{j=i_0}^{i_1-1} \xi_j \right).$$

Therefore the lattices coincide on $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle$, $0 \le i_0 < i_1 \le 3$, $0 \le i'_0 < i'_1 \le 3$, iff

$$\frac{\alpha^k - 1}{\alpha^n - \alpha^k} \prod_{j=i_0}^{i_1-1} \xi_j = \frac{{\alpha'}^{k-1}}{{\alpha'}^n - {\alpha'}^k} \prod_{j=i'_0}^{i'_1-1} \xi'_j, \quad k = 1, 2, \ldots, n-1. \tag{3.12}$$

In Theorem 3.1 it was shown that for $n \ge 3$ the system (3.12) has precisely two solutions and they are given by (3.11). □

REMARK 3.6. *As in Theorem 3.1, we have more degrees of freedom for $n = 2$. In this case, lattices coincide on the common edge iff*

$$\prod_{j=0}^{i_0-1} \xi_j \prod_{j=i_1}^{d} \xi_j \prod_{j=i'_0}^{i'_1-1} \xi'_j = \prod_{j=0}^{i'_0-1} \xi'_j \prod_{j=i'_1}^{d} \xi'_j \prod_{j=i_0}^{i_1-1} \xi_j.$$

By using barycentric coordinates one can similarly show that the lattices of order n, $n \ge 3$, coincide on the common edge

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle, \qquad 0 \le i_0 < i_1 \le 3, \quad 0 \le i'_1 < i'_0 \le 3,$$

iff

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j = \alpha^n \prod_{j=i'_1}^{i'_0-1} {\xi'_j}^{-1} \text{ and } \alpha' = \alpha \right) \text{ or } \left(\prod_{j=i_0}^{i_1-1} \xi_j = \prod_{j=i'_1}^{i'_0-1} {\xi'_j}^{-1} \text{ and } \alpha'\alpha = 1 \right). \tag{3.13}$$

Consider now two four-pencil lattices that share a lattice on a common triangle of tetrahedrons $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ and $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ (see Figure 3.9, e.g.).

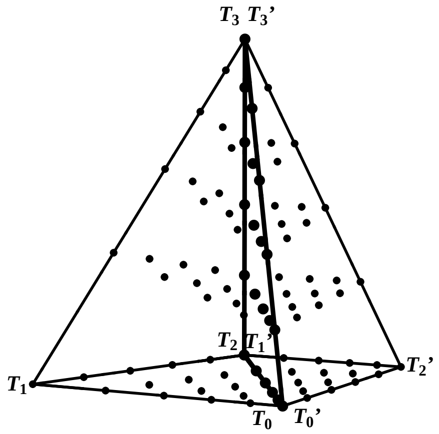


Figure 3.9: Matching of two lattices on a common face of tetrahedrons.

COROLLARY 3.7. *Let $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ and $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ be given tetrahedrons and let $(\boldsymbol{B}_\gamma(\boldsymbol{\xi}))_{\gamma \in I^3_n}$ and $\left(\boldsymbol{B}_\gamma(\boldsymbol{\xi}')\right)_{\gamma \in I^3_n}$ be the barycentric coordinates w.r.t.*

$\triangle$ and $\triangle'$ of four-pencil lattices of order n, $n \geq 3$, on $\triangle$ and $\triangle'$, respectively. Let $\alpha^n = \prod_{j=0}^{3} \xi_j \neq 1$ and let

$$\widetilde{\triangle} := \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \boldsymbol{T}_{i_2} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1}, \boldsymbol{T}'_{i'_2} \rangle,$$

$0 \leq i_0 < i_1 < i_2 \leq 3$, $0 \leq i'_0 < i'_1 < i'_2 \leq 3$, be the common triangle of tetrahedrons. Then the lattices coincide on $\widetilde{\triangle}$ iff

$$\prod_{j=i_k}^{i_{k+1}-1} \xi_j = \prod_{j=i'_k}^{i'_{k+1}-1} \xi'_j, \quad k = 0, 1, \quad \text{and} \quad \alpha' = \alpha. \tag{3.14}$$

Proof. The lattices coincide on $\widetilde{\triangle}$ iff they match on all three edges of $\widetilde{\triangle}$. Recall (3.11). Then for the first possibility, $\alpha' = \alpha$, the lattices coincide on $\widetilde{\triangle}$ iff (3.14) holds, and for the second one, $\alpha'\alpha = 1$, iff

$$\prod_{j=i_k}^{i_{k+\ell}-1} \xi_j = \alpha^n \prod_{j=i'_k}^{i'_{k+\ell}-1} \xi'_j, \qquad k = 0, 1, \quad \ell = 1, \ldots, 2-k. \tag{3.15}$$

But from (3.15) we obtain $\alpha = \alpha^2$, which is a contradiction, since $\alpha \neq 1$. $\square$

REMARK 3.8. *Note that with the assumption $\alpha = 1$ some further analysis could be easier but we would lose a degree of freedom.*

REMARK 3.9. *If $n = 2$, the lattices coincide on $\widetilde{\triangle}$ iff*

$$\prod_{j=0}^{i_k-1} \xi_j \prod_{j=i_{k+1}}^{d} \xi_j \prod_{j=i'_k}^{i'_{k+1}-1} \xi'_j = \prod_{j=0}^{i'_k-1} \xi'_j \prod_{j=i'_{k+1}}^{d} \xi'_j \prod_{j=i_k}^{i_{k+1}-1} \xi_j, \quad k = 0, 1, \quad \text{and} \quad \alpha' = \alpha.$$

The following corollary will be important for dealing with tetrahedral partitions.

COROLLARY 3.10. *Let $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ be a tetrahedron and let $(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}))_{\boldsymbol{\gamma} \in I_n^3}$ be the barycentric coordinates w.r.t. $\triangle$ of a four-pencil lattice of order n, $n \geq 3$, on $\triangle$. Further, let $\left(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')\right)_{\boldsymbol{\gamma} \in I_n^3}$ be the barycentric coordinates w.r.t. $\triangle'$ of the lattice on $\triangle'$, where*

$$\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle := \langle \boldsymbol{T}_{\sigma(0)}, \boldsymbol{T}_{\sigma(1)}, \boldsymbol{T}_{\sigma(2)}, \boldsymbol{T}_{\sigma(3)} \rangle, \quad \sigma \in C_4,$$

and C_4 is the cyclic group of order 4. Let $\alpha^n = \prod_{j=0}^{3} \xi_j \neq 1$. Then the lattices coincide on $\triangle$ iff

$$\xi'_i = \xi_{\sigma(i)}, \quad i = 0, 1, 2, 3. \tag{3.16}$$

Proof. The lattices coincide on $\triangle$ iff they coincide on all edges of $\triangle$. Using relations (3.11) and (3.13), Corollary 3.7, and the fact $|\sigma(i) - \sigma(j)| = |i - j|$, lattices match on the edges $\langle \boldsymbol{T}'_j, \boldsymbol{T}'_{j+1} \rangle = \langle \boldsymbol{T}_{\sigma(j)}, \boldsymbol{T}_{\sigma(j+1)} \rangle$, $j = 0, 1, 2$, and $\langle \boldsymbol{T}'_0, \boldsymbol{T}'_2 \rangle = \langle \boldsymbol{T}_{\sigma(0)}, \boldsymbol{T}_{\sigma(2)} \rangle$ iff (3.16) holds. It is then straightforward to verify the matching of the lattices on the other two edges. $\square$

In the following, our goal is to extend the global four-pencil lattice from two adjacent tetrahedrons to a regular simply connected tetrahedral partition. This extension should be done in such a way that the lattices on any two adjacent tetrahedrons coincide on common faces of tetrahedrons (Figure 3.9). We will prove that such an extension exists as well as that a lattice obtained in this way has the maximal possible number of free parameters. More precisely, we will prove Conjecture 3.4 for $d = 3$. Firstly, consider a particular case of a tetrahedral partition, i.e., a star in $\mathbb{R}^3$. Recall that a star of degree m in $\mathbb{R}^3$ is a tetrahedral partition with exactly one inner vertex (of the degree m). Note that such a star has $m + 1$ vertices all together. We will now only consider lattices of order $n \geq 3$. Similar analysis can be done for $n = 2$.

LEMMA 3.11. *Let $\mathcal{S}$ be a star of tetrahedrons of degree $V - 1$. Then there exists a global four-pencil lattice on $\mathcal{S}$ and there are V degrees of freedom to construct it.*

Proof. Let us first prove the lemma for the minimal star $\mathcal{S}_0 \in \mathbb{R}^3$, which consists of four tetrahedrons (see Figure 3.10). Then we will show how an arbitrary star can be obtained

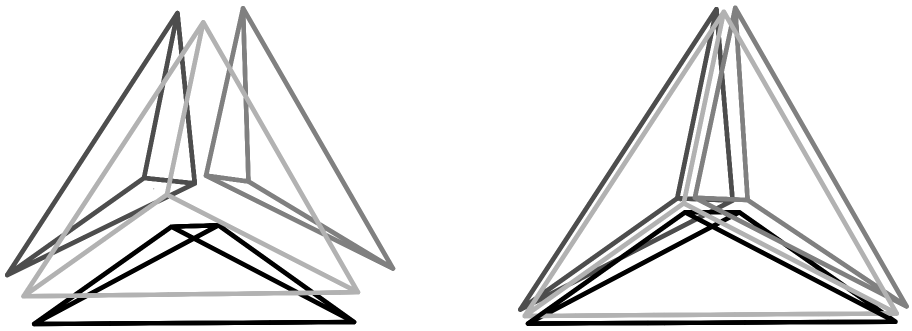


Figure 3.10: A minimal star in $\mathbb{R}^3$ obtained by gluing together four tetrahedrons.

from $\mathcal{S}_0$. The triangle

$$\langle \boldsymbol{T}_{j_0}, \boldsymbol{T}_{j_1}, \boldsymbol{T}_{j_2} \rangle_i, \quad j_0, j_1, j_2 \in \{0, 1, 2, 3\},$$

will denote the facet of the i-th tetrahedron with vertices $\boldsymbol{T}_{j_0}, \boldsymbol{T}_{j_1}, \boldsymbol{T}_{j_2}$. Let $\widetilde{\mathcal{S}}$ be the triangulation obtained from a star $\mathcal{S}$ by removing the interior point of $\mathcal{S}$ and all its incident edges. Further let the inner point of a star for all tetrahedrons in $\mathcal{S}$ be labeled by $\boldsymbol{T}_3$ and let the other vertices of tetrahedrons in $\mathcal{S}_0$ be ordered as in Figure 3.11. Here and throughout the proof the most important is to assure that the common triangle of any two adjacent tetrahedrons (i-th and i'-th) is of the form

$$\langle \boldsymbol{T}_{j_0}, \boldsymbol{T}_{j_1}, \boldsymbol{T}_{j_2} \rangle_i = \langle \boldsymbol{T}_{j_0'}, \boldsymbol{T}_{j_1'}, \boldsymbol{T}_{j_2'} \rangle_{i'}, \qquad j_0 < j_1 < j_2,\ j_0' < j_1' < j_2'. \tag{3.17}$$

Note that we could also use some other ordering of vertices in $\widetilde{\mathcal{S}}_0$, which satisfies (3.17) for all common triangles. Since $\mathcal{S}_0$ is a star of degree 4, we have to prove that there are 5 degrees of freedom to construct a lattice on it. Let the lattice on the i-th tetrahedron be determined by parameters $\xi_j^{(i)}$, $j = 0, 1, 2, 3$. We have to assure the matching of lattices

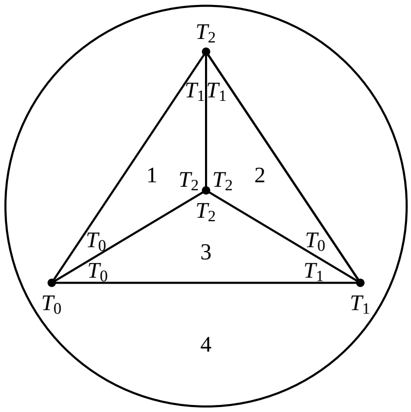


Figure 3.11: Triangulation $\widetilde{\mathcal{S}}_0$, obtained from a minimal star $\mathcal{S}_0$ by removing the interior point $\boldsymbol{T}_3$ and all its incident edges, embedded in a sphere.

on the following common triangles (see Figure 3.11)

$$\begin{aligned}
&\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_1 = \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_4, \quad \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_2 = \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_4,\\
&\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_3 = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_4, \quad \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 = \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2,\\
&\langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2 = \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_3, \quad \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 = \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_3.
\end{aligned}$$

By Corollary 3.7, all parameters $\xi_j^{(i)}$, $j = 0,1,2,3$, $i = 1,2,3,4$, are determined by 5 parameters $\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}$ and $\xi_0^{(2)}$ as

$$\begin{aligned}
\boldsymbol{\xi}^{(1)} &= \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(2)} = \left(\xi_0^{(2)}, \xi_1^{(1)}, \xi_2^{(1)}, \frac{\xi_0^{(1)}\xi_3^{(1)}}{\xi_0^{(2)}}\right)^T,\\
\boldsymbol{\xi}^{(3)} &= \left(\frac{\xi_0^{(1)}}{\xi_0^{(2)}}, \xi_1^{(1)}\xi_0^{(2)}, \xi_2^{(1)}, \xi_3^{(1)}\right)^T, \ \boldsymbol{\xi}^{(4)} = \left(\frac{\xi_0^{(1)}}{\xi_0^{(2)}}, \xi_0^{(2)}, \xi_1^{(1)}\xi_2^{(1)}, \xi_3^{(1)}\right)^T. \qquad (3.18)
\end{aligned}$$

Let E denote the number of edges, F the number of triangles, and V_k the number of vertices of degree k in $\widetilde{\mathcal{S}}$. Since

$$2E = \sum_k kV_k \quad \text{and} \quad 3F = 2E, \tag{3.19}$$

the Euler formula implies

$$\sum_k V_k(6-k) = 12. \tag{3.20}$$

Therefore, $\widetilde{\mathcal{S}}$ must have a vertex of degree at most five. Because every edge of $\widetilde{\mathcal{S}}$ must lie on two distinct triangles, each vertex has degree greater than two. Thus there is at least one vertex of degree 3, 4 or 5 in $\widetilde{\mathcal{S}}$. Let now $\mathcal{S}'$ denote the star of degree $V-2$. Any new star $\mathcal{S}$ of degree $V-1$ can be obtained from $\mathcal{S}'$ by one of the following operations. Add a new vertex into $\widetilde{\mathcal{S}}'$ to split

(a) one tetrahedron into three tetrahedrons (Figure 3.12),

(b) two tetrahedrons into four tetrahedrons (Figure 3.13),

(c) three tetrahedrons into five tetrahedrons (Figure 3.14).

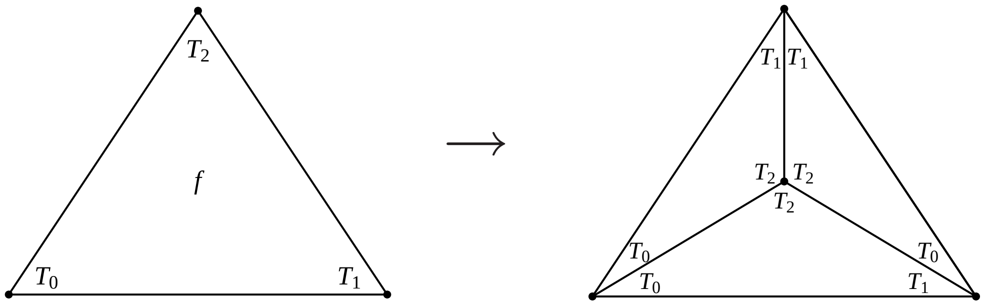


Figure 3.12: Adding a new vertex in order to split one tetrahedron into three tetrahedrons (Clough-Tocher split).

All these operations add one new vertex to $\mathcal{S}'$, so we have to prove that for each operation the number of free parameters increases by one. The relations that determine the parameters of three new tetrahedrons after the operation (a) (Figure 3.12) are similar to the relations in (3.18) and thus this operation brings up one new parameter. Let us now prove the same for the operation (b). Without loss of generality we can assume that the faces $f_1, f_2 \in \widetilde{\mathcal{S}}'$, of two selected tetrahedrons, as also the newly obtained tetrahedrons are ordered as in Figure 3.13. After a new vertex is added we have to assure the matching

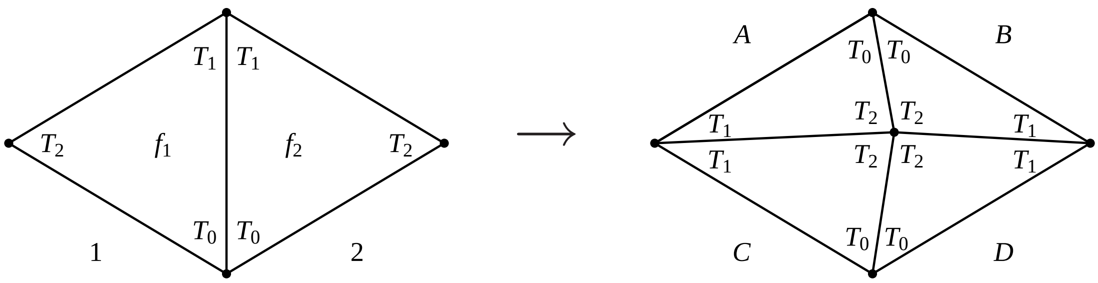


Figure 3.13: A new vertex splits two tetrahedrons into four tetrahedrons.

on common triangles

$$
\begin{aligned}
\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_A, & \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_C, \\
\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_B, & \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_D, \\
\langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_A &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_B, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_A &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_C, \\
\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_B &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_D, & \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_C &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_D.
\end{aligned}
$$

Using Corollary 3.7, the number of degrees of freedom increases again by one. Indeed

$$\boldsymbol{\xi}^{(1)} = \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(2)} = \left(\xi_0^{(1)}, \xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}}, \xi_3^{(1)}\right)^T,$$

$$\boldsymbol{\xi}^{(A)} = \left(\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(B)} = \left(\xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right)^T,$$

$$\boldsymbol{\xi}^{(C)} = \left(\xi_0^{(1)}\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(D)} = \left(\xi_0^{(1)}\xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\right)^T.$$

The operation (c) splits three tetrahedrons into five tetrahedrons (Figure 3.14). Again, without loss of generality, we can assume that the triangles f_1, f_2 and f_3 as also the newly obtained tetrahedrons are ordered as in Figure 3.14. Now there are 10 common

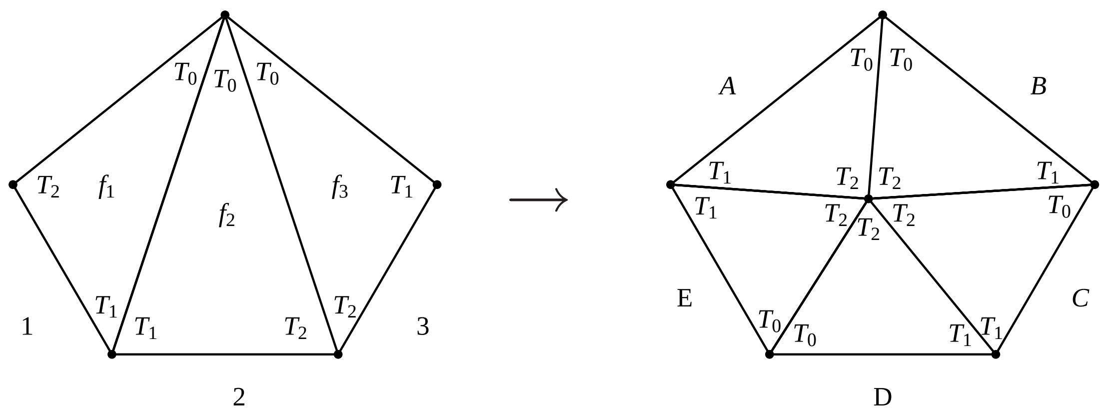


Figure 3.14: Three tetrahedrons are replaced with five tetrahedrons.

triangles where the matching has to be assured,

$$\begin{aligned}
\langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_A, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_E,\\
\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_D, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_3 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_C,\\
\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_3 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_B, & \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_A &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_B,\\
\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_B &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_C, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_C &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_D,\\
\langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_D &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_E, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_E &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_A,
\end{aligned}$$

and again Corollary 3.7 proves the desired fact. Indeed

$$\boldsymbol{\xi}^{(1)} = \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(2)} = \left(\xi_0^{(1)}, \xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}}, \xi_3^{(1)}\right)^T,$$

$$\boldsymbol{\xi}^{(3)} = \left(\xi_0^{(3)}, \frac{\xi_0^{(1)}\xi_1^{(2)}}{\xi_0^{(3)}}, \frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(A)} = \left(\xi_0^{(1)}\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\right)^T,$$

$$\boldsymbol{\xi}^{(B)} = \left(\xi_0^{(3)}, \frac{\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(A)}}{\xi_0^{(3)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(C)} = \left(\frac{\xi_0^{(1)}\xi_1^{(2)}}{\xi_0^{(3)}}, \frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\xi_0^{(3)}\right)^T,$$

$$\boldsymbol{\xi}^{(D)} = \left(\xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right)^T, \qquad \boldsymbol{\xi}^{(E)} = \left(\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right)^T.$$

It is straightforward to verify that all other orderings of the triangles f_1, f_2 and f_3 before the operations (b) and (c) and orderings of newly obtained tetrahedrons after these operations provide the same results as soon as these orderings are such that (3.17) holds for all common triangles. □

We are now able to generalize Theorem 3.3 and thus to prove Conjecture 3.4 for $d = 3$.

THEOREM 3.12. *Let $\mathcal{T}$ be a regular simply connected tetrahedral partition with V vertices. Then there exists a global four-pencil lattice on $\mathcal{T}$ which is determined by V parameters.*

Proof. By Corollary 3.7 the theorem obviously holds for two tetrahedrons and by Lemma 3.11 for a star. Suppose now that the lattice exists on a subpartition $\mathcal{T}'$ of the tetrahedral partition $\mathcal{T}$ and is determined by V' parameters, where V' is the number of vertices of $\mathcal{T}'$. Now take a vertex $\boldsymbol{T}$ at the boundary of $\mathcal{T}'$. Our goal is to prove the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$, where $\mathcal{S}$ is a star around the vertex $\boldsymbol{T}$. We have to use the procedure of Lemma 3.11 for $\mathcal{S}$ in such a way that the lattices on $\mathcal{T}'$ and $\mathcal{S}$ will match on $\mathcal{S}' := \mathcal{S} \cap \mathcal{T}' \neq \emptyset$. Let the inner point of $\mathcal{S}$ be labeled by $\boldsymbol{T}_3$ for all tetrahedrons in $\mathcal{S}$. Since the tetrahedrons in $\mathcal{S}'$ already have prescribed order of vertices, we have to use Corollary 3.10 to reorder these vertices such that the inner point of $\mathcal{S}$ becomes $\boldsymbol{T}_3$ also for all tetrahedrons in $\mathcal{S}'$. Furthermore, Lemma 3.11 shows that in order to prove the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$, we only have to find such an ordering of the vertices of tetrahedrons in $\mathcal{S}\backslash\mathcal{S}'$, that assures (3.17) for all common triangles in $\mathcal{S}$ (the relation (3.17) already holds for all common triangles in $\mathcal{S}'$). Note that if such an ordering exists then it can obviously be produced by the procedure described in Lemma 3.11. Let us describe one of the possible orderings of vertices of tetrahedrons from $\mathcal{S}\backslash\mathcal{S}'$, which satisfies the given requirements. Recall that the inner vertex of $\mathcal{S}$ should be labeled by $\boldsymbol{T}_3$ for all tetrahedrons in $\mathcal{S}\backslash\mathcal{S}'$. Suppose now that $\mathcal{S}$ is obtained from $\mathcal{S}'$ by adding a tetrahedron at a time, such that the instantaneous $\mathcal{S}$ is always simply connected. In each step a new tetrahedron can have a vertex that is not yet a part of the temporary $\mathcal{S}$. If so, then this vertex should be labeled by $\boldsymbol{T}_2$, while the remaining two vertices have to be ordered in such a way that (3.17) holds for all common triangles. If not so, then the vertex which has been added as the last one has to be labeled by $\boldsymbol{T}_2$, while again the same as before holds for the other two vertices. Thus we have proved the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$, and since $\mathcal{T}' \cap \mathcal{S} = \mathcal{S}'$, the number of parameters that describe the lattice increases exactly by the number of vertices added to the subpartition $\mathcal{T}'$. By continuing this process we finally prove the existence of the lattice on $\mathcal{T}$, which is determined by V parameters. □

3.3. Lattices on simplicial partitions

In this section, a $(d+1)$-pencil lattice will be extended from a simplex to a regular simply connected simplicial partition in $\mathbb{R}^d$. Since the results from the previous sections can not be directly generalized to a d-dimensional case, several additional tools and properties of lattices will be presented. Finally, Conjecture 3.4 will be proved in general.

3.3.1 Operations on $(d+1)$-pencil lattices

In this subsection, some necessary tools for extending a $(d+1)$-pencil lattice from a simplex to a simplicial partition are provided. Note that they pave a way to an important part of numerical analysis, computer algorithms. Several theorems, which are closely related to each other, are presented. The most important for the extension of a lattice from a simplex to a simplicial partition is Theorem 3.18 together with its corollaries. But the basis for all results in this subsection is the following theorem, which reveals a restriction of a lattice to a face of a simplex (Figure 3.15). Recall first (2.5) and (2.6).

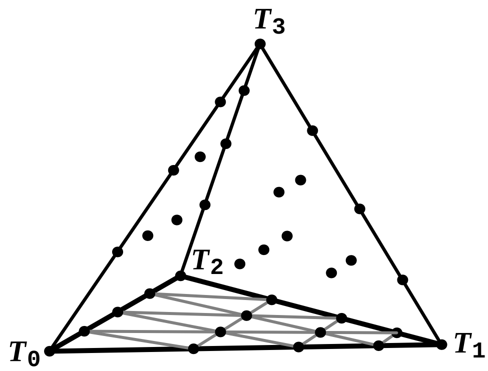


Figure 3.15: A restriction of a four-pencil lattice on a tetrahedron to a three-pencil lattice on one of its facets.

THEOREM 3.13. *Let a $(d+1)$-pencil lattice on a d-simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ be given in the barycentric form by the parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ as in (2.29). Let the indices*

$$\boldsymbol{i} = (i_0, i_1, \ldots, i_r)^T, \quad 0 \leq i_j \leq d, \quad \text{where } i_k \neq i_j \ \text{ if } k \neq j, \ \ r \leq d, \ \ w(\boldsymbol{i}) = 1,$$

determine an r-face $\triangle' = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \dots, \boldsymbol{T}_{i_r} \rangle \subset \triangle$. A restriction of the lattice to $\triangle'$ is an $(r+1)$-pencil lattice on $\triangle'$, with the barycentric coordinates w.r.t. $\triangle'$ determined by $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \dots, \xi'_r)^T$, where

$$\xi'_j = \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)}, \quad j = 0, 1, \dots, r, \tag{3.21}$$

and $\boldsymbol{\ell} = (\ell_j)_{j=0}^{r+1} = u\left((i_0, i_1, \dots, i_r, i_0)^T\right)$ (see Figure 3.16).

REMARK 3.14. *The product $\prod_{k=0}^{d} \xi_k$ of all parameters of a $(d+1)$-pencil lattice on a simplex $\triangle$ is equal to the product $\prod_{k=0}^{r} \xi'_k$ of all parameters of any $(r+1)$-pencil lattice, which is a restriction of the $(d+1)$-pencil lattice to an r-face of $\triangle$.*

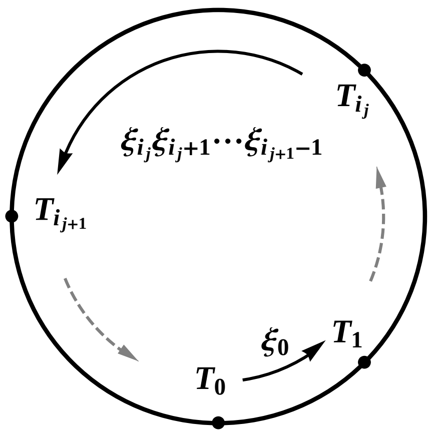


Figure 3.16: Parameters that determine a lattice on an r-face $\triangle' \subset \triangle$.

Proof. Recall Theorem 2.6 first. The notation $(v)_k$ will throughout the proof denote the k-th component of a vector v. Let $\boldsymbol{\gamma} \in \mathbb{N}_0^{r+1}$, $|\boldsymbol{\gamma}| = n$, be an index vector of a lattice point generated by $\boldsymbol{\xi}'$ over $\triangle'$. The map

$$\phi : \mathbb{N}_0^{r+1} \to \mathbb{N}_0^{d+1},$$

$$(\phi(\boldsymbol{\gamma}))_{k+1} = \begin{cases} \gamma_j, & i_j = k,\ 0 \le j \le r \\ 0, & \text{otherwise} \end{cases}, \quad k = 0, 1, \dots, d,$$

gives a relation between the index vectors of a particular point expressed in both lattices. Thus

$$\left(\boldsymbol{B}_{\phi(\boldsymbol{\gamma})}(\boldsymbol{\xi})\right)_{k+1} = 0, \quad k \ne i_j,\ 0 \le j \le r,$$

and one has to verify that

$$\left(\boldsymbol{B}_{\phi(\boldsymbol{\gamma})}(\boldsymbol{\xi})\right)_{i_j+1} = \left(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')\right)_{j+1}, \quad j = 0, 1, \dots, r, \tag{3.22}$$

only. Let $\alpha^n = \prod_{k=0}^{d} \xi_k$, and ${\alpha'}^n = \prod_{k=0}^{r} \xi'_k$. Note that

$$\left[(\phi(\boldsymbol{\gamma}))_{i_j+1}\right]_\alpha = [\gamma_j]_\alpha\,,$$

so by (2.29)

$$D_{\phi(\boldsymbol{\gamma}),\boldsymbol{\xi}} \cdot \left(\boldsymbol{B}_{\phi(\boldsymbol{\gamma})}\left(\boldsymbol{\xi}\right)\right)_{i_j+1}$$

simplifies to

$$\left(\prod_{k=0}^{i_j-1} \xi_k\right) \alpha^{n-\sum_{t=0}^{i_j}(\phi(\boldsymbol{\gamma}))_{t+1}} \; [\gamma_j]_\alpha\,. \tag{3.23}$$

Suppose the relations (3.21) hold. Then

$${\alpha'}^n = \prod_{j=0}^{r} \xi'_j = \prod_{j=0}^{r} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)}.$$

But the assertion $w(\boldsymbol{i}) = 1$ implies the existence of a unique s, $0 \le s \le r$, such that

$$0 \le \underbrace{i_{s+1} < i_{s+2} < \cdots < i_r <}_{r-s} \; \underbrace{i_0 < i_1 < \cdots < i_s}_{s+1} \le d,$$

and

$$\prod_{\substack{j=0\\ j\neq s}}^{r} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \left(\prod_{k=\ell_0}^{\ell_s-1} \xi_{m(k)}\right)\left(\prod_{k=\ell_{s+1}}^{\ell_{r+1}-1} \xi_{m(k)}\right) = \prod_{k=i_{s+1}}^{i_s-1} \xi_k,$$

with

$$\prod_{k=\ell_s}^{\ell_{s+1}-1} \xi_{m(k)} = \left(\prod_{k=i_s}^{d} \xi_k\right)\left(\prod_{k=0}^{i_{s+1}-1} \xi_k\right).$$

Therefore

$$\alpha' = \alpha.$$

Similarly,

$$\prod_{k=0}^{j-1} \xi'_k = \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)},$$

and so

$$D_{\boldsymbol{\gamma},\boldsymbol{\xi}'} \cdot \left(\boldsymbol{B}_{\boldsymbol{\gamma}}\left(\boldsymbol{\xi}'\right)\right)_{j+1} = \left(\prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)}\right) \alpha^{n-\sum_{t=0}^{j}\gamma_t} \; [\gamma_j]_\alpha\,. \tag{3.24}$$

Note that (3.22) follows from (2.29) if the quotient of the expressions (3.23) and (3.24) does not depend on j. A brief look at (3.23) at $j = 0$ reveals this quotient as

$$c = \left(\prod_{k=0}^{i_0-1} \xi_k\right) \alpha^{-\sum_{t=0}^{i_0-1}(\phi(\boldsymbol{\gamma}))_{t+1}}\,.$$

Indeed, the constant c is a quotient of (3.23) and (3.24) if

$$\frac{1}{c} \cdot \left(\prod_{k=0}^{i_j-1} \xi_k \right) \alpha^{n-\sum_{t=0}^{i_j} (\phi(\boldsymbol{\gamma}))_{t+1}} = \left(\prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)} \right) \alpha^{n-\sum_{t=0}^{j} \gamma_t} \tag{3.25}$$

for $0 \leq j \leq r$. To begin with, suppose that $0 \leq j \leq s$. Then $i_k = \ell_k$, $0 \leq k \leq j$, $i_0 < i_1 < \cdots < i_j$, and the left-hand side of the equation (3.25) simplifies to

$$\left(\prod_{k=i_0}^{i_j-1} \xi_k \right) \alpha^{n-\sum_{t=i_0}^{i_j} (\phi(\boldsymbol{\gamma}))_{t+1}} = \left(\prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)} \right) \alpha^{n-\sum_{t=0}^{j} \gamma_t},$$

as required. Now let $j > s$. Thus $i_j < i_0$ and the left-hand side of (3.25) simplifies to

$$\left(\prod_{k=i_j}^{i_0-1} \xi_k^{-1} \right) \alpha^{n+\sum_{t=i_j+1}^{i_0-1} (\phi(\boldsymbol{\gamma}))_{t+1}}.$$

Since

$$\left(\prod_{k=i_j}^{i_0-1} \xi_k^{-1} \right) \alpha^n = \left(\prod_{k=0}^{i_j-1} \xi_k \right) \left(\prod_{k=i_0}^{d} \xi_k \right) = \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)}$$

and

$$\sum_{t=i_j+1}^{i_0-1} (\phi(\boldsymbol{\gamma}))_{t+1} = n - \sum_{t=i_0}^{d} (\phi(\boldsymbol{\gamma}))_{t+1} - \sum_{t=0}^{i_j} (\phi(\boldsymbol{\gamma}))_{t+1} = n - \sum_{t=0}^{j} \gamma_t,$$

the proof is completed. □

As an example, let us consider a four-pencil lattice on a tetrahedron $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$, determined by parameters ξ_0, ξ_1, ξ_2 and ξ_3. A restriction of the lattice to a triangle $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$ (see Figure 3.15) is then determined by three parameters

$$\xi_0' = \xi_0, \quad \xi_1' = \xi_1, \quad \xi_2' = \xi_2 \xi_3.$$

Recall that the product $\alpha^n = \xi_0 \xi_1 \xi_2 \xi_3$ is equal to the product $\alpha'^n = \xi_0' \xi_1' \xi_2'$.

Let us now apply Theorem 3.13 in a particularly simple way: a restriction of a lattice to a line segment $\triangle' = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle$. Quite clearly, $w\left((i_0, i_1)^T\right) = 1$. Thus

$$\boldsymbol{\xi}' = (\xi_0', \xi_1')^T = \left(\xi_0', \frac{\alpha^n}{\xi_0'} \right)^T = \left(\prod_{k=\ell_0}^{\ell_1-1} \xi_{m(k)}, \prod_{k=\ell_1}^{\ell_2-1} \xi_{m(k)} \right)^T \tag{3.26}$$

and

$$\xi_0' = \begin{cases} \prod_{k=i_0}^{i_1-1} \xi_k, & i_0 < i_1, \\ \alpha^n \prod_{k=i_1}^{i_0-1} \xi_k^{-1}, & i_0 > i_1. \end{cases} \tag{3.27}$$

By (2.29), the barycentric coordinates of the lattice points on $\triangle'$ are

$$\left(\frac{[n]_\alpha - [n-\gamma_0]_\alpha}{[n]_\alpha - [n-\gamma_0]_\alpha + [n-\gamma_0]_\alpha\, \xi_0'}, \frac{[n-\gamma_0]_\alpha\, \xi_0'}{[n]_\alpha - [n-\gamma_0]_\alpha + [n-\gamma_0]_\alpha\, \xi_0'}\right)^T, \quad \gamma_0 = n, n-1, \ldots, 0, \tag{3.28}$$

as already obtained in previous sections. However, if the lattice points (3.28) are prescribed, the corresponding ξ_0', ξ_1', and $\alpha = \sqrt[n]{\xi_0'\xi_1'}$ are not unique, even for $n \geq 3$ (Theorem 3.1). In the latter case, there are precisely two pairs of parameters,

$$(\xi_0', \xi_1')^T, \quad \left(\frac{1}{\xi_1'}, \frac{1}{\xi_0'}\right)^T, \tag{3.29}$$

that generate the same lattice points (3.28). This is straightforward to deduce from identities

$$\frac{1}{\alpha^{2n-1-\gamma_0}}\left([n]_\alpha - [n-\gamma_0]_\alpha\right) = [n]_{\frac{1}{\alpha}} - [n-\gamma_0]_{\frac{1}{\alpha}},$$
$$\frac{1}{\alpha^{2n-1-\gamma_0}}\,[n-\gamma_0]_\alpha = \frac{1}{\alpha^n}\,[n-\gamma_0]_{\frac{1}{\alpha}},$$

or, alternatively, it can be proved by using relations (3.2).

In order to simplify further discussion, let us artificially define a two-pencil lattice. With a two-pencil lattice of order n on $\langle \boldsymbol{T}_0, \boldsymbol{T}_1 \rangle$ we will denote any restriction of a $(d+1)$-pencil lattice of order n to some of its edges. Accordingly to (3.28), the barycentric coordinates of two-pencil lattice points w.r.t. $\langle \boldsymbol{T}_0, \boldsymbol{T}_1 \rangle$ are determined by $(\xi_0, \xi_1)^T$ as

$$\left(\frac{[n]_\alpha - [n-\gamma_0]_\alpha}{[n]_\alpha - [n-\gamma_0]_\alpha + [n-\gamma_0]_\alpha\, \xi_0}, \frac{[n-\gamma_0]_\alpha\, \xi_0}{[n]_\alpha - [n-\gamma_0]_\alpha + [n-\gamma_0]_\alpha\, \xi_0}\right)^T, \quad \gamma_0 = n, n-1, \ldots, 0,$$

where $\alpha^n = \xi_0\xi_1$ (see Figure 3.17).

Figure 3.17: Two-pencil lattices with $\xi_0 = 1,\ \xi_1 = \frac{1}{5}, 1, 5$ (left), and $\xi_0 = \frac{1}{5}, 1, 5,\ \xi_1 = 1$ (right).

Now let us extend the previous consideration to line segments of an edge cycle

$$\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle, \quad k = 0, 1, \ldots, r, \quad i_{r+1} := i_0,$$

with $\boldsymbol{i} = (i_k)_{k=0}^r$, and

$$(\ell_k)_{k=0}^{r+1} = u\left((i_0, i_1, \ldots, i_r, i_0)^T\right).$$

Let $\left(\xi_{0,k}', \xi_{1,k}'\right)^T$ denote parameters of the restriction of a lattice to $\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle$. From (3.26) and (3.27) one obtains

$$\prod_{k=0}^{r} \xi_{0,k}' = \prod_{k=0}^{r} \prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \prod_{t=\ell_0}^{\ell_{r+1}-1} \xi_{m(t)} = \alpha^{n\cdot w(\boldsymbol{i})}, \tag{3.30}$$

that gives the value α in terms of parameters $\xi'_{0,k}$ only. Consider the lattice points at a particular edge $\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle$. By (3.29) they could be generated as a restriction of at most two different classes of lattices, the one with

$$\alpha = \sqrt[n]{\xi'_{0,k}\xi'_{1,k}},$$

or the additional one, having

$$\alpha = \frac{1}{\sqrt[n]{\xi'_{0,k}\xi'_{1,k}}}.$$

In order to further explore the second possibility, let $\varsigma_k, 0 < \varsigma_k < 1$, be the first barycentric coordinate of a lattice point given by (3.28) on $\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle$ at $\gamma_0 = n-1$. Such a lattice point exists for any $n \geq 2$. Then

$$\xi'_{0,k} = \xi'_{0,k}(\alpha) := \frac{1-\varsigma_k}{\varsigma_k}\left([n]_\alpha - 1\right),$$

and (3.30) simplifies to

$$\prod_{k=0}^{r} \xi'_{0,k}(\alpha) = \left(\prod_{k=0}^{r} \frac{1-\varsigma_k}{\varsigma_k}\right)\left([n]_\alpha - 1\right)^{r+1} = \alpha^{n\cdot w(\boldsymbol{i})}.$$

The equation

$$f(\rho) := [n]_\rho - 1 - c\,\rho^{\frac{n\cdot w(\boldsymbol{i})}{r+1}} = 0, \quad c := \left(\prod_{k=0}^{r} \frac{1-\varsigma_k}{\varsigma_k}\right)^{-\frac{1}{r+1}} > 0,$$

has at least one positive solution, $\rho = \alpha$, by the assumption. But f is a polynomial in $\sqrt[r+1]{\rho}$,

$$f(\rho) = -c\left(\sqrt[r+1]{\rho}\right)^{n\cdot w(\boldsymbol{i})} + \sum_{i=1}^{n-1}\left(\sqrt[r+1]{\rho}\right)^{i\cdot(r+1)},$$

and the Descartes's rule of signs (Lemma 3.2) shows that there are at most two zeros of f in $(0,\infty)$. If there are two, then by the observation for a particular edge the zeros are necessarily ρ and $1/\rho$, and an elimination of c from

$$f(\rho) = 0, \quad f\left(\frac{1}{\rho}\right) = 0,$$

yields

$$\frac{[n]_\rho - 1}{\rho^{\frac{n\cdot w(\boldsymbol{i})}{r+1}}} = \frac{[n]_{1/\rho} - 1}{\rho^{-\frac{n\cdot w(\boldsymbol{i})}{r+1}}}. \tag{3.31}$$

However,

$$[n]_\rho - 1 = \sum_{i=0}^{n-1}\rho^i - 1 = \sum_{i=1}^{n-1}\rho^i = \rho\,[n-1]_\rho,$$

$$[n]_{1/\rho} - 1 = \sum_{i=1}^{n-1}\rho^{-i} = \rho^{-(n-1)}\sum_{i=0}^{n-2}\rho^i = \rho^{-(n-1)}\,[n-1]_\rho,$$

and (3.31) reduces to

$$\rho^{n\left(\frac{2\cdot w(\boldsymbol{i})}{r+1}-1\right)} - 1 = 0,$$

that can only be satisfied for a positive ρ, $\rho \neq 1$, iff

$$w(\boldsymbol{i}) = \frac{r+1}{2}.$$

Thus we obtain the following observation. Suppose that the restriction of a $(d+1)$-pencil lattice of order n with a barycentric representation determined by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ is known for some edge of a simplex. By (3.28) we can then determine whether the corresponding $\alpha = \sqrt[n]{\prod_{k=0}^{d} \xi_k} = 1$ or $\alpha \neq 1$. Indeed, let the barycentric coordinates of lattice points on an edge be known, and suppose that they are, w.r.t. this edge, of the form

$$(b_{\gamma_0}, 1 - b_{\gamma_0})^T, \quad \gamma_0 = n, n-1, \ldots, 0.$$

Since all b_{γ_0} are by (3.28) determined with two parameters ξ_0', ξ_1', with $\alpha^n = \xi_0'\xi_1'$, it is enough to consider only two of them (e.g., the ones with $\gamma_0 = 1$ and $\gamma_0 = n-1$). Therefore,

$$\frac{\alpha^{n-1}}{\alpha^{n-1} + [n-1]_\alpha \, \xi_0'} = b_1, \quad \frac{[n]_\alpha - 1}{[n]_\alpha - 1 + \xi_0'} = b_{n-1}$$

is a system of two equations for the unknowns ξ_0' and ξ_1'. It follows that

$$\alpha = 1 \;\Leftrightarrow\; \frac{(1-b_1)b_{n-1}}{b_1(1-b_{n-1})} = (n-1)^2.$$

If $\alpha \neq 1$ (thus $\alpha \neq 1/\alpha$) there could be two classes of lattices, having the same restriction to this edge. The following theorem shows that in this case the restriction to a particular edge cycle has to be known, in order to determine the corresponding α uniquely (see Figure 3.18).

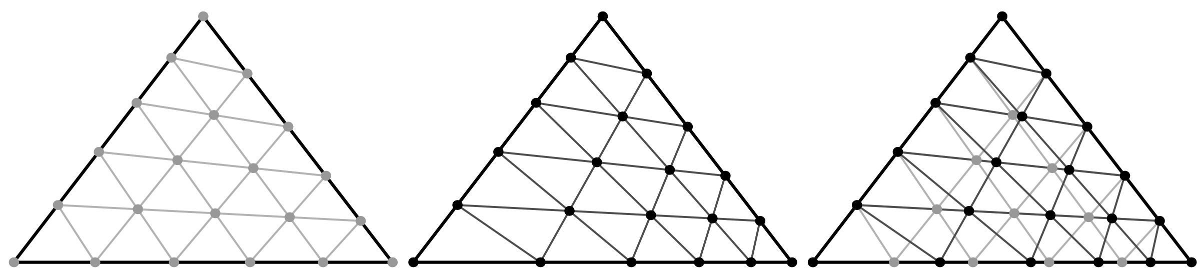


Figure 3.18: Two different lattices, which have the same restriction to two edges of a triangle. They are determined by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2)^T$ and $\boldsymbol{\xi}' = \left(\dfrac{1}{\xi_1\xi_2}, \xi_1, \dfrac{1}{\xi_0\xi_1}\right)^T$.

THEOREM 3.15. *Let the barycentric representation of a $(d+1)$-pencil lattice of order n on a d-simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ be given by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ and let $\prod_{k=0}^{d} \xi_k \neq 1$. A restriction of the lattice to a cycle*

$$\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle, \ k = 0, 1, \ldots, r, \quad i_{r+1} := i_0, \quad \boldsymbol{i} = (i_k)_{k=0}^{r},$$

determines the corresponding $\alpha = \sqrt[n]{\prod_{k=0}^{d} \xi_k}$ uniquely iff

$$w(\boldsymbol{i}) \neq \frac{r+1}{2}.$$

It is obvious that a $(d+1)$-pencil lattice on a simplex $\triangle$ is uniquely determined, if its restrictions to all edges of $\triangle$ are known. But, as we will see, only particular $d+1$ edges are actually needed (see Figure 3.19). For simplicity, let $\mathcal{G}(S)$ denote a graph induced by vertices and edges of a simplicial complex S (see Figure 3.20). Here S is a union of some arbitrarily dimensional faces of a simplex. Moreover, the subgraph $\mathcal{G}(S_1)$ spans the graph $\mathcal{G}(S)$ if the sets of vertices of both graphs coincide.

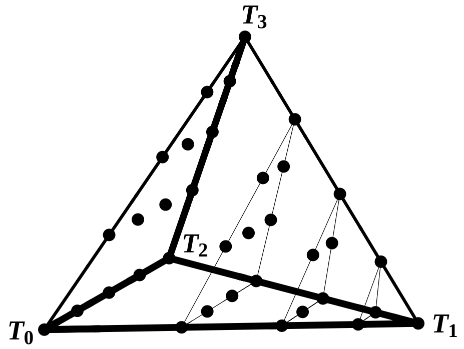


Figure 3.19: A restriction of a lattice to $d+1$ edges that uniquely determines the lattice on a simplex.

THEOREM 3.16. *A $(d+1)$-pencil lattice on a simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ with parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ is uniquely determined by restrictions to distinct edges*

$$e_k = \langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{j_k} \rangle, \quad k = 0, 1, \ldots, d,$$

iff the graph $g := \mathcal{G}\left(\bigcup_{k=0}^{d} e_k\right)$ spans the graph $\mathcal{G}(\triangle)$ and

(a) $\prod_{k=0}^{d} \xi_k = 1$ *or*

(b) g *contains a cycle*

$$e_{t_q} = \langle \boldsymbol{T}_{i_{t_q}}, \boldsymbol{T}_{j_{t_q}} \rangle, \quad q = 0, 1, \ldots, r,$$

with $i_{t_{q+1}} = j_{t_q}$, $q = 0, 1, \ldots, r-1$, $j_{t_r} = i_{t_0}$, such that

$$w\left(\left(i_{t_q}\right)_{q=0}^{r} \right) \neq \frac{r+1}{2}. \tag{3.32}$$

Proof. If g does not span $\mathcal{G}(\triangle)$, one can find a vertex $\boldsymbol{T}_t \in \mathcal{G}(\triangle)$ such that

$$\{e_k\}_{k=0}^d \subset \triangle' = \langle \boldsymbol{T}_0, \dots, \boldsymbol{T}_{t-1}, \boldsymbol{T}_{t+1}, \dots, \boldsymbol{T}_d \rangle.$$

Let the lattice on $\triangle$ be given by $\boldsymbol{\xi} = (\xi_k)_{k=0}^d$. By Theorem 3.13, its restriction to $\triangle'$ is determined by parameters

$$(\xi_0, \dots, \xi_{t-2}, \xi_{t-1} \cdot \xi_t, \dots, \xi_d)^T.$$

This makes impossible to recover both ξ_{t-1} and ξ_t, since only the product $\xi_{t-1} \cdot \xi_t$ is pinned down. Suppose now that g spans $\mathcal{G}(\triangle)$. Let $e' \in \mathcal{G}(\triangle)$ be any edge such that $e' \notin g$. Then there exists a cycle in

$$\mathcal{G}\left(\left(\cup_{k=0}^d e_k\right) \cup e'\right)$$

that contains e'. The restriction of the lattice to e' is determined by (3.30) iff $\alpha^n = \prod_{k=0}^d \xi_k$ is known. But the latter is assured by the assumptions (a) or (b) and Theorem 3.15. Thus a restriction of the lattice to any edge is determined, and restrictions to the edges $\langle \boldsymbol{T}_k, \boldsymbol{T}_{k+1} \rangle$, $k = 0, 1, \dots, d$, yield parameters $\boldsymbol{\xi}$. The proof is completed. □

Note that this result covers also the smallest cycle, i.e., $\langle \boldsymbol{T}_i, \boldsymbol{T}_j \rangle$, $\langle \boldsymbol{T}_j, \boldsymbol{T}_i \rangle$.

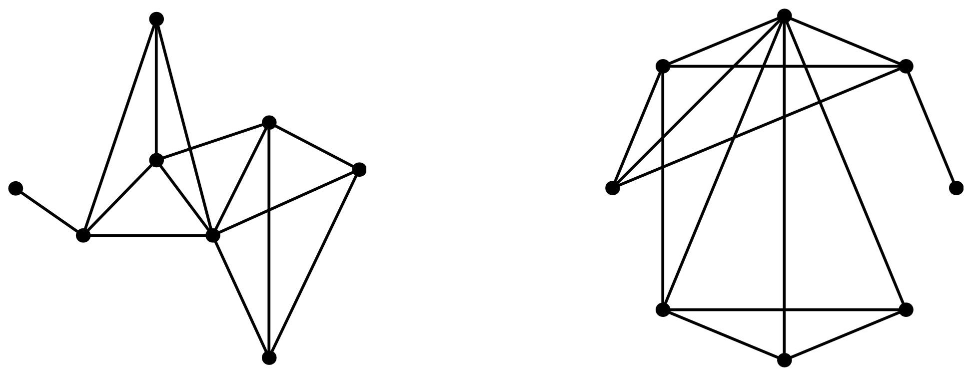


Figure 3.20: A simplicial complex $S \subseteq \mathbb{R}^3$ (left) and a graph $\mathcal{G}(S)$ induced by vertices and edges of S (right).

The assumption (3.32) in Theorem 3.16 is clearly used to determine the product α^n uniquely. But if this product is known, Theorem 3.16 simplifies to the following corollary, which needs no additional proof.

COROLLARY 3.17. *Suppose that the product*

$$\alpha^n = \prod_{k=0}^d \xi_k,$$

that corresponds to the barycentric representation of a $(d+1)$-pencil lattice with parameters $\boldsymbol{\xi}$ on a simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \dots, \boldsymbol{T}_d \rangle$, is known. The lattice is determined by restrictions to distinct edges

$$e_k = \langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{j_k} \rangle, \quad k = 1, 2, \dots, d,$$

iff the graph $g := \mathcal{G}\left(\bigcup_{k=1}^d e_k\right)$ spans the graph $\mathcal{G}(\triangle)$.

Our aim is now to generalize Theorem 3.1, Theorem 3.5 and Corollary 3.7. We will consider relations between two $(d+1)$-pencil lattices of order n that share a common face. Since this face is a simplex too, the first step is to determine when two lattices defined over the same simplex are equivalent, i.e., when they have the same lattice points. As expected, the choice of centers is inherent to equivalent lattices.

THEOREM 3.18. *Let $\triangle$ be a given simplex, with vertices ordered as*

$$\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \dots, \boldsymbol{T}_d \rangle, \tag{3.33}$$

and reordered according to an index vector $\boldsymbol{i} = (i_0, i_1, \dots, i_d)^T$ as

$$\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \dots, \boldsymbol{T}'_d \rangle = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \dots, \boldsymbol{T}_{i_d} \rangle. \tag{3.34}$$

Suppose that on the simplices $\triangle$ and $\triangle'$ there are given two $(d+1)$-pencil lattices of order n, with barycentric coordinates determined by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_d)^T$ and $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \dots, \xi'_d)^T$ w.r.t. the vertex sequences (3.33) and (3.34), respectively. Both lattices share the same lattice points iff one of the following possibilities holds:

(a) $w(\boldsymbol{i}) = 1$ *and* $\xi'_j = \xi_{i_j}$, $j = 0, 1, \dots, d$;

(b) $w(\boldsymbol{i}) = d$ *and* $\xi'_j = \dfrac{1}{\xi_{i_{j+1}}}$, $j = 0, 1, \dots, d$;

(c) $1 < w(\boldsymbol{i}) < d$ *and* $\prod_{j=0}^{d} \xi_j = 1$,

$$\xi'_j = \begin{cases} \displaystyle\prod_{k=i_j}^{i_{j+1}-1} \xi_k, & i_j < i_{j+1}, \\ \displaystyle\prod_{k=i_{j+1}}^{i_j-1} \frac{1}{\xi_k}, & i_j > i_{j+1}, \end{cases} \qquad j = 0, 1, \dots, d.$$

Proof. It is straightforward to verify the assertion for $d = 1$. Suppose now that $d > 1$. Then there is a 3-cycle along the edges of simplices $\triangle$ and $\triangle'$. So, by Theorem 3.15, the products $\alpha^n = \prod_{j=0}^{d} \xi_j$ and ${\alpha'}^n = \prod_{j=0}^{d} \xi'_j$ are determined uniquely. Let us consider a restriction of the lattice determined by $\boldsymbol{\xi}$ to $\langle \boldsymbol{T}_{i_j}, \boldsymbol{T}_{i_{j+1}} \rangle$, and let us denote $\boldsymbol{\ell} = (\ell_j)_{j=0}^{d+1} = u\left((i_0, i_1, \dots, i_d, i_0)^T \right)$. The lattice points of both lattices should coincide. Theorem 3.13 and the relation (3.29) reveal two possible choices, i.e.,

$$\xi'_j = \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} \tag{3.35}$$

if $\alpha' = \alpha$, and

$$\xi'_j = \frac{1}{\alpha^n} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} \tag{3.36}$$

if $\alpha'\alpha = 1$. Of course, ξ'_j can always be determined from (3.35) or (3.36). However, the relation between α and α' should not be violated. Let us multiply the left-hand sides and the right-hand sides of these equations, respectively, for all possible j. From (3.35) we obtain

$$\prod_{j=0}^{d} \xi'_j = \alpha'^{\,n} = \alpha^n = \prod_{j=0}^{d} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \alpha^{n\cdot w(\boldsymbol{i})}.$$

This relation could only be satisfied if $w(\boldsymbol{i}) = 1$ (the assertion (a)), or $\alpha = \alpha' = 1$. Similarly,

$$\prod_{j=0}^{d} \xi'_j = \alpha'^{\,n} = \frac{1}{\alpha^n} = \prod_{j=0}^{d} \frac{1}{\alpha^n} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \alpha^{n\cdot(w(\boldsymbol{i})-(d+1))}$$

confirms the assertion (b). If $1 < w(\boldsymbol{i}) < d$, only the possibility $\alpha = \alpha' = 1$ is left, and a brief look at (3.27) completes the necessary part of the proof. But if either one of the possibilities (a), (b) or (c) holds, the lattices agree on all edges of $\triangle$, i.e., $\langle \boldsymbol{T}'_j, \boldsymbol{T}'_k \rangle = \langle \boldsymbol{T}_{i_j}, \boldsymbol{T}_{i_k} \rangle$, $j < k$, and therefore on the whole simplex. □

If $\alpha = \alpha' = 1$, both lattices can coincide for any winding number of the index vector $\boldsymbol{i}$. But consequently a restriction on lattice parameters is obtained. Theorem 3.18 clearly suggests how a lattice known at some face should be extended to a whole simplex if one is not prepared to lose a degree of freedom with the assumption $\alpha = 1$.

COROLLARY 3.19. *Let $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_r \rangle$ be a given face, with the lattice determined by $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_r)^T$. The lattice can be extended to*

$$\triangle' = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_i, \boldsymbol{T}', \boldsymbol{T}_{i+1}, \ldots, \boldsymbol{T}_r \rangle \subset \mathbb{R}^{r+1}$$

by parameters

$$\boldsymbol{\xi}' = \left(\xi_0, \xi_1, \ldots, \xi_{i-1}, \eta, \frac{\xi_i}{\eta}, \xi_{i+1}, \ldots, \xi_r\right)^T,$$

where $\eta > 0$ is an additional free parameter.

Now we can consider two $(d+1)$-pencil lattices of order n that share a lattice on a common face of simplices (see Figure 3.21, e.g.). By combining Theorem 3.13 and Theorem 3.18 one obtains the following corollary.

COROLLARY 3.20. *Let*

$$\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle, \quad \triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \ldots, \boldsymbol{T}'_d \rangle$$

be given simplices, and let the lattices be determined by parameters

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T, \quad \boldsymbol{\xi}' = (\xi'_0, \xi'_1, \ldots, \xi'_d)^T,$$

respectively. Suppose that

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_r} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1}, \ldots, \boldsymbol{T}'_{i'_r} \rangle, \quad 1 \le r \le d,$$

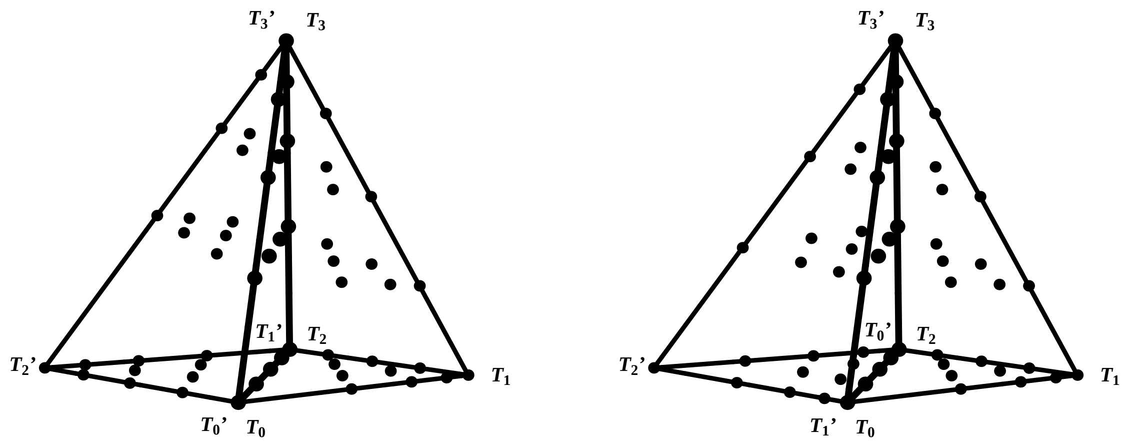


Figure 3.21: Matching of two lattices on a common face of simplices for both possibilities in Corollary 3.20, with $\xi_0 = \frac{4}{3}, \xi_1 = \frac{1}{2}, \xi_2 = 1, \xi_3 = \frac{1}{3}$, and the additional free parameter $\xi_1' = \frac{4}{5}$.

$0 \le i_0 < i_1 < \cdots < i_r \le d$, *is a common* r*-face of* $\triangle$ *and* $\triangle'$*, with corresponding vertices*

$$\boldsymbol{T}_{i_k} = \boldsymbol{T}'_{i'_k}, \quad k = 0, 1, \ldots, r.$$

Let $(\ell_0, \ldots, \ell_{r+1})^T = u\left((i_0, \ldots, i_r, i_0)^T\right)$ *and* $\left(\ell'_0, \ldots, \ell'_{r+1}\right)^T = u\left((i'_0, \ldots, i'_r, i'_0)^T\right)$*. If* $\alpha^n = \prod_{i=0}^d \xi_i \neq 1$*, the lattices coincide at the common* r*-face iff one of the following possibilities holds (see Figure 3.21):*

(a) $w(\boldsymbol{i}') = 1$ *and*

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \ldots, r\,; \tag{3.37}$$

(b) $w(\boldsymbol{i}') = r$ *and*

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \alpha^n \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \ldots, r.$$

3.3.2 Extension to a simplicial partition

We are now ready to prove the Conjecture 3.4 in general. The following theorem and its proof provide an approach to the construction of a global $(d+1)$-pencil lattice over a regular simply connected simplicial partition. This leads to an efficient computer algorithm for the design of a lattice. One of the main ideas of the proof is the assumption, that the product of lattice parameters should be equal to the same constant α^n for all local lattices on simplices of a partition. Namely, it would be to complicated to control the behavior of a global lattice if we would allow both α and $\frac{1}{\alpha}$ to interchange.

THEOREM 3.21. *Let $\mathcal{T}$ be a regular simply connected simplicial partition in $\mathbb{R}^d$ with $V \geq d+1$ vertices*

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{V-1}. \tag{3.38}$$

Then there exists a global $(d+1)$-pencil lattice on $\mathcal{T}$ with precisely V degrees of freedom.

Proof. For any simplex $\triangle \in \mathcal{T}$, let us order the vertices similarly as in (3.38), i.e.,

$$\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle := \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_d} \rangle, \quad 0 \leq i_0 < i_1 < \cdots < i_d \leq V-1,$$

and let us choose the local barycentric representation of a lattice on each of the simplices accordingly (see Figure 3.22).

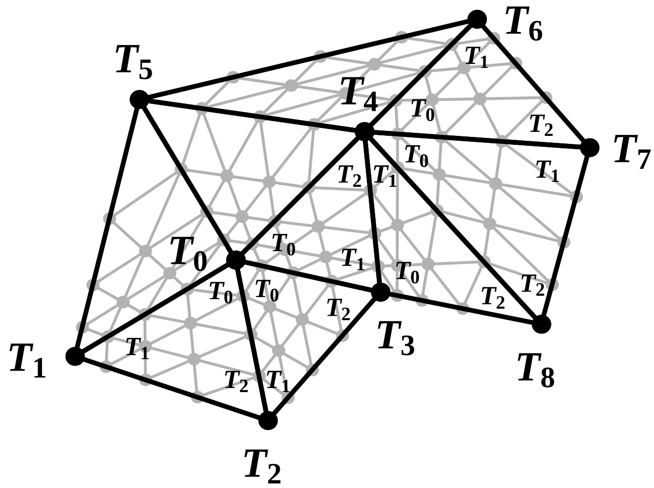


Figure 3.22: A global labeling of all vertices implies the order of vertices (and consequently the positions of control points) for all local simplices.

Note that this choice of local lattice control points assures that any pair of simplices $\triangle, \triangle' \in \mathcal{T}$,

$$\triangle = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_d} \rangle, \quad \triangle' = \langle \boldsymbol{T}_{i'_0}, \boldsymbol{T}_{i'_1}, \ldots, \boldsymbol{T}_{i'_d} \rangle,$$

with a common r-face, denoted in $\triangle$ as

$$\langle \boldsymbol{T}_{i_{j_0}}, \boldsymbol{T}_{i_{j_1}}, \ldots, \boldsymbol{T}_{i_{j_r}} \rangle, \quad i_{j_0} < i_{j_1} < \cdots < i_{j_r},$$

and corresponding vertices in $\triangle'$ given by

$$\boldsymbol{T}_{i'_{j'_k}} = \boldsymbol{T}_{i_{j_k}}, \quad k = 0, 1, \ldots, r,$$

satisfies

$$w\left((i_{j_0}, i_{j_1}, \ldots, i_{j_r})^T\right) = w\left(\left(i'_{j'_0}, i'_{j'_1}, \ldots, i'_{j'_r}\right)^T\right) = 1. \tag{3.39}$$

The proof proceeds by the induction on the number of simplices in a simplicial partition $\mathcal{T}' \subset \mathcal{T}$, with an additional assertion that a product of local barycentric lattice parameters for each simplex considered is equal to the same constant α^n. Since $\mathcal{T}$ is regular, we may, without loss of generality, assume that $\mathcal{T}'$ grows from a single simplex to $\mathcal{T}$ in such a way that each simplex added has F, $1 \leq F \leq d$, facets in common with simplices in the instantaneous partition $\mathcal{T}'$. If $\mathcal{T}' = \{\triangle\}$, then by (2.29) the lattice has $d+1$ free parameters $\boldsymbol{\xi} = (\xi_i)_{i=0}^d$, defining $\alpha^n = \prod_{i=0}^d \xi_i$. The number of degrees of freedom is clearly equal to the number of vertices of $\mathcal{T}'$. Thus the assertion holds true. Suppose now that it holds true for $\mathcal{T}'$, and let us show that it holds also for

$$\mathcal{T}' \cup \{\triangle'\},\ \triangle' = \langle \boldsymbol{T}_{i'_0}, \boldsymbol{T}_{i'_1}, \ldots, \boldsymbol{T}_{i'_d} \rangle \notin \mathcal{T}'.$$

Let the local barycentric lattice representation on $\triangle'$ depend on parameters $\boldsymbol{\xi}' = (\xi'_i)_{i=0}^d$, and let $\{f_1, f_2, \ldots, f_F\}$ be the set of all distinct facets of $\triangle'$ that are shared with simplices in $\mathcal{T}'$. Since (3.39) holds, the relations (3.37) confirm that the lattice can be extended from the common face f_1 to $\triangle'$ provided particular d relations concerning $\boldsymbol{\xi}'$ are satisfied. With an index r uniquely determined by $\boldsymbol{T}_{i'_r} \in \triangle' \setminus f_1$, these relations determine d values

$$\left(\xi'_0, \xi'_1, \ldots, \xi'_{r-2}, \xi'_{r-1} \cdot \xi'_r, \xi'_{r+1}, \xi'_{r+2}, \ldots, \xi'_d\right)^T,$$

and assure $\prod_{i=0}^d \xi'_i = \alpha^n$. If $F = 1$, $\boldsymbol{T}_{i'_r} \notin \mathcal{T}'$. So $\mathcal{T}' \to \mathcal{T}' \cup \{\triangle'\}$ brings up precisely one additional vertex as well as one additional free parameter, and the induction step in the case $F = 1$ is concluded. Let now $2 \leq F \leq d$. The number of vertices in $\mathcal{T}' \cup \{\triangle'\}$ is equal to the number of vertices in $\mathcal{T}'$. At least one of the edges $\langle \boldsymbol{T}_{i'_{r-1}}, \boldsymbol{T}_{i'_r} \rangle$ and $\langle \boldsymbol{T}_{i'_r}, \boldsymbol{T}_{i'_{r+1}} \rangle$ belongs to f_2. Let us denote it by e. Since α has already been determined, a restriction of the lattice to the edge e determines the last free parameter in $\boldsymbol{\xi}'$ uniquely. Note that the lattice given by $\boldsymbol{\xi}'$ by the construction agrees with any lattice on f_2, inherited from $\mathcal{T}'$, on $f_1 \cap f_2$ and e. But $\mathcal{G}((f_1 \cap f_2) \cup e)$ spans $\mathcal{G}(f_2)$, so by Corollary 3.17 both lattices have to coincide on all of f_2. Similarly, $\mathcal{G}((f_1 \cap f_j) \cup (f_2 \cap f_j))$ spans $\mathcal{G}(f_j)$ for any j, $3 \leq j \leq d$, and the lattice given by $\boldsymbol{\xi}'$ agrees with inherited lattice on any f_j. The induction step in the case $F > 1$ is concluded too, and the proof is completed. □

In the last theorem and in similar theorems in previous sections it was always assumed that $\mathcal{G}(\mathcal{T})$ is d-vertex connected.

DEFINITION 3.22. *A graph $\mathcal{G}$ is said to be ℓ-vertex connected if the cardinality of a smallest set of vertices, whose removal renders $\mathcal{G}$ disconnected, is greater or equal to ℓ.*

Let us extend Theorem 3.21 to such simplicial partitions.

COROLLARY 3.23. *Let $\mathcal{T}$ be a regular simply connected simplicial partition in $\mathbb{R}^d$ with V vertices, such that $\mathcal{G}(\mathcal{T})$ is 2-vertex connected. Then there exists a global $(d+1)$-pencil lattice on $\mathcal{T}$ with precisely V degrees of freedom.*

Proof. As in Theorem 3.21, we will assume that for all local lattices the product of lattice parameters is equal to the same constant α^n. Clearly, $\mathcal{G}(\mathcal{T})$ is at most d-vertex connected. If $\mathcal{G}(\mathcal{T})$ is d-vertex connected, then the proof follows by Theorem 3.21. Let now $\mathcal{G}(\mathcal{T})$ be k-vertex connected, $2 \leq k < d$, but not $(k+1)$-vertex connected (see Figure 3.23, e.g.). Then there exists a d-simplex $\triangle$, which has only i-simplices, $i \leq k-1$, in common with adjacent d-simplices. Let $\triangle'$ be a simplex, such that $\triangle \cap \triangle'$ is a $(k-1)$-simplex. Since there are $d+1-k$ vertices of $\triangle'$ that are not vertices of $\triangle$, we have exactly $d-(k-1)$ free parameters to determine the rest of the lattice on $\triangle'$. Note that a lattice on an r-simplex is determined by r parameters, since α^n is known. Thus, there are again V degrees of freedom to construct a lattice on such a simplicial partition. $\square$

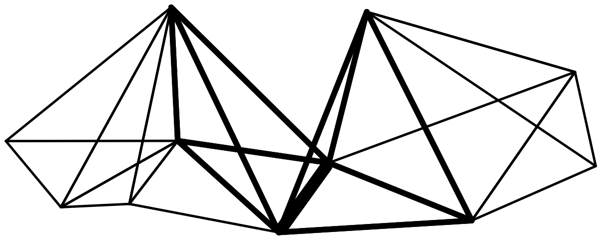


Figure 3.23: An example of a tetrahedral partition $\mathcal{T}$ in $\mathbb{R}^3$, such that $\mathcal{G}(\mathcal{T})$ is not 3-vertex connected.

REMARK 3.24. *It is straightforward to verify, that for a simply connected regular simplicial partition $\mathcal{T}$ in $\mathbb{R}^d$ with V vertices, such that $\mathcal{G}(\mathcal{T})$ is not 2-vertex connected, the number of degrees of freedom equals $V+m$, where m is the cardinality of a set of vertices, which have a property, that removing any of them causes $\mathcal{G}(\mathcal{T})$ to be disconnected (see Figure 3.24).*

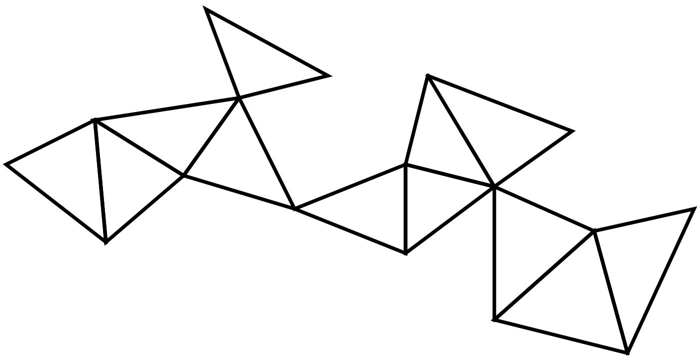


Figure 3.24: An example of a triangulation with 17 vertices, where a global three-pencil lattice with 20 shape parameters can be constructed.

Let us conclude this and previous two sections by considering an example of a star in $\mathbb{R}^3$ with $2m-2$, $m \geq 3$, tetrahedrons, where m and $m-2$ tetrahedrons are glued together in such a way, that they share a common edge, respectively (see Figure 3.25). This example also covers the minimal possible star in $\mathbb{R}^3$ (see Figure 3.10) with 4 tetrahedrons ($m=3$). Our aim is to explicitly express $(d+1)(2m-2) = 8(m-1)$ parameters

$$\xi_j^{(i)} > 0, \quad j = 0,1,2,3, \quad i = 1,\ldots,2m-2,$$

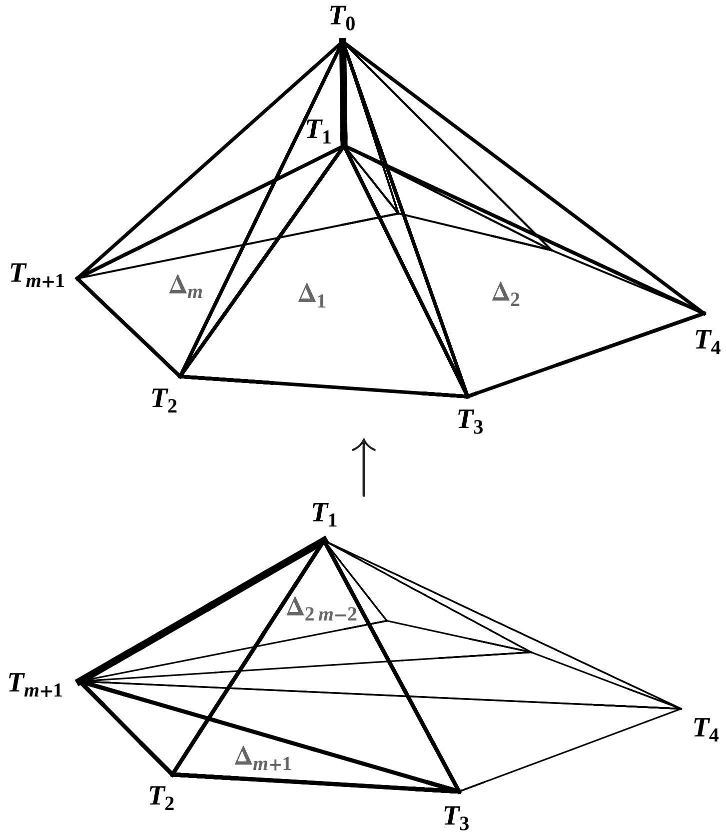


Figure 3.25: The star with $2m-2$ tetrahedrons, where m and $m-2$ tetrahedrons have a common edge, respectively.

with $V = m+2$ independent free parameters that determine the global lattice on this simplicial partition with V vertices and $2m-2$ tetrahedrons. Here $\xi_j^{(i)}$ is the parameter that determines the control point $\boldsymbol{P}_j^{(i)}$ of a lattice on the i-th tetrahedron $\triangle_i$. Let us label the vertices of the simplicial partition with $\boldsymbol{T}_i$, $i = 0, 1, \ldots, m+1$, and let us denote the simplices by (see Figure 3.25)

$$\triangle_i := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_{i+1}, \boldsymbol{T}_{i+2} \rangle, \quad i = 1, \ldots, m, \quad \boldsymbol{T}_{m+2} := \boldsymbol{T}_2,$$

and

$$\triangle_i := \langle \boldsymbol{T}_1, \boldsymbol{T}_{i-m+1}, \boldsymbol{T}_{i-m+2}, \boldsymbol{T}_{m+1} \rangle, \quad i = m+1, \ldots, 2m-2.$$

The construction in the proof of Theorem 3.21 gives us the relations between the parameters $\xi_j^{(i)}$ so that the lattice points on all common faces of the star agree. Let us consider how the parameters for the lattices on $\triangle_1$ and $\triangle_2$ are related. Since $w\big((0,1,3)^T\big) = w\big((0,1,2)^T\big) = 1$, the lattice on $\triangle_2$ is by Corollary 3.20 determined by parameters $\xi_j^{(1)}$, $j = 0, 1, 2, 3$, and the additional one $\xi_2^{(2)}$ as

$$\xi_0^{(2)} = \xi_0^{(1)}, \;\; \xi_1^{(2)} = \xi_1^{(1)} \xi_2^{(1)}, \;\; \xi_2^{(2)} = \xi_2^{(2)}, \;\; \xi_3^{(2)} = \frac{\xi_3^{(1)}}{\xi_2^{(2)}}.$$

Using a similar approach for all simplices $\triangle_i$, all parameters can be expressed by V parameters

$$\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}, \xi_2^{(2)}, \xi_2^{(3)}, \dots, \xi_2^{(m-1)}$$

as

$$\xi_0^{(i)} = \xi_0^{(1)}, \quad \xi_1^{(i)} = \xi_1^{(1)}\xi_2^{(1)} \prod_{j=2}^{i-1} \xi_2^{(j)}, \quad \xi_2^{(i)} = \xi_2^{(i)}, \quad \xi_3^{(i)} = \frac{\xi_3^{(1)}}{\prod_{j=2}^{i} \xi_2^{(j)}},$$

for $i = 2, 3, \dots, m-1$, and

$$\xi_0^{(m)} = \xi_0^{(1)}, \;\; \xi_1^{(m)} = \xi_1^{(1)}, \;\; \xi_2^{(m)} = \xi_2^{(1)} \prod_{j=2}^{m-1} \xi_2^{(j)}, \;\; \xi_3^{(m)} = \frac{\xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}},$$

$$\xi_0^{(m+1)} = \xi_1^{(1)}, \;\; \xi_1^{(m+1)} = \xi_2^{(1)}, \;\; \xi_2^{(m+1)} = \prod_{j=2}^{m-1} \xi_2^{(j)}, \;\; \xi_3^{(m+1)} = \frac{\xi_0^{(1)}\xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}},$$

$$\xi_0^{(m+i)} = \xi_1^{(1)}\xi_2^{(1)} \prod_{j=2}^{i-1} \xi_2^{(j)}, \;\; \xi_1^{(m+i)} = \xi_2^{(i)}, \;\; \xi_2^{(m+i)} = \prod_{j=i+1}^{m-1} \xi_2^{(j)}, \;\; \xi_3^{(m+i)} = \frac{\xi_0^{(1)}\xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}},$$

for $i = 2, 3, \dots, m-2$.

3.4. Lattices on simplicial partitions which are not simply connected

In this section, $(d+1)$-pencil lattices on non-simply connected simplicial partitions in $\mathbb{R}^d$ will be studied. Since such simplicial partitions appear quite often in practice, it is important to consider them too. A straightforward but naive construction of a lattice would enlarge the original partition $\mathcal{T}$ to a simply connected one, construct a lattice over it, and restrict it to $\mathcal{T}$. But such an approach would clearly neglect the structure of the original partition. For this reason, we will study the lattice construction over the original partition, and show that the additional degrees of freedom obtained can be used to increase the flexibility of the lattice. Furthermore, the data on the boundary of holes may be of a particular importance, and a lattice, which does not consider the holes, may not be appropriate.

In Theorem 3.21 and in Corollary 3.23 it was shown, that for a regular simply connected simplicial partition $\mathcal{T} \subseteq \mathbb{R}^d$ with V vertices, there exists a global $(d+1)$-pencil lattice on $\mathcal{T}$ with precisely V degrees of freedom. In this section, this result will be extended to more general simplicial partitions.

Recall two basic ideas of Theorem 3.21 that will be important later on. First, the product of lattice parameters should be equal to the same constant α^n for all local lattices on simplices of a partition. Furthermore, by (3.37), we have to assure, that for both index vectors on every common face the winding number is equal to 1. Recall, that this can be assured if we label all vertices of a simplicial partition by $\boldsymbol{T}_0, \boldsymbol{T}_1, \dots, \boldsymbol{T}_{V-1}$ and order the vertices of any local simplex $\triangle \in \mathcal{T}$ as (see Figure 3.22)

$$\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \dots, \boldsymbol{T}_d \rangle := \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \dots, \boldsymbol{T}_{i_d} \rangle, \;\; 0 \le i_0 < i_1 < \dots < i_d \le V-1. \quad (3.40)$$

Before extending Theorem 3.21 to a more general case, we have to answer the following question. By Corollary 3.17, the lattice on a simplex $\triangle$, with known α^n, is uniquely determined if its restriction to at least two facets of $\triangle$ is known. Suppose now, that d-pencil lattices on r facets of $\triangle$ are given, where $r \in \{2, 3, \ldots, d+1\}$, such that they coincide on common faces. Are they a restriction of some $(d+1)$-pencil lattice on $\triangle$?

LEMMA 3.25. *Suppose that the product $\alpha^n = \prod_{k=0}^{d} \xi_k$, which corresponds to the barycentric representation of a $(d+1)$-pencil lattice with parameters $\boldsymbol{\xi}$ on a simplex $\triangle \subseteq \mathbb{R}^d$, is known. Let d-pencil lattices with the same α^n be given on r facets*

$$\{f_i\}_{i=1}^{r} \subseteq \triangle, \quad r \in \{2, 3, \ldots, d+1\},$$

such that they coincide on common faces. If $d \geq 3$ or ($d = 2$ and $r = 2$), then there exists a unique $(d+1)$-pencil lattice on $\triangle$, such that its restriction to $\{f_i\}_{i=1}^{r}$ coincides with given d-pencil lattices.

Proof. By (3.37), a d-pencil lattice given on f_1 can be extended to a $(d+1)$-pencil lattice on $\triangle$ with one free parameter. As the case $d = 2$, $r = 2$ is straightforward, let $d \geq 3$. Since a lattice on $f_1 \cap f_2$ is a $(d-1)$-pencil lattice, which can be extended to the d-pencil lattice on f_2 by one additional parameter, the $(d+1)$-pencil lattice on $\triangle$ is uniquely determined by lattices on f_1 and f_2. Furthermore, since f_i have a common facet with both f_1 and f_2, $3 \leq i \leq d+1$, the restriction of the lattice on f_i to $f_i \cap f_k$, $k = 1, 2$, is known by lattices on f_1 and f_2. Thus, by Corollary 3.17, the whole lattice on f_i is given by lattices on f_1 and f_2, for $3 \leq i \leq d+1$. □

Result of Lemma 3.25 does not hold for the planar case with $r = 3$. This is confirmed by the following example. Let $\triangle_i$, $i = 1, 2, 3, 4$, be triangles as in Figure 3.26 and let a lattice on $\triangle_i$ be determined by parameters $\boldsymbol{\xi}^{(i)}$, for all i. Further, let the product of

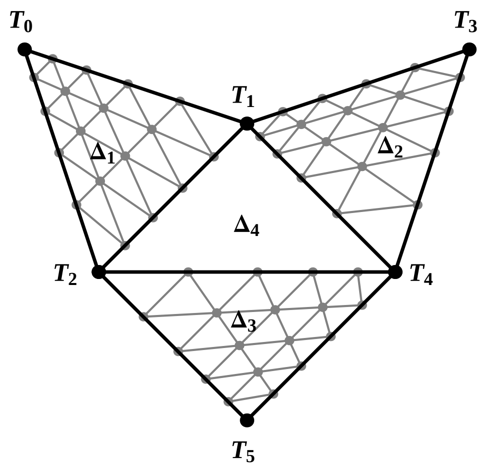


Figure 3.26: The planar case, where the lattice is predetermined on all facets of a triangle.

local lattice parameters be equal to the same constant α^n for all four lattices. Let us first determine the lattices on $\triangle_i$, $i = 1, 2, 3$. They are given by 7 parameters as

$$\boldsymbol{\xi}^{(1)} := \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}\right)^T, \ \boldsymbol{\xi}^{(2)} := \left(\xi_0^{(2)}, \xi_1^{(2)}, \frac{\alpha^n}{\xi_0^{(2)}\xi_1^{(2)}}\right)^T, \ \boldsymbol{\xi}^{(3)} := \left(\xi_0^{(3)}, \xi_1^{(3)}, \frac{\alpha^n}{\xi_0^{(3)}\xi_1^{(3)}}\right)^T, \tag{3.41}$$

where $\alpha^n = \xi_0^{(1)}\xi_1^{(1)}\xi_2^{(1)}$. By these three lattices a lattice on $\triangle_4$ is already determined on all three edges. If Lemma 3.25 would hold for this case too, then the lattice on $\triangle_4$ would now be uniquely determined. But, by Theorem 3.21, all four lattices should be determined by only 6 parameters, since there are only 6 vertices in this partition. Thus, the 7 parameters in (3.41) must be dependent, in order to be able to construct a lattice also on $\triangle_4$. If we label the vertices of the partition as in Figure 3.26, then (3.37) and (3.40) imply

$$\xi_0^{(4)} := \xi_1^{(1)}, \ \xi_1^{(4)} := \xi_0^{(3)} \quad \text{and} \quad \xi_0^{(4)}\xi_1^{(4)} := \xi_0^{(2)}\xi_1^{(2)}.$$

Therefrom, the relation between the parameters in (3.41) is

$$\xi_0^{(3)} := \frac{\xi_0^{(2)}\xi_1^{(2)}}{\xi_1^{(1)}}.$$

We have seen, that if we would like to construct a lattice on $\triangle_4$, we have to adjust one of the lattices on other triangles. Note that if we would add an additional vertex inside of $\triangle_4$ in order to split the lattice on $\triangle_4$ into three lattices, we would still have to adjust one of the lattices on $\triangle_i$, $i = 1, 2, 3$, although now the number of vertices of the partition would coincide with the number of parameters in (3.41). This will be more precisely proved in Corollary 3.33. Let us now extend Theorem 3.21.

THEOREM 3.26. *Let $\mathcal{T}$ be a connected regular simplicial partition in $\mathbb{R}^d$ with $V \geq d+1$ vertices and H interior holes, which are homeomorphic to a simplicial ball. Further let $\mathcal{G}(\mathcal{T})$ be 2-vertex connected. Then there exists a global $(d+1)$-pencil lattice on $\mathcal{T}$ with precisely*

$$V + \delta_{d,2}H$$

degrees of freedom, where $\delta_{i,j}$ is given in (2.1).

Proof. As in Theorem 3.21, we will assume that for all local lattices the product of lattice parameters is equal to the same constant α^n. If $H = 0$, then by Theorem 3.21 there are V degrees of freedom. Suppose now that $H \neq 0$. Without loss of generality, we may assume that $H = 1$ and that the hole, which can be identified with a simplicial partition homeomorphic to a simplicial ball, consists of one simplex $\triangle$. Let first $d = 2$. There are 3 free parameters to determine 2-pencil lattices on all facets of $\triangle$ (one for each lattice, since α^n is given in advance). But that is one more than we have available to determine the lattice on the whole $\triangle$ (since α^n is known). Therefore, each hole brings up one new parameter. Let now $d \geq 3$. Since there are d-pencil lattices, which coincide on common faces, given on all facets of $\triangle$, the lattice on the whole $\triangle$ is by Lemma 3.25 with these lattices uniquely determined. Thus, a hole implies no additional degrees of freedom. $\square$

3.5. Extension over holes

Suppose that a lattice constructed on $\mathcal{T}$ has already been used in an interpolation process. But then, some slight changes in the topology of $\mathcal{T}$ appear. As a model problem, one may think of a diffusion process over a partition $\mathcal{T}$ with many holes. During the process it may happen that the substance will break into some of the holes. One would clearly tend to preserve the original data, particularly if the evaluation of the function being interpolated is very expensive, but the new data (over these holes) have to be interpolated too. This gives an impetus to study how a lattice on $\mathcal{T}$ can be extended over some particular holes. Obviously the extension should preserve at least the continuity of the interpolant and consequently the lattice points should agree on all common faces of simplices.

Let $\mathcal{T}$ be a connected regular simplicial partition which is not simply connected and let us extend the global lattice from $\mathcal{T}$ over some holes. In order to do this, we have to somehow bound and partition the hole into simplices, such that it becomes a simplicial partition, which will be denoted by $\mathcal{H}$. A lattice is predetermined on some parts of the boundary of $\mathcal{H}$ and we have to determine the rest of it on the whole $\mathcal{H}$. We will further need some definitions (see Figure 3.27).

- An *interior facet* of a simplicial partition $\mathcal{T}$ in $\mathbb{R}^d$ is a $(d-1)$-simplex, which is a facet of two simplices in $\mathcal{T}$. Otherwise it is called a *boundary facet* of $\mathcal{T}$. The set of all boundary facets of $\mathcal{T}$ will be denoted by $\mathcal{B}(\mathcal{T})$.

- An *interior vertex* of a simplicial partition $\mathcal{T}$ is a vertex, which has a property that all $(d-1)$-simplices, containing it as a vertex, are interior facets of $\mathcal{T}$. Otherwise it is called a *boundary vertex* of $\mathcal{T}$.

- An *interior simplex* of a simplicial partition $\mathcal{T}$ is a simplex with all facets being interior facets of $\mathcal{T}$. Otherwise it is called a *boundary simplex* of $\mathcal{T}$.

- An *interior hole* of a simplicial partition $\mathcal{T}$ is a simplicial partition $\mathcal{H}$, such that all boundary facets of $\mathcal{H}$ are interior facets of $\mathcal{T} \cup \mathcal{H}$.

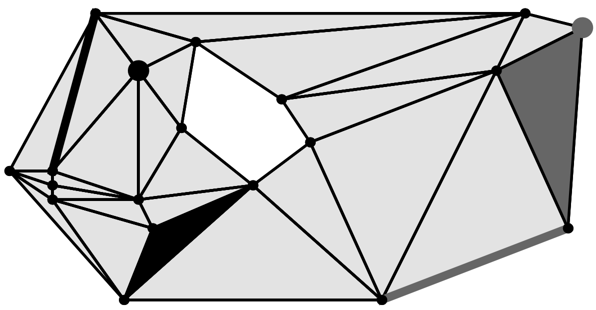


Figure 3.27: A triangulation $\mathcal{T}$ with an interior hole. Examples of an interior vertex, interior facet and interior simplex of $\mathcal{T}$ are colored black, while the examples of the boundary ones are colored dark gray.

From Lemma 3.25 we can conjecture that the planar case will differ from higher dimensional cases. This is perhaps due to the fact that $(d+1)$-pencil lattices are defined differently for $d = 1$.

3.5.1 The planar case

In the planar case, all possible holes of a connected triangulation are homeomorphic to the 2-dimensional simplicial ball. Therefore, all holes can be considered in a same manner. Let $\mathcal{H}$ be a hole with V_I interior vertices and let $\triangle$ be an arbitrary triangle in $\mathcal{H}$. From (3.37) and Lemma 3.25 it follows that there are V_I degrees of freedom to extend the lattice over $\mathcal{H}\backslash\{\triangle\}$. Therefore, we can further assume that the hole consists of only one triangle $\triangle_{\mathcal{H}}$ (Figure 3.29, left).

In the example in the previous section, we have seen, that it is not always possible to extend a lattice over a hole $\triangle_{\mathcal{H}}$, without adjusting some parts of the lattice on $\mathcal{T}\backslash\{\triangle_{\mathcal{H}}\}$.

DEFINITION 3.27. *A sequence of triangles $\triangle_1, \triangle_2, \ldots, \triangle_k$, such that two consecutive triangles $\triangle_i$ and $\triangle_{i+1}$ have an edge in common, is called a strip of triangles (see Figure 3.28).*

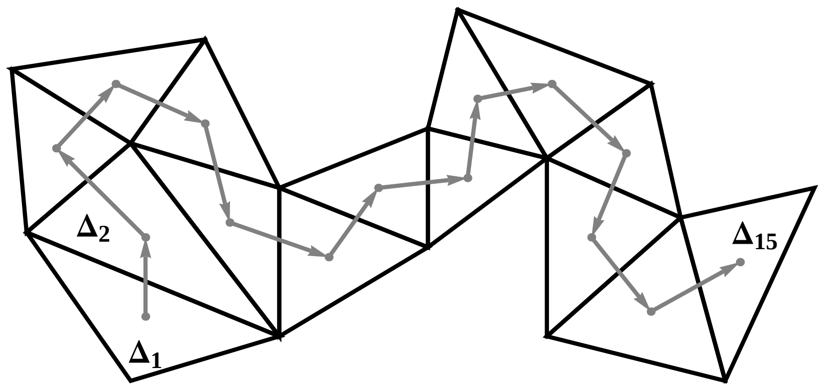


Figure 3.28: An example of a strip of triangles.

THEOREM 3.28. *Let $\mathcal{T}$ be a connected regular triangulation in $\mathbb{R}^2$ with an interior triangular hole $\triangle_{\mathcal{H}}$ and let $\mathcal{L}$ be a global $(d+1)$-pencil lattice on $\mathcal{T}$. Then there exists a global $(d+1)$-pencil lattice $\mathcal{L}'$ on $\mathcal{T}' := \mathcal{T} \cup \{\triangle_{\mathcal{H}}\}$, such that $\mathcal{L}$ and $\mathcal{L}'$ differ only on one strip of triangles: $\triangle_{\mathcal{H}}, \triangle_2, \triangle_3, \ldots, \triangle_m$, where $\triangle_m$ is any boundary triangle of $\mathcal{T}'$ (Figure 3.30).*

Proof. Let the construction of the lattice $\mathcal{L}'$ start at $\triangle_{\mathcal{H}}$. By Lemma 3.25, a lattice on any triangle is uniquely determined by lattices on two adjacent triangles, which share a common edge with this triangle. Thus, in order to construct a lattice on $\triangle_{\mathcal{H}}$ we have to choose an edge of $\triangle_{\mathcal{H}}$ where this lattice will not coincide with $\mathcal{L}$ (Figure 3.29, right). Let us further construct a lattice on the adjacent triangle $\triangle_2$ containing this edge. Again we have to choose on which of its remaining edges the lattice will not coincide with $\mathcal{L}$. Next, the construction of $\mathcal{L}'$ proceeds on the adjacent triangle $\triangle_3$, which contains this edge. We continue with this procedure until we reach a triangle $\triangle_m$, $m \geq 2$, with at least one of its edges in $\mathcal{B}(\mathcal{T}')$. Furthermore, let $\mathcal{L}'$ be the same as $\mathcal{L}$ on

$$\mathcal{T} \backslash \left\{ \bigcup_{i=2}^{m} \triangle_i \right\}.$$

Therefore $\mathcal{L}'$ and $\mathcal{L}$ differ only on the strip of triangles $\triangle_{\mathcal{H}}, \triangle_2, \triangle_3, \ldots, \triangle_m$. □

Clearly, there are many possible ways to choose a strip, where the lattices $\mathcal{L}'$ and $\mathcal{L}$ will differ (see Figure 3.30, e.g.). Let us define an optimal strip as a strip containing the minimal number of triangles among all appropriate strips. Since in practice not all strips are available (it may not be allowed to change the lattice on some particular triangles), we have to choose one of the optimal strips among all available. Optimal strips can be found by searching for the shortest paths in the dual graph of the graph $\mathcal{G}(\mathcal{T}')$ from the vertex corresponding to $\triangle_{\mathcal{H}}$ to the one corresponding to $\mathbb{R}^2 \backslash \mathcal{G}(\mathcal{T}')$.

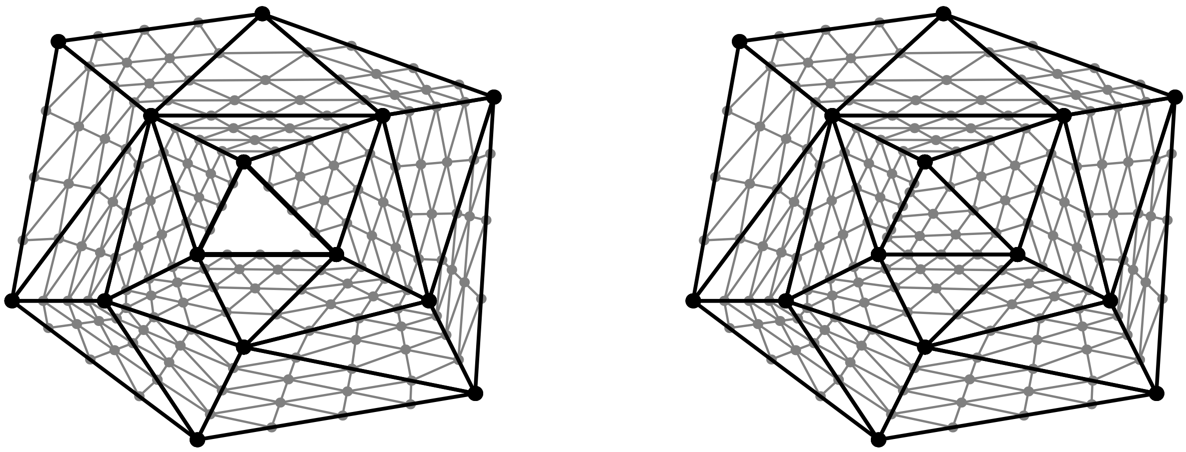


Figure 3.29: A global lattice on a triangulation, which is not simply connected (left) and an extension of the lattice over a hole before the adjustment of the existent lattice (right).

In Figure 3.31 (top) an example of a more realistic triangulation $\mathcal{T}$ is presented. The triangulation $\mathcal{T}$ contains three holes. Suppose that $\mathcal{L}$ is a global lattice on $\mathcal{T}$, which is further extended over all three holes to a global lattice $\mathcal{L}'$. An example of strips of triangles, where both lattices $\mathcal{L}$ and $\mathcal{L}'$ would differ, is shown in Figure 3.31 (bottom). We can see that in practice the number of triangles, where the initial global lattice has to be adjusted in order to be extended over the holes, is small in comparison to the number of all triangles in a triangulation.

3.5.2 The case $d \geq 3$

In higher dimensions ($d \geq 3$), the problem how to extend a lattice over a hole becomes much more complicated. We shall prove two important topological lemmas first, which will be needed later on.

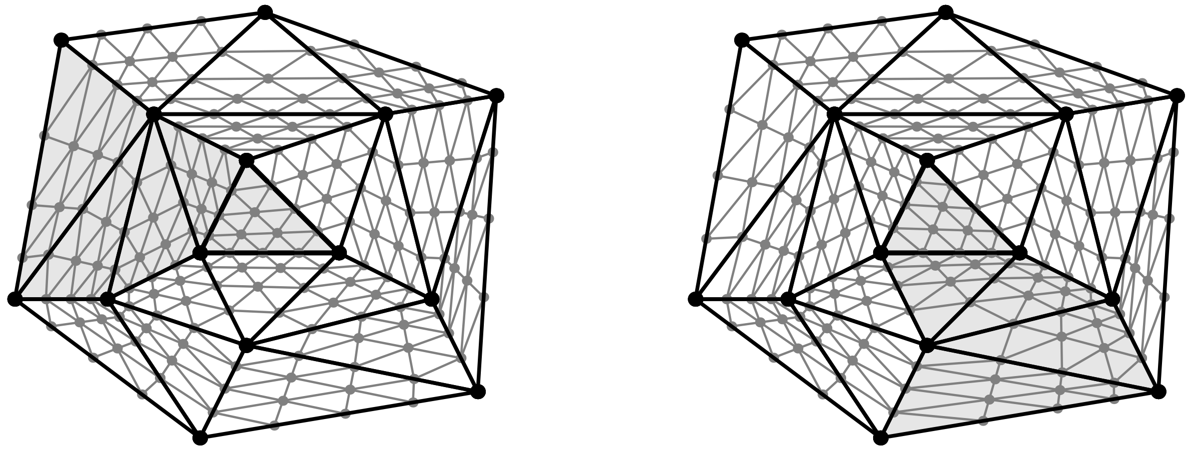


Figure 3.30: Two different optimal adjustments of the lattice given in Figure 3.29, left. Both global lattices differ with the one in Figure 3.29 only on the strip of gray triangles.

LEMMA 3.29. *Let $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 2$, be a simplicial partition homeomorphic to a simplicial ball with no interior vertices. Then there exists a simplex in $\mathcal{T}$ with at least two of its facets in $\mathcal{B}(\mathcal{T})$ (see Figure 3.32, left).*

Proof. Suppose that $\mathcal{T}$ is constructed by adding one simplex at a time, in such a way that the current simplicial partition $\mathcal{T}'$ is homeomorphic to a simplicial ball with no interior vertices at each step. We may assume, without loss of generality, that in each step one of the simplices, which are adjacent to most simplices in $\mathcal{T}'$, is added. Since $\mathcal{T}$ is homeomorphic to a simplicial ball and contains no star as a subpartition, $\mathcal{T}'$ grows from a single simplex to $\mathcal{T}$ in such a way that each simplex added has F, $1 \leq F \leq d-1$, facets in common with simplices in the instantaneous partition $\mathcal{T}'$. Therefore, each newly added simplex has at least two facets in $\mathcal{B}(\mathcal{T}')$. Since the same holds for $\mathcal{T}' = \mathcal{T}$, the proof is completed. □

LEMMA 3.30. *Let $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 2$, be a simplicial partition homeomorphic to a simplicial ball with V_I interior vertices. If all simplices in $\mathcal{T}$ have at most one facet in $\mathcal{B}(\mathcal{T})$, then there exists a simplex in $\mathcal{T}$ having exactly one facet in $\mathcal{B}(\mathcal{T})$ and containing one of the interior vertices of $\mathcal{T}$ as the remaining vertex (see Figure 3.32, right).*

Proof. Suppose first that $V_I = 1$ and let the only interior vertex of $\mathcal{T}$ be denoted by $\boldsymbol{T}_0$. Then $\boldsymbol{T}_0$ is necessarily the interior vertex of some star $\mathcal{S}_0 \subseteq \mathcal{T}$. But it is straightforward to see that in this case $\mathcal{T}$ itself is necessarily a star. Indeed, the next possible partition with the property, that it does not contain any simplex with at least two facets in $\mathcal{B}(\mathcal{T})$, is a union of two stars $\mathcal{S}_0 \cup \mathcal{S}_1$. But $\mathcal{S}_0 \cup \mathcal{S}_1$ already contains two interior vertices $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$. Thus, inductively, the partition $\mathcal{T}$, which satisfies the assumptions of the lemma, is necessarily of the form

$$\mathcal{S}_0 \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{V_I - 1},$$

where $\mathcal{S}_i$ is a star with the interior vertex $\boldsymbol{T}_i$. Take now a star $\mathcal{S}_j$, which contains at least one simplex $\triangle$ that is a boundary simplex of $\mathcal{T}$. Clearly, such a star exists. Then $\triangle$ has exactly one facet in $\mathcal{B}(\mathcal{T})$ and $\boldsymbol{T}_j$ as the remaining vertex, which completes the proof. □

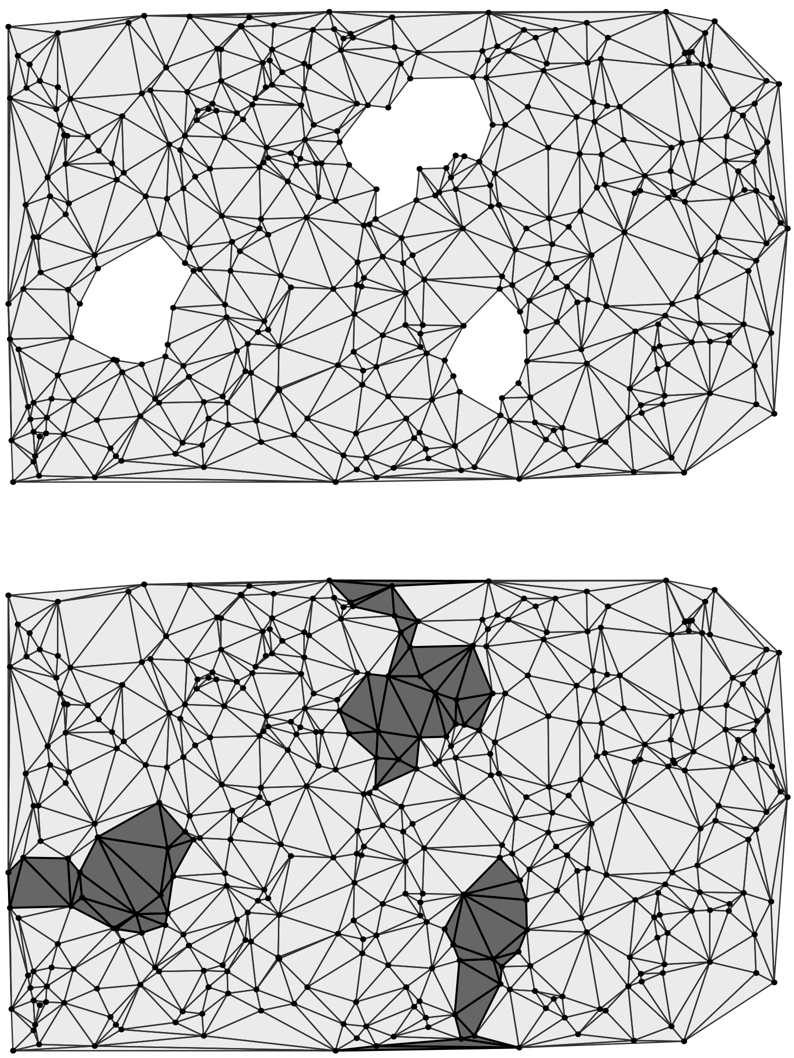


Figure 3.31: A triangulation $\mathcal{T}$ which is not simply connected (top), and $\mathcal{T}$ with gray colored triangles, where the global lattice, extended over holes, differs with the initial global lattice on $\mathcal{T}$ (bottom).

We are now able to prove the following theorem, which unfortunately does not hold in the planar case (see Theorem 3.28).

THEOREM 3.31. *Let $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 3$, be a simplicial partition homeomorphic to a simplicial ball with V_I interior vertices. Furthermore, let d-pencil lattices be given on all its boundary facets, such that they coincide on all common faces. Suppose that a product of local lattice parameters is equal to the same constant α^n for all simplices in $\mathcal{B}(\mathcal{T})$. Then there exists a global lattice on $\mathcal{T}$, whose restriction coincides with the given lattices on $\mathcal{B}(\mathcal{T})$. Moreover, there are V_I degrees of freedom to construct the lattice on $\mathcal{T}$.*

Proof. The proof proceeds by the induction on the number of simplices in $\mathcal{T}$. Suppose that $|\mathcal{T}| = 1$. This implies $\mathcal{T} = \{\triangle\}$. Since the lattice on $\triangle$ is predetermined on all facets of $\triangle$, then by Lemma 3.25 the lattice is uniquely determined on $\triangle$. Thus, there

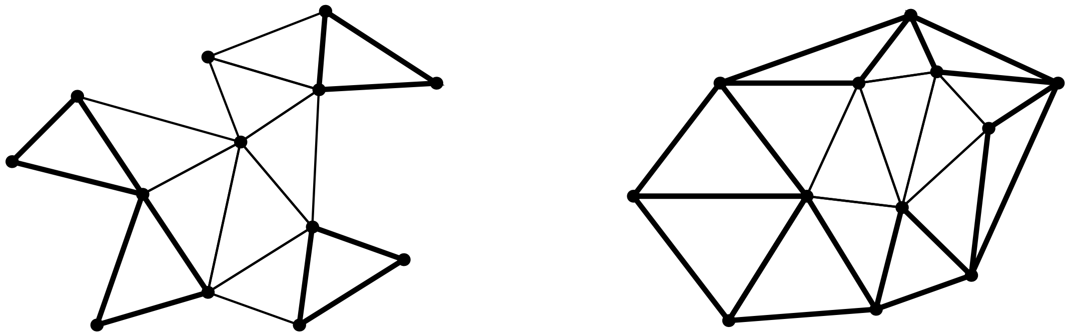


Figure 3.32: A triangulation $\mathcal{T}$ with no interior vertices (left), and a triangulation $\mathcal{T}'$ with no triangles, containing two boundary edges of $\mathcal{T}'$.

are no degrees of freedom, which corresponds to no interior vertices in $\triangle$. Let now $|\mathcal{T}| > 1$. This case will be proved by the induction on the number of interior vertices V_I. Suppose first that $V_I = 0$. By Lemma 3.29, there exists a simplex $\triangle$, which has a lattice predetermined on at least two of its facets. If there are more such simplices, let $\triangle$ be one of the simplices with a lattice predetermined on most of its facets. Thus $\mathcal{T}' := \mathcal{T}\backslash\{\triangle\}$ is still homeomorphic to a simplicial ball. By Lemma 3.25, the lattice is completely determined on $\triangle$ and we can construct it uniquely. Since for $\mathcal{T}'$, $|\mathcal{T}'| = |\mathcal{T}| - 1$, the result follows by induction, we have no degrees of freedom to construct a lattice on $\mathcal{T}$. Let now $V_I > 0$ and consider the pair $(\mathcal{T}, V_I)$. We can, by Lemma 3.25, uniquely construct a lattice on each simplex in $\mathcal{T}$, which has at least two facets in $\mathcal{B}(\mathcal{T})$. Thus, we are left with the pair $(\mathcal{T}', V_I)$, $|\mathcal{T}'| \leq |\mathcal{T}|$. Since $V_I > 0$, there exists at least one interior vertex $\boldsymbol{T}$. Lemma 3.30 assures the existence of a simplex $\triangle$, having exactly one facet in $\mathcal{B}(\mathcal{T}')$ and $\boldsymbol{T}$ as the remaining vertex. Thus, by (3.37), there is one degree of freedom to construct a lattice on $\triangle$. The result follows now by the induction for $(\mathcal{T}'', V_I - 1)$, $\mathcal{T}'' = \mathcal{T}'\backslash\{\triangle\}$, and the proof is completed. $\square$

While the only possible holes in connected regular simplicial partitions in $\mathbb{R}^2$ are interior holes homeomorphic to a simplicial ball, this is not the case for $d \geq 3$. Let us therefore consider the second most natural kind of a hole in $\mathbb{R}^3$: a hole, which is not an interior hole, and can be bounded and partitioned into simplices in such a way, that it becomes homeomorphic to a simplicial ball. Therefrom, the hole becomes a simplicial partition where the lattice is not predetermined on all its boundary facets. An example of a partition with such a hole is a simplicial torus. To consider this kind of holes, we will need the following lemma.

LEMMA 3.32. *Let $\mathcal{T}$ be a triangulation in $\mathbb{R}^2$ homeomorphic to a simplicial ball and let $\mathcal{L}$ be a lattice on $\mathcal{T}$. Moreover, let $\mathcal{T}$ have exactly three boundary facets $\{e_i\}_{i=1}^3$ and let $\triangle_B$ be a triangle with edges $\{e_i\}_{i=1}^3$. Suppose that a product of local lattice parameters is equal to the same constant for all triangles in $\mathcal{T}$. Then there exists a lattice $\mathcal{L}_B$ on $\triangle_B$, which coincides with $\mathcal{L}$ on $\{e_i\}_{i=1}^3$ (see Figure 3.33, e.g.).*

Proof. Suppose first that $\mathcal{T}$ is a star of degree 3 consisting of triangles $\{\triangle_i\}_{i=1}^3$ and let $\mathcal{L}_i$ be a lattice on $\triangle_i$, $i = 1, 2, 3$. Since $\mathcal{T} \cup \{\triangle_B\}$ can be identified with a tetrahedron in

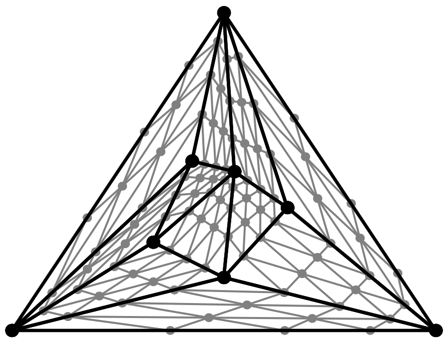


Figure 3.33: A triangulation with exactly three boundary edges (left) and a unique extension to a three-pencil lattice on a triangle $\triangle_B$ consisting of these three boundary edges (right).

$\mathbb{R}^3$ (Figure 3.34, left), the lattices $\mathcal{L}_3$ and $\mathcal{L}_B$ are by Lemma 3.25 uniquely determined by $\{\mathcal{L}_i\}_{i=1}^2$ and $\mathcal{L}_B$ coincides with $\mathcal{L} = \cup_{i=1}^3 \mathcal{L}_i$ on all common edges. Therefore, the lemma holds for a star of degree 3. Consider now two adjacent triangles $\triangle_1$ and $\triangle_2$ with lattices

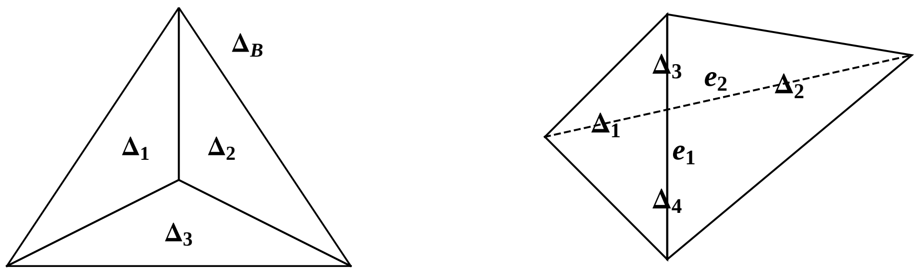


Figure 3.34: A star of degree 3 (left), and two adjacent triangles together with their dual triangles (right).

$\mathcal{L}_1$ and $\mathcal{L}_2$, which coincide on a common edge e_1 (Figure 3.34, right). Construct two new adjacent triangles $\triangle_3$ and $\triangle_4$ by connecting the vertices of $\triangle_1 \cup \triangle_2$, which are not on e_1, by an edge e_2. Since $\{\triangle_i\}_{i=1}^4$ can again be identified with a tetrahedron, lattices $\mathcal{L}_3$ and $\mathcal{L}_4$ on $\triangle_3$ and $\triangle_4$ are by Lemma 3.25 uniquely determined by $\{\mathcal{L}_i\}_{i=1}^2$, and they coincide on e_2. We will call $\{\triangle_3, \triangle_4\}$ the dual triangles of $\{\triangle_1, \triangle_2\}$ and $\{\mathcal{L}_3, \mathcal{L}_4\}$ the dual lattices of $\{\mathcal{L}_1, \mathcal{L}_2\}$. Let now $\mathcal{T}$ be an arbitrary triangulation satisfying the assumptions of the lemma. We can apply the following procedure for reducing $\mathcal{T}$ and $\mathcal{L}$ to a star of degree 3 with a lattice $\mathcal{L}'$, such that $\mathcal{L}$ coincides with $\mathcal{L}'$ on $\{e_i\}_{i=1}^3$. At each step of the procedure some of the following operations are used.

- Replace two adjacent triangles together with lattices with their dual triangles and dual lattices.
- Replace a subtriangulation $\mathcal{T}'$, which is a star of degree 3, with a triangle, whose edges are the boundary edges of $\mathcal{T}'$, and construct a lattice on it.
- Replace a triangle with a star of degree 3 (Clough-Tocher split) and construct a lattice on it.

Clearly all operations do not change the lattice on $\{e_i\}_{i=1}^3$. Note, that the last operation is inverse to the second operation and is needed only for very special triangulations. Using these three operations one can reduce a star of degree 3 to a triangle, a star of degree 4 to two adjacent triangles, and a star of degree 5 to a strip of three triangles. Since by (3.19) and (3.20), for each such triangulation there exists at least one vertex with the degree 3, 4 or 5, we finally obtain a lattice on a star of degree 3 (see Figure 3.35, e.g.), for which the lemma already holds. □

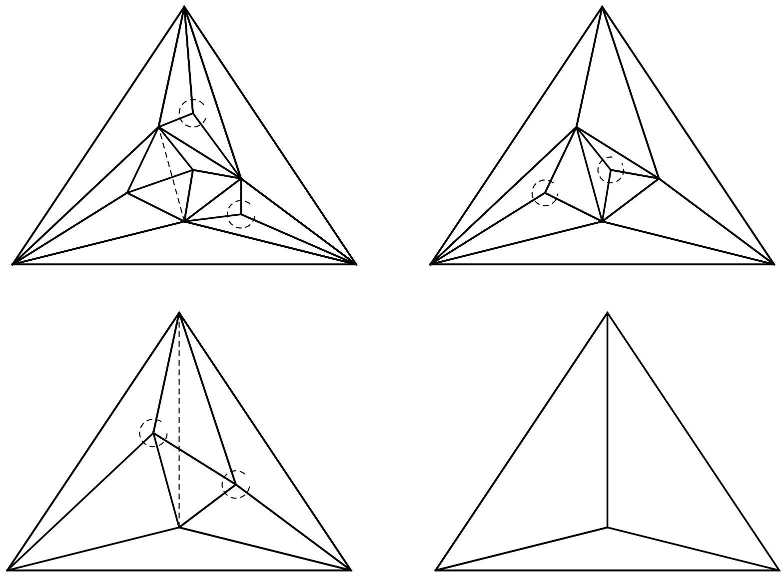


Figure 3.35: An example of the reduction procedure on the triangulation.

The following corollary follows directly from this lemma.

COROLLARY 3.33. *Let $\triangle$ be a triangle in $\mathbb{R}^2$ and let 2-pencil lattices (with the same constant α^n) be predetermined on all three edges of $\triangle$. If these lattices can not be extended to the 3-pencil lattice on the whole $\triangle$, then they can also not be extended to a global 3-pencil lattice on a regular simply connected triangulation obtained from $\triangle$ by adding some additional vertices to the interior of $\triangle$ (see Figure 3.35, e.g.).*

Now we are able to prove the following theorem.

THEOREM 3.34. *Let $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 3$, be a simplicial partition homeomorphic to a simplicial ball. Furthermore, let d-pencil lattices be given on $\mathcal{T}' \subset \mathcal{B}(\mathcal{T})$, such that they coincide on all common faces (see Figure 3.36, e.g.). Suppose that a product of local lattice parameters is equal to the same constant for all simplices in $\mathcal{B}(\mathcal{T})$. Then there exists a lattice on $\mathcal{B}(\mathcal{T})$, which coincides with the given lattices on $\mathcal{T}'$. Moreover, there are V_B degrees of freedom to extend the lattice from $\mathcal{T}'$ to $\mathcal{B}(\mathcal{T})$, where V_B is the number of those boundary vertices, which are not the vertices of $\mathcal{T}'$.*

Proof. Without loss of generality, we can assume that $\mathcal{G}(\mathcal{B}(\mathcal{T})\backslash\mathcal{T}')$ is d-vertex connected. Since $\mathcal{B}(\mathcal{T})\backslash\mathcal{T}'$ can be identified with a simplicial partition in $\mathbb{R}^{d-1}$, homeomorphic to

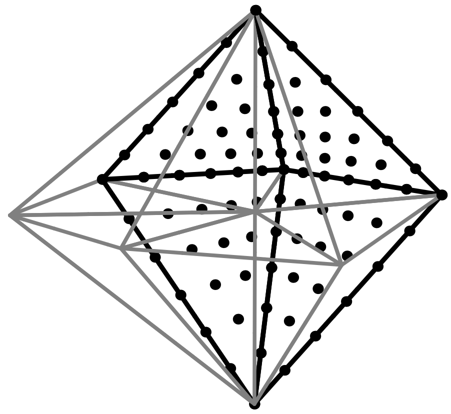


Figure 3.36: A simplicial partition with a lattice predetermined on some of its boundary facets.

a simplicial ball, where the lattice is predetermined on all its boundary facets, Theorem 3.31 confirms the theorem for $d > 3$. Let now $d = 3$. The whole $\mathcal{B}(\mathcal{T})$ can be identified with a particular simplicial partition $\mathcal{P}$ in $\mathbb{R}^2$, which has exactly three "boundary" facets e_1, e_2 and e_3 composing a triangle $\triangle$, which is also a part of $\mathcal{P}$ (see Figure 3.35, e.g.). Since the lattice is predetermined on $\mathcal{P}' \subset \mathcal{P}$, where $\mathcal{G}(\mathcal{P}\backslash\mathcal{P}')$ is d-vertex connected, the theorem will follow by (3.37) and Lemma 3.25 as soon as we prove it for $\mathcal{P}\backslash\mathcal{P}' = \{\triangle\}$. Thus, we can assume that the lattice is given on $\mathcal{P}\backslash\{\triangle\}$. We need to prove the existence of a lattice on $\triangle$, which coincides with the lattice on $\mathcal{P}\backslash\{\triangle\}$ on common edges $\{e_i\}_{i=1}^{3}$. But now, Lemma 3.32 completes the proof. $\square$

Theorem 3.31 and Theorem 3.34 can be combined to obtain the following corollary.

COROLLARY 3.35. *Let $\mathcal{T}$ be a simplicial partition in $\mathbb{R}^d$, $d \geq 3$, homeomorphic to a simplicial ball. Let a lattice be predetermined on some simplices $\mathcal{T}' \subseteq \mathcal{B}(\mathcal{T})$. Then there exists a global $(d+1)$-pencil lattice on the whole $\mathcal{T}$. Furthermore, the extension is determined by $V_I + V_B$ degrees of freedom, where V_I is the number of all interior vertices of $\mathcal{T}$ and V_B the number of those boundary vertices of $\mathcal{T}$, which are not vertices of $\mathcal{T}'$.*

We are now able to answer the question how to extend a lattice over a hole $\mathcal{H}$ of a simplicial partition $\mathcal{T}$. First, we have to partition the hole into simplices, such that $\mathcal{H}$ becomes a bounded simplicial partition. Next, we have to extend $\mathcal{H}$ in such a way, that it becomes homeomorphic to a simplicial ball (add some simplices of $\mathcal{T}\backslash\mathcal{H}$ to $\mathcal{H}$). Now $\mathcal{H}$ has a lattice predetermined on some boundary facets of $\mathcal{H}$. If $d \geq 3$, the lattice can be extended over $\mathcal{H}$ by Corollary 3.35. If $d = 2$, the extension is assured by only some small corrections of a lattice on $\mathcal{T}\backslash\mathcal{H}$ (see Theorem 3.28).

Let us conclude the section by considering an example for the case $d = 3$. Let $\mathcal{T}$ be a tetrahedral partition in $\mathbb{R}^3$ as in Figure 3.37. Suppose that $\mathcal{T}$ has 3-pencil lattices predetermined on all its boundary facets (triangles). These lattices coincide on all common edges of adjacent boundary triangles. Since $\mathcal{T}$ has only one interior vertex, we have to show, that these lattices can be extended to a global lattice on the whole

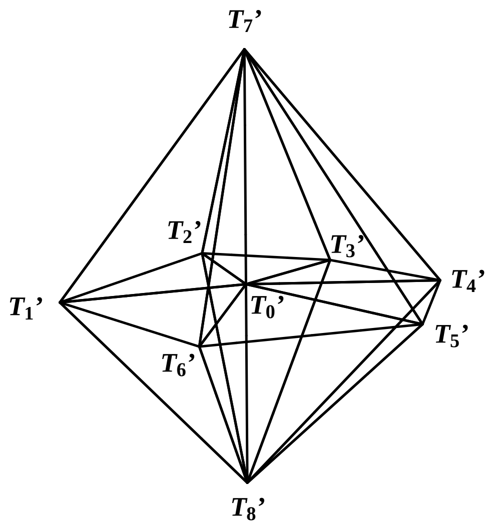


Figure 3.37: A tetrahedral partition with a chosen labeling of vertices.

$\mathcal{T}$ by an additional free parameter, in order to certify Theorem 3.31 on this particular example.

Let us label the vertices of $\mathcal{T}$ with $\boldsymbol{T}'_0, \boldsymbol{T}'_1, \ldots, \boldsymbol{T}'_8$ as in Figure 3.37 and let us denote the simplices

$$\begin{aligned}
\triangle_i &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_i := \langle \boldsymbol{T}'_0, \boldsymbol{T}'_i, \boldsymbol{T}'_{i+1}, \boldsymbol{T}'_7 \rangle, \qquad i = 1, 2, \ldots, 5,\\
\triangle_i &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_i := \langle \boldsymbol{T}'_0, \boldsymbol{T}'_{i-6}, \boldsymbol{T}'_{i-5}, \boldsymbol{T}'_8 \rangle, \quad i = 7, 8, \ldots, 11,\\
\triangle_i &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_i := \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_6, \boldsymbol{T}'_{6+\frac{i}{6}} \rangle, \qquad i = 6, 12.
\end{aligned}$$

Moreover, let the boundary triangles of $\mathcal{T}$ be denoted as

$$f_i = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle_i := \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_{\triangle_i}, \quad i = 1, 2, \ldots, 12,$$

where $\langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_{\triangle_i}$ is the facet of $\triangle_i$ on vertices $\boldsymbol{T}_1, \boldsymbol{T}_2$ and $\boldsymbol{T}_3$. Let a lattice on f_i be determined by $\boldsymbol{\eta}^{(i)} := \left(\eta_0^{(i)}, \eta_1^{(i)}, \eta_2^{(i)}\right)^T$ and a lattice on $\triangle_i$ by $\boldsymbol{\xi}^{(i)} := \left(\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)}, \xi_3^{(i)}\right)^T$. Since the lattices, which are predetermined on $\mathcal{B}(\mathcal{T})$, coincide on all common faces of boundary triangles, (3.37) implies

$$\begin{aligned}
\eta_1^{(i+1)} &= \frac{\eta_1^{(i)}}{\eta_0^{(i+1)}}, \quad i \in \{1, 2, \ldots, 10\} \backslash \{5, 6\},\\
\eta_0^{(i+6)} &= \eta_0^{(i)}, \quad i = 1, 2, \ldots, 6,\\
\eta_1^{(i+1)} &= \eta_1^{(i)}, \quad i = 5, 11,\\
\eta_0^{(i)}\eta_1^{(i)} &= \eta_0^{(i-5)}\eta_1^{(i-5)}, \quad i = 6, 12.
\end{aligned} \tag{3.42}$$

Since the product of local lattice parameters is equal to the same constant

$$\alpha^n := \prod_{j=0}^{2} \eta_j^{(1)}$$

for all lattices, it is easy to show from (3.42) that the predetermined lattices on $\mathcal{B}(\mathcal{T})$ are given by 8 parameters

$$\eta_0^{(1)},\, \eta_0^{(2)},\, \eta_0^{(3)},\, \eta_0^{(4)},\, \eta_0^{(5)},\, \eta_1^{(1)}, \eta_2^{(1)},\, \eta_1^{(7)}$$

as

$$\boldsymbol{\eta}^{(i+6k)} := \left(\eta_0^{(i)},\, \frac{\eta_1^{(1+6k)}}{\prod_{j=2}^{i} \eta_0^{(j)}},\, \frac{\alpha^n \prod_{j=2}^{i-1} \eta_0^{(j)}}{\eta_1^{(1+6k)} \prod_{j=i}^{1} \eta_0^{(j)}} \right)^T, \quad i = 1, 2, \ldots, 5, \;\; k = 0, 1,$$

$$\boldsymbol{\eta}^{(6+6k)} := \left(\prod_{j=1}^{5} \eta_0^{(j)},\, \frac{\eta_1^{(1+6k)}}{\prod_{j=2}^{5} \eta_0^{(j)}},\, \frac{\alpha^n}{\eta_0^{(1)} \eta_1^{(1+6k)}} \right)^T, \quad k = 0, 1.$$

Now we can extend the lattice to $\mathcal{T}$ by an additional parameter $\xi_0^{(1)}$ as

$$\boldsymbol{\xi}^{(i+6k)} := \left(\xi_0^{(1)} \prod_{j=1}^{i-1} \eta_0^{(j)},\, \eta_0^{(i)},\, \frac{\eta_1^{(1+6k)}}{\prod_{j=2}^{i} \eta_0^{(j)}},\, \frac{\alpha^n}{\xi_0^{(1)} \eta_0^{(1)} \eta_1^{(1+6k)}} \right)^T, \quad i = 1, 2, \ldots, 5, \;\; k = 0, 1,$$

$$\boldsymbol{\xi}^{(6+6k)} := \left(\xi_0^{(1)},\, \prod_{j=1}^{5} \eta_0^{(j)},\, \frac{\eta_1^{(1+6k)}}{\prod_{j=2}^{5} \eta_0^{(j)}},\, \frac{\alpha^n}{\xi_0^{(1)} \eta_0^{(1)} \eta_1^{(1+6k)}} \right)^T, \quad k = 0, 1.$$

Chapter 4

Newton-Cotes cubature rules

In this chapter, Newton-Cotes cubature rules over $(d+1)$-pencil lattices on simplices and simplicial partitions will be studied. The multivariate integration has been quite a challenge in numerical analysis since integrals, encountered in many mathematical models, can rarely be calculated analytically. Multivariate integration appears in practical applications, such as finite elements methods, statistical models, computer graphics, financial mathematics, etc. A cubature rule over a simplex $\triangle \subset \mathbb{R}^d$ of the form

$$Q_{\triangle}(f) = \sum_{\boldsymbol{\gamma}} \omega_{\boldsymbol{\gamma}} f(\boldsymbol{X}_{\boldsymbol{\gamma}}), \quad \boldsymbol{X}_{\boldsymbol{\gamma}} \in \triangle, \tag{4.1}$$

where $f(\boldsymbol{X}_{\boldsymbol{\gamma}})$ are values of a function f at points $\boldsymbol{X}_{\boldsymbol{\gamma}}$, $\omega_{\boldsymbol{\gamma}}$ are weights, and $\boldsymbol{\gamma}$ is a multiindex, is one of the usual ways how to approximate a multivariate integral over a compact domain in $\mathbb{R}^d$, partitioned into simplices. The choice of points $\boldsymbol{X}_{\boldsymbol{\gamma}}$ and weights $\omega_{\boldsymbol{\gamma}}$ usually does not depend on the function f. There are several criteria to classify cubature rules based on their behavior for specific classes of functions (see [18], e.g.). Probably the most often used rules of the form (4.1) are polynomial-based ones, which are exact for a particular set of polynomials. In this case, the points $X_{\boldsymbol{\gamma}}$ should provide a basis for the correct interpolation with the polynomial class concerned. If integration points are to be determined in advance, as is the case of Newton-Cotes cubature rules, this is not a trivial job in the multivariate case. The principal lattices lead to the Newton-Cotes cubature rules that can be viewed as a straightforward generalization of the equidistant univariate case. These Newton-Cotes rules can already be found in [46]. Even though $(d+1)$-pencil lattices are nowadays quite important in multivariate polynomial interpolation, their impact on Newton-Cotes numerical integration was not well understood. This is perhaps due to the fact that it was not clear how to continuously extend a lattice from a particular simplex to its neighbours.

Newton-Cotes cubature rules over principal lattices are here carried over to $(d+1)$-pencil lattices. The generalization is based upon a simple form of the Lagrange basis polynomials in the barycentric representation (see Theorem 2.7). A similar form of the Newton basis polynomials enables us to derive a closed form of the error remainder too.

Moreover, it is possible to efficiently extend the rules to global $(d+1)$-pencil lattices on simplicial partitions. Since usually most of the lattice points lie on facets of simplices, it is therefore very important to evaluate the function f at these points only once. As a bonus, if the function as a mapping is known too, we can improve the approximation by using an adaptive algorithm. Therefore a subdivision step that refines a $(d+1)$-pencil lattice on a simplex to its subsimplices is presented. Moreover, if the number of function evaluations is at stake, the additional freedom of $(d+1)$-pencil lattices can be exploited to obtain a more efficient adaptive algorithm over simplicial partitions.

The extended Newton-Cotes cubature rules are useful in many practical applications. Suppose that the function values over a $(d+1)$-pencil lattice on a simplicial partition are known in advance (for example, they were computed for the construction of a continuous interpolant over the lattice). Then these values should be used also for the numerical integration over the simplicial partition. We can further apply an adaptive algorithm based on the extended Newton-Cotes rules in order to improve the obtained approximation. Moreover, the cubature rules over $(d+1)$-pencil lattices can be used if the evaluation of a function is much more expensive over some particular parts of a simplicial partition. The additional freedom of $(d+1)$-pencil lattices can be used to diminish the number of points on the undesired parts.

4.1. Newton-Cotes cubature rules over a simplex

Let $S_\triangle(f)$ denote the integral of a scalar field $f : \triangle \to \mathbb{R}$ over a simplex $\triangle$. The cubature rules will be based on the barycentric form. As expected, this enables us to extend the rules to an arbitrary simplex in $\mathbb{R}^d$ by a simple transformation only. Let

$$Q(f) := Q^{(n)}(f;\boldsymbol{\xi}) := \sum_{\boldsymbol{\gamma}\in\mathcal{I}_n^d} \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) f_{\boldsymbol{\gamma}}, \tag{4.2}$$

where $f_{\boldsymbol{\gamma}}$ is the value of a function f at the point with the barycentric coordinates $\boldsymbol{B}_{\boldsymbol{\gamma}}$, denote a cubature rule of degree n in the barycentric form over the standard simplex $\triangle_{d+1}^d \subseteq \mathbb{R}^{d+1}$, given in (2.2). Since Newton-Cotes rules are interpolatory, the weights $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ are determined as

$$\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = S_{\triangle_{d+1}^d}(\mathcal{L}_{\boldsymbol{\gamma}}), \quad \boldsymbol{\gamma} \in \mathcal{I}_n^d, \tag{4.3}$$

where $\mathcal{L}_{\boldsymbol{\gamma}}$ are the Lagrange basis polynomials in the barycentric form. They have been explicitly determined in Theorem 2.7. In this chapter it will be more convenient to write hyperplanes $h_{i,j,\boldsymbol{\gamma}}$ as (see (2.36))

$$h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}) := \frac{h_{i,j}(\boldsymbol{x};\boldsymbol{\xi})}{h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi})},$$

where

$$h_{i,j}(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{t=i}^{i+d} a_{t,j}(\boldsymbol{\xi})\, x_t, \quad a_{t,j}(\boldsymbol{\xi}) = \begin{cases} [n-j]_\alpha\,, & t=i, \\ \left([n-j]_\alpha - [n]_\alpha\right)\left(\prod_{k=i}^{t-1}\xi_k\right)^{-1}, & t>i. \end{cases} \tag{4.4}$$

Note that indices of $a_{t,j}(\boldsymbol{\xi})$ are not taken modulo $d+1$. Recall that

$$h_{i,j}\left(\boldsymbol{x};\boldsymbol{\xi}\right)=0$$

is the equation of the hyperplane $H_{i,j}$ in the barycentric form, based upon the center $\boldsymbol{C}_{i+1}$ and lattice points

$$\boldsymbol{B}_{\boldsymbol{\gamma}'}(\boldsymbol{\xi}),\quad \boldsymbol{\gamma}'\in\mathcal{I}_n^d,\ \gamma_i'=j.$$

Therefore

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi})=\prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1}\frac{h_{i,j}(\boldsymbol{x};\boldsymbol{\xi})}{h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi})},\qquad \boldsymbol{x}:=(x_i)_{i=0}^d\in\mathbb{R}^{d+1},\quad \sum_{i=0}^{d}x_i=1.$$

Let us introduce the sets of indices $\Lambda_{\boldsymbol{\gamma}}^i$, $i=0,1,\ldots,d$, and $\Lambda_{\boldsymbol{\gamma}}$ (see Figure 4.1). If $\gamma_i\neq 0$, let

$$\Lambda_{\boldsymbol{\gamma}}^i:=\left\{\boldsymbol{\lambda}^i:=\left(\lambda_0^i,\lambda_1^i,\ldots,\lambda_{\gamma_i-1}^i\right)^T,\ \lambda_0^i=i,\ \lambda_j^i\in\{i,i+1,\ldots,i+d\},\ 0<j\leq\gamma_i-1\right\},$$

and $\left|\Lambda_{\boldsymbol{\gamma}}^i\right|=(d+1)^{\gamma_i-1}$, otherwise $\Lambda_{\boldsymbol{\gamma}}^i:=\emptyset$. Further,

$$\Lambda_{\boldsymbol{\gamma}}:=\left\{\boldsymbol{\lambda}:=\left(\lambda_0^0,\lambda_1^0,\ldots,\lambda_{\gamma_0-1}^0,\ldots,\lambda_0^d,\lambda_1^d,\ldots,\lambda_{\gamma_d-1}^d\right)^T\in\mathbb{N}_0^n,\ \left(\lambda_0^i,\ldots,\lambda_{\gamma_i-1}^i\right)^T\in\Lambda_{\boldsymbol{\gamma}}^i\right\},$$

and clearly

$$|\Lambda_{\boldsymbol{\gamma}}|=\prod_{i=0}^{d}(d+1)^{\max\{0,\gamma_i-1\}}. \tag{4.5}$$

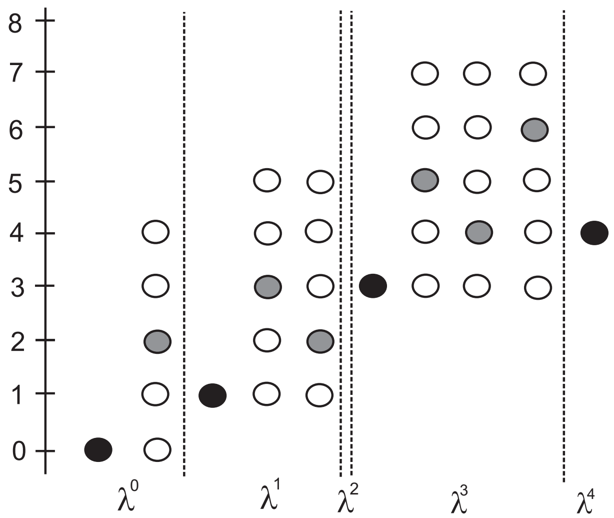


Figure 4.1: Index vector $\boldsymbol{\lambda}=(0,2,1,3,2,3,5,4,6,4)^T\in\Lambda_{\boldsymbol{\gamma}}$ is an example for $d=4$, $n=10$ and $\boldsymbol{\gamma}=(2,3,0,4,1)^T$. Any other selection of grey points would determine another index vector in $\Lambda_{\boldsymbol{\gamma}}$, what corresponds to (4.5).

We can now state the following theorem.

THEOREM 4.1. *The weights $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ of the cubature rule (4.2) are*

$$\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = K(\boldsymbol{\xi}) \cdot \sum_{\boldsymbol{\lambda}\in\Lambda_{\boldsymbol{\gamma}}} \left(\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} a_{\lambda^i_j,j}(\boldsymbol{\xi}) \right) \frac{\boldsymbol{k_\lambda}!}{(n+d)!}, \tag{4.6}$$

where

$$K(\boldsymbol{\xi}) := \left(\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi}) \right)^{-1}, \quad \boldsymbol{k_\lambda} := (k_{\boldsymbol{\lambda},0}, k_{\boldsymbol{\lambda},1}, \ldots, k_{\boldsymbol{\lambda},d})^T,$$

and $k_{\boldsymbol{\lambda},i}$ denotes the frequency of i in $\boldsymbol{\lambda}$.

Proof. By (4.3),

$$\begin{aligned}
\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) &= S_{\triangle^d_{d+1}}(\mathcal{L}_{\boldsymbol{\gamma}}) = K(\boldsymbol{\xi}) \cdot \int_{\triangle^d_{d+1}} \left(\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} h_{i,j}(\boldsymbol{x};\boldsymbol{\xi}) \right) \mathrm{d}\boldsymbol{x} = \\
&= K(\boldsymbol{\xi}) \cdot \int_{\triangle^d_{d+1}} \prod_{i=0}^{d} \left(\prod_{j=0}^{\gamma_i-1} \sum_{t=i}^{i+d} a_{t,j}(\boldsymbol{\xi}) x_t \right) \mathrm{d}\boldsymbol{x} = \\
&= K(\boldsymbol{\xi}) \cdot \int_{\triangle^d_{d+1}} \prod_{i=0}^{d} \left(\sum_{\boldsymbol{\lambda}^i\in\Lambda^i_{\boldsymbol{\gamma}}} \prod_{j=0}^{\gamma_i-1} a_{\lambda^i_j,j}(\boldsymbol{\xi})\, x_{\lambda^i_j} \right) \mathrm{d}\boldsymbol{x} = \\
&= K(\boldsymbol{\xi}) \cdot \sum_{\boldsymbol{\lambda}\in\Lambda_{\boldsymbol{\gamma}}} \left(\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} a_{\lambda^i_j,j}(\boldsymbol{\xi}) \cdot \int_{\triangle^d_{d+1}} \prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} x_{\lambda^i_j}\, \mathrm{d}\boldsymbol{x} \right).
\end{aligned}$$

With the help of the notation $k_{\boldsymbol{\lambda},i}$ we count the multiplicity of x_i in the product

$$\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} x_{\lambda^i_j}$$

and obtain

$$\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} x_{\lambda^i_j} = \prod_{i=0}^{d} \left(\prod_{\ell=0}^{d} x_\ell^{k_{\boldsymbol{\lambda}^i,\ell}} \right) = \prod_{\ell=0}^{d} x_\ell^{k_{\boldsymbol{\lambda},\ell}}.$$

Now, $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ becomes

$$\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = K(\boldsymbol{\xi}) \cdot \sum_{\boldsymbol{\lambda}\in\Lambda_{\boldsymbol{\gamma}}} \left(\prod_{i=0}^{d} \prod_{j=0}^{\gamma_i-1} a_{\lambda^i_j,j}(\boldsymbol{\xi}) \int_{\triangle^d_{d+1}} \prod_{i=0}^{d} x_i^{k_{\boldsymbol{\lambda},i}}\, \mathrm{d}\boldsymbol{x} \right). \tag{4.7}$$

Further, with Γ being the gamma function,

$$\int_{\triangle^d_{d+1}} \prod_{i=0}^{d} x_i^{k_{\boldsymbol{\lambda},i}}\, \mathrm{d}\boldsymbol{x} = \frac{\Gamma(k_{\boldsymbol{\lambda},0}+1)\,\Gamma(k_{\boldsymbol{\lambda},1}+1)\cdots\Gamma(k_{\boldsymbol{\lambda},d}+1)}{\Gamma(k_{\boldsymbol{\lambda},0}+k_{\boldsymbol{\lambda},1}+\ldots+k_{\boldsymbol{\lambda},d}+d+1)},$$

where for $\boldsymbol{x} \in \mathbb{R}^{d+1}$, $\sum_{i=0}^{d} x_i = 1$,

$$\int_{\triangle_{d+1}^{d}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} := \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \cdots \int_0^{1-\sum_{i=1}^{d} x_i} f\left((1-\sum_{i=1}^{d} x_i,\, x_1, \dots, x_d)^T\right) \mathrm{d}x_d. \tag{4.8}$$

Since

$$\sum_{i=0}^{d} k_{\boldsymbol{\lambda},i} = n,$$

it follows

$$\int_{\triangle_{d+1}^{d}} \prod_{i=0}^{d} x_i^{k_{\boldsymbol{\lambda},i}}\,\mathrm{d}\boldsymbol{x} = \frac{k_{\boldsymbol{\lambda},0}!\,k_{\boldsymbol{\lambda},1}!\cdots k_{\boldsymbol{\lambda},d}!}{(n+d)!},$$

and the proof is concluded. □

As an example, let us compute the weights for $d = 2$ and $n = 3$. The barycentric form of the cubature rule is given by

$$Q^{(3)}(f;\boldsymbol{\xi}) := \sum_{\boldsymbol{\gamma}\in\mathcal{I}_3^2} \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) f_{\boldsymbol{\gamma}},$$

and $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$, $\boldsymbol{\gamma} \in \mathcal{I}_3^2$, is equal to one of the following possibilities:

$$\boldsymbol{\gamma} \in \left\{(3,0,0)^T, (0,3,0)^T, (0,0,3)^T\right\},\ i := (\gamma_i = 3):$$

$$\frac{1}{20} + \frac{\alpha^3(1+\alpha) + \alpha\xi_{i+1}(1+\xi_{i+1})\left(\alpha^2(1+\alpha) - (1+\alpha(3+\alpha))\xi_i\right)}{60\,(1+\alpha)\xi_i^2\xi_{i+1}^2},$$

$$\boldsymbol{\gamma} \in \left\{(2,1,0)^T, (2,0,1)^T, (0,2,1)^T\right\},\ i := (\gamma_i = 2),\ j := (\gamma_j = 1):$$

$$-\frac{\left(\alpha+\alpha^2+\prod_{t=i}^{j-1}\xi_t\right)^3\left((j-i)^2\alpha^2 + 2\xi_{i+1}\left(\alpha^2-(j-i)(1+\alpha)\xi_i\right)\right)}{(j-i)\,120\,\alpha^2(1+\alpha)(1+\alpha+\alpha^2)\xi_i^2\xi_{i+1}^{j-i}},$$

$$\boldsymbol{\gamma} \in \left\{(1,2,0)^T, (1,0,2)^T, (0,1,2)^T\right\},\ i := (\gamma_i = 2),\ j := (\gamma_j = 1):$$

$$-\frac{\left(\alpha^2+(1+\alpha)\prod_{t=j}^{i-1}\xi_t\right)^3\left(2\alpha^2 + (i-j)\xi_{i+1}\left((i-j)\alpha^2 - 2(1+\alpha)\xi_i\right)\right)}{(i-j)\,120\,\alpha^5(1+\alpha)\left(\alpha-(1+\alpha)^2\right)\xi_j^{i-j}\xi_1^{i-j-1}},$$

$$\boldsymbol{\gamma} = (1,1,1)^T:$$

$$\frac{\left(\alpha^2+\xi_0(\alpha+\xi_1)\right)^3}{120\,\alpha^3\xi_0^2\xi_1}.$$

For two particular choices of parameters $\boldsymbol{\xi}$ and for $I = \{(3,0,0)^T,\ (0,3,0)^T,\ (0,0,3)^T,$

$(2,1,0)^T$, $(2,0,1)^T$, $(0,2,1)^T$, $(1,2,0)^T$, $(1,0,2)^T$, $(0,1,2)^T$, $(1,1,1)^T\}$, the weights are

$$(\xi_0,\xi_1,\xi_2)^T = (1,1,1)^T:$$
$$\{\omega_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}\in I\} = \left\{\frac{1}{60},\frac{1}{60},\frac{1}{60},\frac{3}{80},\frac{3}{80},\frac{3}{80},\frac{3}{80},\frac{3}{80},\frac{3}{80},\frac{9}{40}\right\},$$
$$(\xi_0,\xi_1,\xi_2)^T = \left(\frac{2}{3},1,\frac{3}{2}\right)^T:$$
$$\{\omega_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}\in I\} = \left\{\frac{3}{80},\frac{17}{1080},\frac{17}{1080},-\frac{8}{405},-\frac{8}{405},\frac{1}{20},\frac{343}{5184},\frac{343}{5184},\frac{1}{20},\frac{343}{1440}\right\}.$$

Note that for $n\geq 4$, the weights can be negative also for principal lattices ($\xi_i = 1$ for all i). Furthermore, it is obvious that $\sum_{\boldsymbol{\gamma}\in I}\omega_{\boldsymbol{\gamma}} = \frac{1}{d!} = \frac{1}{2}$.

Consider now a simplex $\triangle = \langle\, \boldsymbol{V}_0, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_d\,\rangle \subset \mathbb{R}^d$ and let

$$\boldsymbol{X}_{\boldsymbol{\gamma}}, \quad \boldsymbol{\gamma}\in\mathcal{I}_n^d,$$

denote the Cartesian coordinates of lattice points, obtained from the barycentric representation as in (2.39). We are now able to state the following corollary.

COROLLARY 4.2. *A Newton-Cotes cubature rule of degree n for a simplex $\triangle\subset\mathbb{R}^d$ is*

$$Q_\triangle(f) := Q_\triangle^{(n)}(f;\boldsymbol{\xi}) := \sum_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}\omega_{\boldsymbol{\gamma},\triangle}(\boldsymbol{\xi})\, f(\boldsymbol{X}_{\boldsymbol{\gamma}}) = d!\,\mathrm{vol}\,(\triangle)\sum_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})\, f(\boldsymbol{X}_{\boldsymbol{\gamma}}),$$

where $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ are the weights given by (4.6) *and* $\mathrm{vol}\,(\triangle)$ *is the volume of the simplex $\triangle$.*

Proof. Let us take the standard simplex $\triangle^d$ and let $\tilde{\boldsymbol{u}} := (\tilde{u}_i)_{i=1}^d \in \triangle^d$. Then the barycentric coordinates of $\tilde{\boldsymbol{u}}$ w.r.t. $\triangle^d$ are

$$\left(1-\sum_{i=1}^d \tilde{u}_i, \tilde{u}_1,\ldots,\tilde{u}_d\right)^T =: (\tilde{u}_0, \tilde{\boldsymbol{u}})^T.$$

Using (4.7) and (4.8), we obtain

$$\omega_{\boldsymbol{\gamma},\triangle^d}(\boldsymbol{\xi}) = \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}).$$

Suppose now that $\triangle\subset\mathbb{R}^d$ is an arbitrary simplex, and let $\boldsymbol{u} := (u_i)_{i=1}^d\in\triangle$. Further, let $\boldsymbol{x}(\boldsymbol{u}) := (x_i(\boldsymbol{u}))_{i=0}^d$ be the barycentric coordinates of $\boldsymbol{u}$ w.r.t. $\triangle$. By the definition of barycentric coordinates,

$$(x_i(\boldsymbol{u}))_{i=0}^d = (\tilde{u}_i)_{i=0}^d.$$

Since

$$\omega_{\boldsymbol{\gamma},\triangle}(\boldsymbol{\xi}) = S_\triangle(\mathcal{L}_{\boldsymbol{\gamma}}) = K(\boldsymbol{\xi})\cdot\sum_{\boldsymbol{\lambda}\in\Lambda_{\boldsymbol{\gamma}}}\left(\prod_{i=0}^d\prod_{j=0}^{\gamma_i-1} a_{\lambda_j^i,j}(\boldsymbol{\xi})\int_\triangle\prod_{i=0}^d x_i(\boldsymbol{u})^{k_{\boldsymbol{\lambda},i}}\,\mathrm{d}\boldsymbol{u}\right)$$

and

$$\int_{\triangle^d} \prod_{i=0}^{d} \tilde{u}_i^{k_{\boldsymbol{\lambda},i}} \, \mathrm{d}\tilde{\boldsymbol{u}} = J \cdot \int_{\triangle} \prod_{i=0}^{d} x_i(\boldsymbol{u})^{k_{\boldsymbol{\lambda},i}} \, \mathrm{d}\boldsymbol{u},$$

with

$$J = \det\left(\frac{\partial \tilde{\boldsymbol{u}}}{\partial \boldsymbol{u}}\right) = \frac{\mathrm{vol}(\triangle^d)}{\mathrm{vol}(\triangle)} = \frac{1}{d!\,\mathrm{vol}(\triangle)},$$

where $\frac{\partial \tilde{\boldsymbol{u}}}{\partial \boldsymbol{u}}$ is the Jacobian matrix, it follows

$$\omega_{\boldsymbol{\gamma},\triangle}(\boldsymbol{\xi}) = \frac{1}{J}\,\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = d!\,\mathrm{vol}(\triangle)\,\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}).$$

The proof is concluded. □

Our next goal is to derive the error term of the cubature rule (4.2) in the barycentric form for a sufficiently smooth function f. Let us recall (2.31). The error is then obtained as

$$S_{\triangle^d_{d+1}}(f - p_n).$$

So we have to derive the interpolation error (see Chapter 1) in a convenient form first. But then the Newton basis polynomials in the barycentric form need to be determined, too. Recall that they are the polynomials of total degrees $|\boldsymbol{\gamma}| \leq n$, that vanish at particular subsets of interpolation points. In order to determine these sets precisely, let us use the abbreviated lattice point indexation introduced in (2.30) (see Figure 2.11). Since the Newton polynomials $\mathcal{N}_{\boldsymbol{\gamma}'}$ satisfy

$$\mathcal{N}_{\boldsymbol{\gamma}'}(\boldsymbol{B}_{\boldsymbol{\beta}'}) = \delta_{\boldsymbol{\gamma}',\boldsymbol{\beta}'}, \quad \forall\, \boldsymbol{\gamma}', \boldsymbol{\beta}' \in \mathbb{N}_0^d,\ |\boldsymbol{\beta}'| \leq |\boldsymbol{\gamma}'| \leq n,$$

they also have a very simple barycentric representation

$$\mathcal{N}_{\boldsymbol{\gamma}'}(\boldsymbol{x}; \boldsymbol{\xi}) = \prod_{i=1}^{d} \prod_{j=0}^{\gamma_i - 1} \frac{h_{i,j}(\boldsymbol{x}; \boldsymbol{\xi})}{h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}'}; \boldsymbol{\xi})}.$$

This follows from the facts that a hyperplane with the equation $h_{i,j} = 0$, given by (4.4), vanishes at lattice points $\boldsymbol{B}_{\boldsymbol{\beta}'}$, $\beta_i = j$, and that for $\boldsymbol{\beta}' \neq \boldsymbol{\gamma}', |\boldsymbol{\beta}'| \leq |\boldsymbol{\gamma}'|$, there exists an index $i \in \{1, 2, \ldots, d\}$, such that $\beta_i < \gamma_i$. Let us recall some notation from Chapter 1 and translate it to the barycentric form.

With any path

$$\underline{\boldsymbol{\mu}} = (\boldsymbol{\mu}'_0, \boldsymbol{\mu}'_1, \ldots, \boldsymbol{\mu}'_n)^T \in \Xi_n,$$

where Ξ_n is given in (1.2), let us associate a set of lattice points $\boldsymbol{B}_{\underline{\boldsymbol{\mu}}}$, a number $\Pi_{\underline{\boldsymbol{\mu}}}$, and a corresponding n-th order differential operator $D^n_{\underline{\boldsymbol{\mu}}}$,

$$\boldsymbol{B}_{\underline{\boldsymbol{\mu}}} := \boldsymbol{B}_{\underline{\boldsymbol{\mu}}}(\boldsymbol{\xi}) := \{\boldsymbol{B}_{\boldsymbol{\mu}'_0}, \boldsymbol{B}_{\boldsymbol{\mu}'_1}, \ldots, \boldsymbol{B}_{\boldsymbol{\mu}'_n}\},$$

$$\Pi_{\underline{\boldsymbol{\mu}}} := \Pi_{\underline{\boldsymbol{\mu}}}(\boldsymbol{\xi}) := \prod_{j=0}^{n-1} \mathcal{N}_{\boldsymbol{\mu}'_j}(\boldsymbol{B}_{\boldsymbol{\mu}'_{j+1}}; \boldsymbol{\xi}),$$

$$D^n_{\underline{\boldsymbol{\mu}}} := D^n_{\underline{\boldsymbol{\mu}}}(\boldsymbol{\xi}) := D_{\boldsymbol{B}_{\boldsymbol{\mu}'_n}-\boldsymbol{B}_{\boldsymbol{\mu}'_{n-1}}} \cdot D_{\boldsymbol{B}_{\boldsymbol{\mu}'_{n-1}}-\boldsymbol{B}_{\boldsymbol{\mu}'_{n-2}}} \cdots D_{\boldsymbol{B}_{\boldsymbol{\mu}'_1}-\boldsymbol{B}_{\boldsymbol{\mu}'_0}}.$$

But the construction of the Newton basis polynomials on $(d+1)$-pencil lattices gives $\Pi_{\underline{\boldsymbol{\mu}}} = 0$ if $\underline{\boldsymbol{\mu}} \notin \widetilde{\Xi}_n$, where

$$\widetilde{\Xi}_n := \{\underline{\boldsymbol{\mu}} \in \Xi_n,\ \boldsymbol{\mu}'_{j+1} = \boldsymbol{\mu}'_j + (\delta_{i,k})_{i=1}^d,\ k \in \{1,2,\ldots,d\},\ j = 0,1,\ldots,n-1\}.$$

This reveals the barycentric form of the interpolation error, given in Theorem 1.21, as

$$f(\boldsymbol{x}) - p_n(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{\underline{\boldsymbol{\mu}} \in \widetilde{\Xi}_n} \mathcal{N}_{\boldsymbol{\mu}'_n}(\boldsymbol{x};\boldsymbol{\xi})\, \Pi_{\underline{\boldsymbol{\mu}}} \int_{[\boldsymbol{B}_{\underline{\boldsymbol{\mu}}},\, \boldsymbol{x}]} D_{\boldsymbol{x}-\boldsymbol{B}_{\boldsymbol{\mu}'_n}} D^n_{\underline{\boldsymbol{\mu}}} f, \qquad \boldsymbol{x} \in \mathbb{R}^{d+1},\ \sum_{i=0}^d x_i = 1,$$

where

$$f(\boldsymbol{x}) := f(\boldsymbol{u}(\boldsymbol{x})), \quad \boldsymbol{u}(\boldsymbol{x}) = \sum_{j=0}^d x_j\, \boldsymbol{V}_j.$$

This proves the following theorem

THEOREM 4.3. *Let $f \in C^{n+1}(\mathbb{R}^d)$. The barycentric form of the error of the cubature rule* (4.2) *is given as*

$$E(f) = S_{\triangle^d_{d+1}} \left(\sum_{\underline{\boldsymbol{\mu}} \in \widetilde{\Xi}_n} \mathcal{N}_{\boldsymbol{\mu}'_n}(\boldsymbol{x};\boldsymbol{\xi})\, \Pi_{\underline{\boldsymbol{\mu}}} \int_{[\boldsymbol{B}_{\underline{\boldsymbol{\mu}}},\, \boldsymbol{x}]} D_{\boldsymbol{x}-\boldsymbol{B}_{\boldsymbol{\mu}'_n}} D^n_{\underline{\boldsymbol{\mu}}} f \right).$$

4.2. Lattice refinement

Using the lattice extension approach, presented in the previous chapter, the cubature rule (4.2) can be efficiently extended from a simplex to a simplicial partition. Since for small enough degrees most of the lattice points lie on facets of simplices, the described extension enables us to evaluate the function at these points only once.

Newton-Cotes cubature rules become really useful in practice when one applies them in an adaptive way. A globally adaptive algorithm over a simplicial partition is usually based upon successive refinements or subdivisions of simplices. Though it is obvious that such a refinement could be carried out for principal lattices, it is far away of being obvious that this can be done for $(d+1)$-pencil lattices too. In this section, a lattice refinement step that is a basis of the adaptive algorithm in the next section, is presented.

A lattice refinement approach is quite useful also in multivariate interpolation. Namely, it can happen that the approximation by the interpolating polynomial is not satisfactory on some simplex of the partition. An obvious remedy is to increase the number of interpolation points on this simplex (Figure 4.2). A natural way to do this is to refine a lattice. Let $\triangle \in \mathcal{T}$ be the simplex where a lattice refinement is needed. In order to retain regularity of a simplicial partition, let us refine the lattice by adding a new vertex into the interior of $\triangle$. The refinement of a lattice on the simplicial partition $\mathcal{T}$ consists of the following steps (Figure 4.2):

- Choose a simplex $\triangle \in \mathcal{T}$, where the refinement is needed.
- Add a new vertex $\boldsymbol{T}$ into the interior of $\triangle$.
- Add $d+1$ edges from $\boldsymbol{T}$ to the vertices of $\triangle$. These edges split the simplex $\triangle$ into $d+1$ new simplices.
- Construct new lattices on these simplices such that two adjacent simplices share the lattice restriction to the common face.

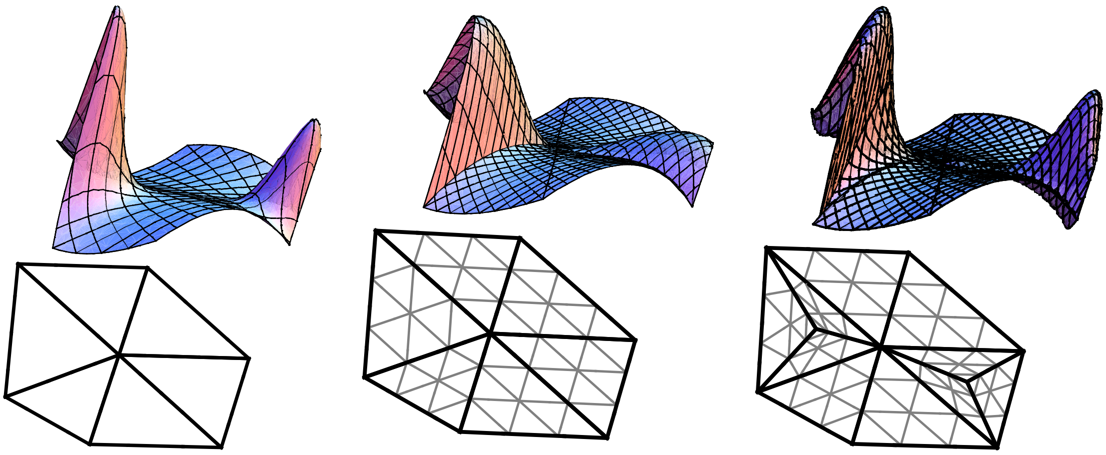


Figure 4.2: A given surface and two different continuous piecewise polynomial interpolants over lattices on underlying triangulations.

The following theorem precisely determines the last step of the lattice refinement. Recall, that $\mathcal{B}(\triangle)$ denotes the boundary (the union of all facets) of a simplex $\triangle$.

THEOREM 4.4. *Let* $(\boldsymbol{B}_\gamma(\boldsymbol{\xi}))_{\gamma\in\mathcal{I}_n^d}$ *be the barycentric coordinates of a* $(d+1)$*-pencil lattice on* $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$*, and let* $\boldsymbol{T}_{d+1}$ *be a vertex in the interior of* $\triangle$ *that splits* $\triangle$ *to* $d+1$ *simplices* $\{\triangle_i\}_{i=1}^{d+1}$*. Then there exist* $(d+1)$*-pencil lattices on* $\{\triangle_i\}_{i=1}^{d+1}$ *which coincide on common faces of* $\{\triangle_i\}_{i=1}^{d+1}$ *and agree with the initial lattice on* $\mathcal{B}(\triangle)$*. Moreover, there is one degree of freedom to construct these lattices (see Figure 4.3).*

Proof. Let us order the vertices of $\triangle_i$, $i = 1, 2, \ldots, d+1$, as

$$\triangle_i = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_d} \rangle, \ 0 \le i_0 < i_1 < \cdots < i_d = d+1. \tag{4.9}$$

Note that the indices of vertices are not taken modulo $d+1$ here. Any pair of simplices $\triangle_i$, $\triangle_j$ has a facet in common. Let this facet be in $\triangle_i$ denoted as

$$\langle \boldsymbol{T}_{i_{r_0}}, \boldsymbol{T}_{i_{r_1}}, \ldots, \boldsymbol{T}_{i_{r_{d-1}}} \rangle, \quad 0 \le i_{r_0} < i_{r_1} < \cdots < i_{r_{d-1}} \le d+1,$$

with the corresponding vertices in $\triangle_j$ given by

$$\boldsymbol{T}_{j_{r_k}} = \boldsymbol{T}_{i_{r_k}}, \quad k = 0, 1, \ldots, d-1.$$

By (4.9),

$$w\left(\left(i_{r_0}, i_{r_1}, \ldots, i_{r_{d-1}}\right)^T\right) = w\left(\left(j_{r_0}, j_{r_1}, \ldots, j_{r_{d-1}}\right)^T\right) = 1. \tag{4.10}$$

Assume that the product of local barycentric lattice parameters on each simplex in $\{\triangle_i\}_{i=1}^{d+1}$ is equal to the product of local barycentric lattice parameters for the lattice on $\triangle$. All simplices $\triangle_i$, $i = 1, 2, \ldots, d+1$, have one facet in common with $\triangle$. Let us first construct the lattice on $\triangle_1$. Since a similar relation as in (4.10) holds on the common facet, the lattice can be extended from this facet to $\triangle_1$ with one additional free parameter (Corollary 3.20). Now the simplices $\triangle_i$, $i = 2, 3, \ldots, d+1$, have two facets in common with $\triangle \cup \triangle_1$. Therefore by the same argument as in the proof of Theorem 3.21 all lattices on $\triangle_2, \triangle_3, \ldots, \triangle_{d+1}$ are uniquely determined and agree with the given one on $\mathcal{B}(\triangle) \cup \mathcal{B}(\triangle_1)$. In order to conclude the proof, it only has to be shown that the lattices agree on common facets

$$\triangle_{ij} := \triangle_i \cap \triangle_j, \quad 2 \le i < j \le d+1.$$

Since $\triangle_{ij}$ are $(d-1)$-simplices, the case $d = 2$ has to be considered separately. For $d \ge 3$, lattices on $\triangle_{ij}$ are already uniquely determined by the lattices on $\triangle \cup \triangle_1$ (Corollary 3.17) and therefore the lattices on $\triangle_i$ and $\triangle_j$ agree on $\triangle_{ij}$. Now let $d = 2$. The same corollary can not be used now, since facets of $(d-1)$-simplices are vertices and they do not include any information about the lattice. However, the idea already used in Lemma 3.32 proves the theorem for the planar case. Namely, all four triangles $\triangle_1, \triangle_2, \triangle_3$ and $\triangle$ together can be identified with a tetrahedron in $\mathbb{R}^3$. Then by Corollary 3.17, the lattices on $\triangle_2$ and $\triangle_3$ are uniquely determined with lattices on $\triangle$ and $\triangle_1$ and they agree on a common edge $\triangle_{23}$. □

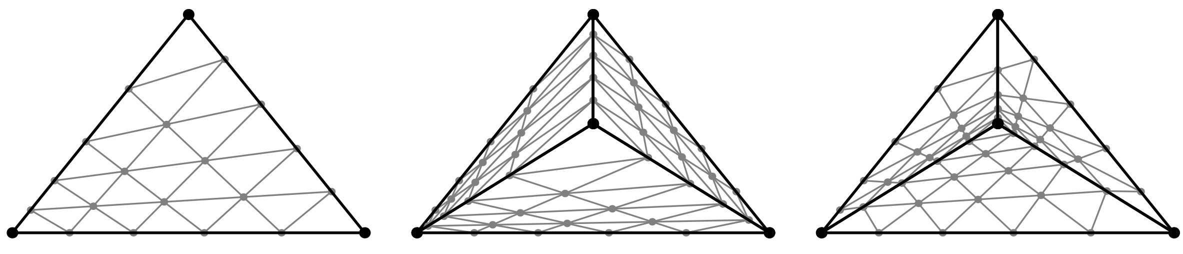


Figure 4.3: A lattice with parameters $\xi_0 = 1/2$, $\xi_1 = 3/5$, $\xi_2 = 4/3$, and two different refinements with the additional shape parameter $\zeta = 7/3$, $1/2$, respectively.

In Figure 4.4, an example of the lattice refinement approach is shown in an adaptive way.

Also some other approaches to refine a lattice on a simplex may be appropriate for some applications. For example, in the Romberg type multivariate integration, a refinement shown in Figure 4.5 may be used.

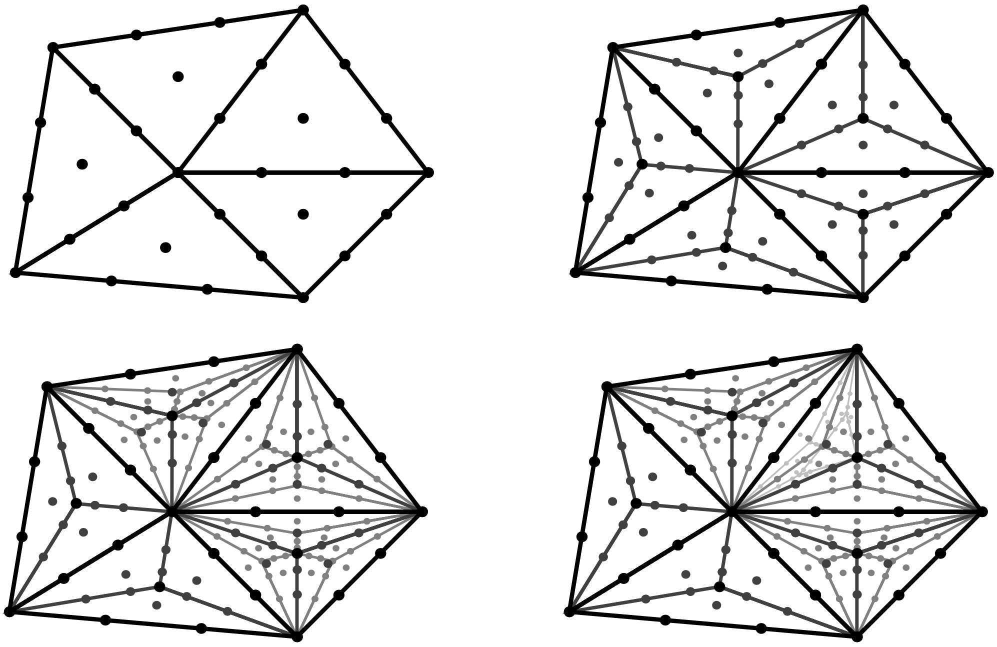


Figure 4.4: An example of the lattice refinement in an adaptive way.

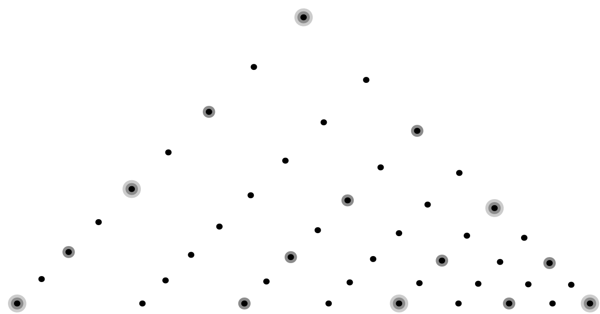


Figure 4.5: Three-pencil lattices of order 2, 4 and 8 with $(\xi_0, \xi_1, \xi_2)^T = \left(\frac{4}{3}, \frac{1}{3}, 1\right)^T$.

4.3. Adaptive cubature rules

In this section, we will study derived cubature rules, applied in an adaptive way. Let us consider the integrals of the form

$$\int_{\mathcal{T}} f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} = \sum_{\triangle\in\mathcal{T}} \int_{\triangle} f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}, \tag{4.11}$$

where $\mathcal{T}$ is a simplicial partition in $\mathbb{R}^d$, using an adaptive algorithm that consists of a sequence of stages, where each stage has the following steps:

(a) from the current simplicial partition $\mathcal{T}'$ (at the beginning $\mathcal{T}' = \mathcal{T}$) select simplices $\triangle$, where the cubature rule does not give a satisfying approximation,

(b) subdivide selected simplices and determine the lattices on the newly obtained simplices,

(c) update the simplicial partition $\mathcal{T}'$ with new simplices, apply a local cubature rule to any new simplex by carefully avoiding extraneous function evaluations, and update the global integral (4.11) for $\mathcal{T}'$.

At the beginning of the algorithm we have to determine a global integral approximation based upon the initial $(d{+}1)$-pencil lattice on the simplicial partition $\mathcal{T}$ and then continue with the step (b). Since the step (c) is straightforward, we only have to describe steps (a) and (b). In the step (a) we select, for a given constant $\epsilon > 0$, collections of simplices

$$\{\triangle_1, \triangle_2, \ldots, \triangle_{d+1}\},$$

for which

$$\left| Q_{\triangle}(f) - \sum_{i=1}^{d+1} Q_{\triangle_i}(f) \right| > \epsilon,$$

where $\triangle$ is the simplex that was split into $\triangle_1, \triangle_2, \ldots, \triangle_{d+1}$ in the previous stage. Clearly, there are several ways how to perform step (b), which requires a subdivision of selected simplices (see [19], e.g.). But since our main goal is to keep the number of function evaluations at new points as low as possible, we will choose the subdivision strategy that will be based upon the lattice refinement approach, presented in the previous section. Recall that in this case we have to determine a subdivision point $\boldsymbol{T}$ in the interior of a simplex, which defines $d+1$ new simplices

$$\triangle_i = \langle \underbrace{\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{d-i}}_{d+1-i}, \underbrace{\boldsymbol{T}_{d-i+2}, \boldsymbol{T}_{d-i+3}, \ldots, \boldsymbol{T}_d}_{i-1}, \boldsymbol{T} \rangle, \quad i = 1, 2, \ldots, d+1. \tag{4.12}$$

Moreover, we have to determine the lattices on the newly obtained simplices, i.e., we have to choose a shape parameter ζ used by the lattice refinement (Figure 4.3). Let us now consider two different possibilities how to determine a subdivision point and the lattices on new simplices.

Algorithm 1. We subdivide a simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ to $d+1$ new simplices (4.12), where

$$\boldsymbol{T} := \frac{1}{d+1} \sum_{i=0}^{d} \boldsymbol{T}_i,$$

and the lattice on each simplex is a principal lattice. This is possible only if the lattices on all simplices in the original simplicial partition are principal lattices. We obtain the standard Newton-Cotes adaptive rule.

Algorithm 2. Using the lattice refinement approach presented in the previous section, we subdivide a simplex $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$, having a $(d+1)$-pencil lattice determined by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$, to $d+1$ simplices $\{\triangle_i\}_{i=1}^{d+1}$ given in (4.12), with lattices determined by parameters

$$
\begin{aligned}
&\boldsymbol{\xi}(\triangle_1) = \left(\xi_0, \xi_1, \ldots, \xi_{d-2}, \zeta, \frac{\xi_{d-1}\xi_d}{\zeta}\right)^T, \\
&\boldsymbol{\xi}(\triangle_j) = \left(\xi_0, \xi_1, \ldots, \xi_{d-j-1}, \xi_{d-j} \cdot \xi_{d-j+1}, \xi_{d-j+2}, \ldots, \xi_{d-1}, \frac{\zeta}{\xi_{d-1}}, \frac{\xi_{d-1}\xi_d}{\zeta}\right)^T, \\
&j = 2, 3, \ldots, d-1, \\
&\boldsymbol{\xi}(\triangle_d) = \left(\xi_0 \cdot \xi_1, \xi_2, \xi_3, \ldots, \xi_{d-1}, \frac{\zeta}{\xi_{d-1}}, \frac{\xi_{d-1}\xi_d}{\zeta}\right)^T, \\
&\boldsymbol{\xi}(\triangle_{d+1}) = \left(\xi_1, \xi_2, \ldots, \xi_{d-1}, \frac{\zeta}{\xi_{d-1}}, \frac{\xi_{d-1}\xi_d\xi_0}{\zeta}\right)^T,
\end{aligned}
\tag{4.13}
$$

where ζ is a free parameter. Suppose now that the number of function evaluations is crucial. The subdivision point

$$\boldsymbol{T} := \boldsymbol{z} = (z_1, z_2, \ldots, z_d)^T$$

and the additional free parameter ζ in (4.13) can then be determined by a particular procedure, which can substantially decrease the number of function evaluations needed. Before we state this procedure, let us introduce the set of

$$\binom{n+d-1}{d-1}$$

monomials of total degree n,

$$\mathcal{P}_n := \left\{\boldsymbol{u}^{\gamma},\ |\boldsymbol{\gamma}| = n,\ \boldsymbol{\gamma} \in \mathbb{N}_0^d\right\},$$

and the closed ball $\Omega_\triangle \subset \triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle \subset \mathbb{R}^d$,

$$\Omega_\triangle := \Omega_\triangle(\boldsymbol{c}, r) := \{\boldsymbol{u} \in \triangle,\ \| \boldsymbol{u} - \boldsymbol{c} \|_2 \le r\}, \quad \boldsymbol{c} := \frac{1}{d+1}\sum_{i=0}^{d} \boldsymbol{T}_i,$$

$$r := \frac{1}{2} \cdot \min_{0 \le i < j \le d} \left\{ \left\| \frac{\boldsymbol{T}_i + \boldsymbol{T}_j}{2} - \boldsymbol{c} \right\|_2 \right\}.$$

procedure ChooseParameters $(d, n, \triangle, \boldsymbol{\xi})$

value$(\boldsymbol{z}, \zeta) := 0$;
for $p \in \mathcal{P}_{n+1}$
  sum$(\boldsymbol{z}, \zeta) = d! \sum_{i=1}^{d+1} \mathrm{vol}(\triangle_i) \left(\sum_{\boldsymbol{\gamma} \in \mathcal{I}_n^d} \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}(\triangle_i))\, p(\boldsymbol{X}_{\boldsymbol{\gamma}}(\triangle_i))\right)$;
  value$(\boldsymbol{z}, \zeta) =$ value$(\boldsymbol{z}, \zeta) + (S_\triangle(p) -$ sum$(\boldsymbol{z}, \zeta))^2$;
end;
$\{\bar{\boldsymbol{z}}, \bar{\zeta}\} := \left(\mathrm{value}(\bar{\boldsymbol{z}}, \bar{\zeta}) = \min_{\boldsymbol{z} \in \Omega_\triangle,\ \frac{1}{4} \le \zeta \le 4} \mathrm{value}(\boldsymbol{z}, \zeta)\right)$;
Return $\{\bar{\boldsymbol{z}}, \bar{\zeta}\}$;

Note that this procedure does not depend on the integrated function f. If our aim is to integrate several functions over the same simplicial partition, we will store all the computed parameters $\{\bar{\boldsymbol{z}}, \bar{\zeta}\}$.

Let us compare both algorithms on several interesting functions (Table 4.1, Figure 4.6). Let

$$\mathcal{T} \subset [-1,2] \times [-3/2, 3/2] \subset \mathbb{R}^2$$

be a star and let $d = 2$ and $n = 3$. As expected, the number of function evaluations is significantly smaller for the second algorithm, since there is a freedom of choosing the subdivision point $\boldsymbol{T}$ and the free parameter ζ at every step. Note that the choice of the parameter ζ brings approximately 20% to this fact.

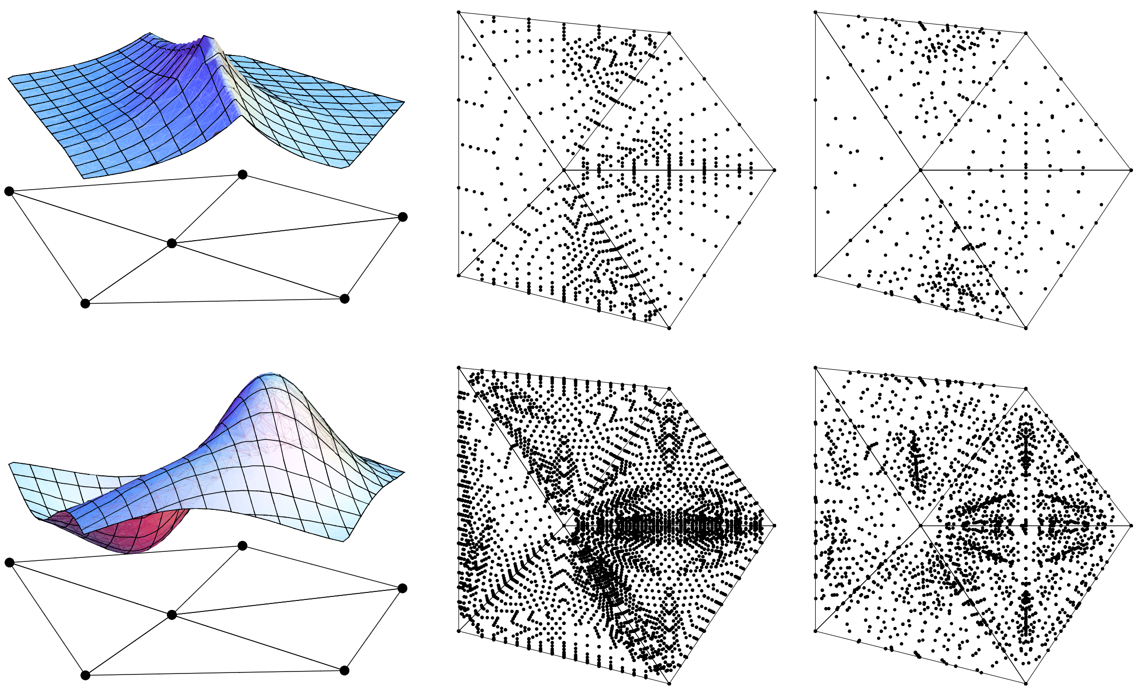


Figure 4.6: The points where the function evaluations are needed in both Newton-Cotes algorithms for the last two rows in Table 4.1.

Let us conclude the section with a brief efficiency comparison between particular Gaussian type adaptive formulae, and the cubature rules outlined in this chapter. However, it should be emphasized that the computational complexity is not the only issue to be kept in mind when one is comparing these two classes of cubature rules. The Gaussian type requires the integrated function more or less to be known in a closed form. On the other hand, Newton-Cotes formulae, which were here extended to $(d+1)$-pencil lattices, are closed form rules based upon the function values evaluated at particular unisolvent sets of points that can be simply generated in any dimension. Thus the data may be supplied in a tabular form only. Also, since it is straightforward to generate the lattice points, a way to make a computer program may be shorter. As for the numerical test, let us recompute the examples by a similar adaptive algorithm, but based upon two different Gaussian rules for $d = 2$ and $n = 3$. In the barycentric form, the weights and

$d=2,\ n=3$			Alg. 1	Alg. 2	Q_G^4	Q_G^6
$f(u_1,u_2)$	$\int f$	$\lvert E(f)\rvert$	number of function evaluations			
$(u_1u_2)^3+(\cos(u_1+u_2))^2$	2.86003	0.005	**724**	**411**	**767**	**245**
$\cos(u_1+u_2)\,\sin(u_1+u_2)$	0.09144	0.0002	**1129**	**351**	**434**	**407**
$\cos(u_1+u_2)\,e^{\sin(u_1+u_2)}$	4.35648	0.0005	**832**	**471**	**308**	**254**
$e^{-\left((0.4\,(u_1-0.3))^2+0.2\,(u_2-0.2)^2\right)}$	5.51291	0.0001	**1345**	**651**	**488**	**317**
$e^{-(2\,\lvert u_1-0.2\rvert+0.6\,\lvert u_2+0.4\rvert)}+1$	8.33514	0.001	**1102**	**411**	**1217**	**1541**
$\frac{3}{2}u_1e^{-2\left((u_1-\frac{1}{2})^2+u_2^2\right)}-\frac{3}{2}u_2e^{-\left(u_1^2+u_2^2\right)}+1$	7.74886	0.0002	**3397**	**2211**	**1487**	**1046**

Table 4.1: The number of function evaluations needed to achieve the error $|E(f)|$ is shown for both Newton-Cotes algorithms and for algorithms based on rules Q_G^4 and Q_G^6.

the points of these two rules are (see [18], e.g.)

$$Q_G^4: \qquad \omega_i=\frac{25}{96},\quad i=1,2,3,\qquad \omega_4=-\frac{9}{32},$$
$$\boldsymbol{B}_1=\left(\frac{3}{5},\frac{1}{5},\frac{1}{5}\right)^T,\ \boldsymbol{B}_2=\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)^T,\ \boldsymbol{B}_3=\left(\frac{1}{5},\frac{1}{5},\frac{3}{5}\right)^T,\ \boldsymbol{B}_4=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^T,$$

$$Q_G^6: \qquad \omega_i=\frac{1}{12},\quad i=1,2,\ldots,6,$$
$$\tau_1:=0.10903900907288,\quad \tau_2:=0.23193336855303,\ \tau_3:=1-\tau_1-\tau_2,$$
$$\boldsymbol{B}_1=(\tau_1,\tau_2,\tau_3)^T,\ \boldsymbol{B}_2=(\tau_1,\tau_3,\tau_2)^T,\ \boldsymbol{B}_3=(\tau_2,\tau_1,\tau_3)^T,$$
$$\boldsymbol{B}_4=(\tau_2,\tau_3,\tau_1)^T,\ \boldsymbol{B}_5=(\tau_3,\tau_1,\tau_2)^T,\ \boldsymbol{B}_6=(\tau_3,\tau_2,\tau_1)^T.$$

The number of function evaluations in the adaptive Gaussian algorithms based upon these two rules are shown in Table 4.1.

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Index

Razširjeni povzetek

Teorija aproksimacije je ena izmed glavnih tem na področju numerične analize. Znotraj nje ima posebno mesto polinomska interpolacija, saj nam da aproksimant v zaključeni obliki, kar lahko s pridom izkoristimo v aplikacijah. Morda je v praksi najpomembnejša polinomska interpolacija v več razsežnostih, saj najdemo njeno uporabo tudi pri kubaturnih pravilih (integracijskih pravilih v več dimenzijah), končnih elementih, rekonstrukciji ploskev, optimizaciji ... V nasprotju s polinomsko interpolacijo v eni dimenziji, ki je dobro raziskana, je večdimenzionalna polinomska interpolacija mnogo kompleksnejša in zato še vedno deležna precejšnje pozornosti.

Za lažje nadaljevanje vpeljimo standardno notacijo, ki se uporablja v večdimenzionalnih problemih. Z d bomo označevali dimenzijo prostora, zapis $\boldsymbol{x} := (x_1, x_2, \ldots, x_d)^T$ nam bo predstavljal točko v $\mathbb{R}^d$, s Π^d bomo identificirali prostor vseh polinomov d spremenljivk z realnimi koeficienti, medtem ko nam bo Π_n^d predstavljal podprostor v Π^d, ki ga tvorijo polinomi oblike

$$\sum_{|\boldsymbol{\alpha}|\leq n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Pri tem zapisu je $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)^T$, $\alpha_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, multiindeksni vektor dolžine $|\boldsymbol{\alpha}| := \sum_{i=1}^d \alpha_i$, $c_{\boldsymbol{\alpha}}$ realno število in $\boldsymbol{x}^{\boldsymbol{\alpha}}$ monom oblike $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. Zlahka dokažemo, da je dimenzija prostora Π_n^d enaka $\binom{n+d}{d}$. V nadaljevanju bomo potrebovali še oznaki

$$\boldsymbol{\beta} \leq \boldsymbol{\alpha} \;\Leftrightarrow\; \beta_i \leq \alpha_i, \quad i = 1, 2, \ldots, d, \qquad \boldsymbol{\alpha}! := \alpha_1!\, \alpha_2! \cdots \alpha_d!.$$

Najpomembnejša med interpolacijskimi problemi sta seveda Lagrangeev in Hermiteov interpolacijski problem. V disertaciji se ukvarjam le s prvim, zato si poglejmo, kako formalno definiramo večdimenzionalen Lagrangeev interpolacijski problem.

DEFINICIJA 1. *Za dane N-dimenzionalen interpolacijski podprostor $\mathcal{P} \subset \Pi^d$, množico različnih interpolacijskih točk $\boldsymbol{x}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$, in vrednosti $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, polinom $p \in \mathcal{P}$, za katerega velja*

$$p(\boldsymbol{x}_i) = y_i, \quad i = 1, 2, \ldots, N,$$

imenujemo Lagrangeev interpolacijski polinom za dane interpolacijski prostor, točke in vrednosti.

Običajno so vrednosti y_i podane s funkcijo $f : \mathbb{R}^d \to \mathbb{R}$, zato lahko definicijo zapišemo tudi nekoliko drugače.

DEFINICIJA 2. *Za dane N-dimenzionalen interpolacijski podprostor $\mathcal{P} \subset \Pi^d$, množico različnih interpolacijskih točk $\boldsymbol{x}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$, in funkcijo $f : \mathbb{R}^d \to \mathbb{R}$, polinom $p \in \mathcal{P}$, za katerega velja*

$$p(\boldsymbol{x}_i) = f(\boldsymbol{x}_i), \quad i = 1, 2, \ldots, N,$$

imenujemo Lagrangeev interpolacijski polinom za dane interpolacijski prostor, točke in funkcijo f.

Zanimivi so predvsem interpolacijski problemi, pri katerih je interpolacijski polinom določen enolično.

DEFINICIJA 3. *Naj bo $\mathcal{P} \subset \Pi^d$ N-dimenzionalen interpolacijski podprostor. Lagrangeev interpolacijski problem na množici N interpolacijskih točk $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N \in \mathbb{R}^d$ imenujemo korekten v $\mathcal{P}$, če za poljubne podatke $y_1, y_2, \ldots, y_N \in \mathbb{R}$ obstaja enoličen polinom $p \in \mathcal{P}$, da velja $p(\boldsymbol{x}_i) = y_i$, $i = 1, 2, \ldots, N$.*

Medtem ko v eni dimenziji $n + 1$ točk vedno interpoliramo s polinomi iz prostora Π_n^1, v več dimenzijah ni več povsem jasno, kateri polinomski podprostor izbrati za dano množico interpolacijskih točk. Namreč, dimenzije standardnih podprostorov, kot so Π_n^d, določajo le neko podmnožico v $\mathbb{N}$. Torej, interpolacijski problem ne bo nujno korekten v takšnem prostoru za poljubno množico interpolacijskih točk. Drugače rečeno, število interpolacijskih točk se mora ujemati z dimenzijo interpolacijskega prostora, če želimo upati na enoličnost interpolanta, žal pa to ni vedno tudi zadosti. Velja namreč naslednji izrek:

IZREK 4. *Lagrangeev interpolacijski problem na množici interpolacijskih točk X je korekten v prostoru $\mathcal{P} \subset \Pi^d$ natanko tedaj, ko interpolacijske točke X ne ležijo na kakšni algebraični hiperploskvi, katere polinom, ki jo podaja v implicitni obliki, pripada prostoru $\mathcal{P}$.*

Iz tega lahko izluščimo ugotovitev, da moramo pri študiju Lagrangeevega interpolacijskega problema obravnavati dva aspekta. Pri prvem imamo vnaprej podane interpolacijske točke, poiskati pa moramo interpolacijski prostor, ki porodi korekten interpolacijski problem. V drugem pristopu imamo podan interpolacijski prostor, običajno Π_n^d, iščemo pa množico interpolacijskih točk, ki nam enolično določa interpolacijski polinom v danem prostoru. V disertaciji se ukvarjam le z drugim pristopom, zato si ga oglejmo podrobneje.

Omejili se bomo na, v praksi, najbolj razširjen interpolacijski prostor, to je Π_n^d. Ker algebraična karakterizacija korektnosti, podana v izreku 4, ni uporabna na primer v aritmetiki premične pike, smo primorani iskati množice interpolacijskih točk, ki bodo korektnost interpolacijskega problema v prostoru Π_n^d zagotavljale vnaprej. Pomembnejši raziskovalci, ki se s tem ukvarjajo, so C. de Boor, J. M. Carnicer, K. C. Chung, M. Gasca, J. Maeztu, G. M. Phillips, T. Sauer, Y. Xu, T. H. Yao ...

Najnaravnejši in najpreprostejši način izbire ustreznih interpolacijskih točk so *osnovne mreže* (slika 1.2 (levo)), kjer so interpolacijske točke dobljene kot preseki $d+1$ šopov po $n+1$ vzporednih hiperravnin. Natančneje, vsaka točka je dobljena kot presek $d+1$ hiperravnin, po ene iz vsakega šopa. V baricentričnih koordinatah, glede na oglišča simpleksa, jih lahko zapišemo kot

$$\left\{\frac{1}{n}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} \in \mathbb{N}_0^{d+1},\ |\boldsymbol{\alpha}| = n\right\}.$$

Za te mreže velja, da zadoščajo pogoju geometrijske karakterizacije, zato porodijo korektnost interpolacijskega problema.

DEFINICIJA 5. *Množica interpolacijskih točk $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ zadošča pogoju geometrijske karakterizacije (GC pogoju), če za vsako točko $\boldsymbol{x}_i \in X$ obstajajo hiperravnine $H_{i,j}$, $j = 1, 2, \ldots, n$, tako da $\boldsymbol{x}_i$ ne leži na nobeni od teh hiperravnin, medtem ko vse ostale točke iz X ležijo na vsaj eni od njih. Natančneje,*

$$\boldsymbol{x}_\ell \in \bigcup_{j=1}^{n} H_{i,j} \Leftrightarrow i \neq \ell, \quad i, \ell = 1, 2, \ldots, N.$$

Včasih bomo GC pogoj označili kot GC_n pogoj, da poudarimo število hiperravnin, asociiranih s posamezno točko.

IZREK 6. *Če množica interpolacijskih točk $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$, $N = \binom{n+d}{d}$, zadošča GC_n pogoju, potem X porodi korektno interpolacijo v prostoru Π_n^d.*

OPOMBA 7. *Če množica interpolacijskih točk zadošča GC pogoju, potem so Lagrangeevi bazni polinomi produkti linearnih polinomov. Čeprav je to vedno res v eni dimenziji, v več dimenzijah velja le takrat, ko množica interpolacijskih točk zadošča GC pogoju. Je pa to vsekakor pomembna prednost pri implementaciji.*

Osnovne mreže lahko posplošimo na tako imenovane *posplošene osnovne mreže*:

DEFINICIJA 8. *Posplošena osnovna mreža reda n v $\mathbb{R}^d$ (slika 1.3 (desno)) je množica $\binom{n+d}{d}$ točk*

$$X = \{\boldsymbol{x}_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathbb{N}_0^{d+1},\ |\boldsymbol{\alpha}| = n\},$$

za katere obstaja $d+1$ šopov po $n+1$ hiperravnin $(H_{i,r})_{r=0}^{n}$, $i = 0, 1, \ldots, d$, tako da je vsaka točka $\boldsymbol{x}_{\boldsymbol{\alpha}} \in X$ dobljena kot

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \bigcap_{i=0}^{d} H_{i,\alpha_i}.$$

Pomemben poseben primer teh mrež so *mreže $d+1$ šopov* (slika 1.4 (levo)), kjer se hiperravnine v posameznem šopu sekajo v tako imenovanem *centru*, ki je hiperravnina kodimenzije dva. S temi mrežami se natančneje ukvarjam v nadaljevanju disertacije.

Ko govorimo o interpolaciji, je seveda vedno potrebno obravnavati tudi vprašanje o napaki interpolanta. Obstaja več različnih pristopov, najpomembnejši pa je najbrž pristop preko končnih diferenc, ki sta ga uvedla T. Sauer in Y. Xu v [44].

V disertaciji so podrobno obravnavane mreže $d+1$ šopov, saj je ta tip mrež uporaben v številnih aplikacijah. Sprva se osredotočim na mreže šopov na posameznem simpleksu, kjer izpeljem baricentrične koordinate točk takšne mreže. Podana je tudi formula v zaključeni obliki za Lagrangeev interpolant nad temi mrežami. V nadaljevanju mreže šopov posplošim na globalne mreže na simplicialnih particijah. Pokažem, da na enostavno povezani simplicialni particiji z V vozlišči obstaja globalna mreža $d+1$ šopov, ki ima V prostostnih stopenj. Ta mreža je takšna, da je odsekoma polinomski interpolant nad to mrežo avtomatično zvezen. Analizo mrež na simplicialnih particijah zaključim z obravnavo le-teh na particijah, ki niso enostavno povezane, ter s sorodnim problemom, kako razširiti mrežo čez izbrane luknje v simplicialni particiji. Disertacijo nadaljujem s študijem Newton-Cotesovih kubaturnih pravil nad mrežami $d+1$ šopov na simpleksih. Kubaturna pravila nato prenesem še na simplicialne particije in izpeljem adaptivni algoritem, s katerim pokažem, kako lahko dodatna svoboda, ki jo imamo pri teh mrežah, pripomore k občutnemu zmanjšanju potrebnih izračunov funkcijskih vrednosti.

Mreže $d+1$ šopov

V tem poglavju obravnavam mreže $d+1$ šopov na simpleksu v $\mathbb{R}^d$. Vpeljimo za začetek nekaj definicij, ki jih bomo potrebovali skozi celotno disertacijo. Simpleks v $\mathbb{R}^d$ je konveksna ovojnica $d+1$ točk $\boldsymbol{T}_i$, $i = 0, 1, \ldots, d$. Ker bo zame pomemben tudi vrstni red oglišč simpleksa, mi bo oznaka $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ predstavljala simpleks s predpisanim vrstnim redom oglišč. Pomembno vlogo bo igral d-simpleks z oglišči $\boldsymbol{T}_i = (\delta_{i,j})_{j=0}^{d}$, $i = 0, 1, \ldots, d$, kjer je $\delta_{i,j}$ Kroneckerjev delta, in ga bomo označevali z $\triangle_{d+1}^{d} \subseteq \mathbb{R}^{d+1}$. Mreža $d+1$ šopov reda n sestoji iz $\binom{n+d}{d}$ točk, določenih s preseki $d+1$ šopov po $n+1$ hiperravnin. Vse hiperravnine v posameznih šopih se sekajo v tako imenovanih *centrih* $\boldsymbol{C}_i$, ki so hiperravnine kodimenzije dva. Izkaže se, da je celotna mreža pravzaprav določena z $d+1$ afino neodvisnimi *kontrolnimi točkami*

$$\boldsymbol{P}_i \in \mathbb{R}^d, \quad i = 0, 1, \ldots, d,$$

kjer kontrolna točka $\boldsymbol{P}_i$ leži na premici skozi oglišči $\boldsymbol{T}_i$ in $\boldsymbol{T}_{i+1}$, a izven daljice $\boldsymbol{T}_i\boldsymbol{T}_{i+1}$ (slika 2.2). Kontrolne točke nato enolično določajo vse centre, pri čemer je center $\boldsymbol{C}_i$ določen s kontrolnimi točkami $\boldsymbol{P}_i, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_{i+d-2}$. Velja tudi

$$\{\boldsymbol{P}_{i+1}, \boldsymbol{P}_{i+2}, \ldots, \boldsymbol{P}_{i+d-2}\} \subseteq \boldsymbol{C}_i \cap \boldsymbol{C}_{i+1}.$$

Opazimo (slika 2.1), da so v ravninskem primeru centri $\boldsymbol{C}_i$ kar enaki kontrolnim točkam $\boldsymbol{P}_i$, medtem ko za $d > 2$ temu ni več tako (slika 2.2). Tukaj in tekom celotne disertacije je privzeto, da se indekse oglišč simpleksa, kontrolnih točk, centrov, točk mreže, parametrov mreže ... razume po modulu $d+1$. Kjer je potrebno, to posebej poudarim s funkcijo $m(i) := i \bmod (d+1)$.

Pomembno vlogo bosta v nadaljevanju igrali tudi naslednji preslikavi. Prva je bijektivna vložitev $u : \mathbb{Z}_{d+1}^{r+1} \to \mathbb{N}_0^{r+1}$,

$$u\left((i_j)_{j=0}^{r} \right) := \left(i_j + (d+1) \sum_{k=0}^{j-1} \chi\left(i_k - i_{k+1}\right) \right)_{j=0}^{r},$$

kjer je

$$\chi(s) := \begin{cases} 1, & s > 0, \\ 0, & \text{sicer}, \end{cases}$$

običajna Heavisideova stopnična funkcija. Druga pa je preslikava $w : \mathbb{Z}_{d+1}^{r+1} \to \mathbb{N}$, definirana kot

$$w\left((i_j)_{j=0}^r \right) := \sum_{k=0}^{r-1} \chi\left(i_k - i_{k+1}\right) + \chi\left(i_r - i_0\right).$$

Sliko te preslikave imenujemo *ovojno število* indeksnega vektorja $(i_j)_{j=0}^r$. Obe preslikavi sta prikazani na sliki 2.4. V nadaljevanju bomo zapis skrajšali tudi z oznako

$$[j]_\alpha := \sum_{i=0}^{j-1} \alpha^i = \begin{cases} j, & \alpha = 1, \\ \dfrac{1-\alpha^j}{1-\alpha}, & \text{sicer}, \end{cases} \qquad j \in \mathbb{N}_0.$$

Izpeljimo sedaj baricentrične koordinate točk mreže $d+1$ šopov. Naredimo to najprej za ravninski primer, ki je tudi najpomembnejši v praksi. V tem posebnem primeru lahko baricentrične koordinate, glede na oglišča trikotnika, pišemo kot

$$\boldsymbol{B}_{n-k-j,k,j}, \quad k, j \geq 0, \quad k + j \leq n,$$

z danimi koordinatami oglišč trikotnika

$$\boldsymbol{B}_{n,0,0} = (1,0,0)^T, \quad \boldsymbol{B}_{0,n,0} = (0,1,0)^T, \quad \boldsymbol{B}_{0,0,n} = (0,0,1)^T.$$

Zapišimo baricentrične koordinate centrov (ki se v tem primeru ujemajo s kontrolnimi točkami) v obliki

$$\boldsymbol{C}_0 = \begin{pmatrix} \dfrac{1}{1-\xi_0} \\ -\dfrac{\xi_0}{1-\xi_0} \\ 0 \end{pmatrix}, \quad \boldsymbol{C}_1 = \begin{pmatrix} 0 \\ \dfrac{1}{1-\xi_1} \\ -\dfrac{\xi_1}{1-\xi_1} \end{pmatrix}, \quad \boldsymbol{C}_2 = \begin{pmatrix} -\dfrac{\xi_2}{1-\xi_2} \\ 0 \\ \dfrac{1}{1-\xi_2} \end{pmatrix}, \tag{1}$$

kjer so $\xi_i > 0$, $i = 0, 1, 2$, prosti parametri (slika 2.5). To posebno obliko uporabimo z namenom, da pokrijemo tudi primere vzporednih premic ($\xi_i = 1$). Opomnimo, da območje $0 < \xi_i < 1$ pokrije poltrak od premice v neskončnosti do oglišča $\boldsymbol{T}_i$, medtem ko območje $1 < \xi_i < \infty$ pokrije poltrak od $\boldsymbol{T}_{i+1}$ do premice v neskončnosti (slika 2.6). Prvi korak pri določitvi baricentričnih koordinat $\boldsymbol{B}_{n-k-j,k,j}$ je naslednja lema.

LEMA 9. *Naj bodo $\boldsymbol{B}_{n-k-j,k,j}$, $k, j \geq 0$, $k + j \leq n$, baricentrične koordinate točk mreže treh šopov, določene s centri $\boldsymbol{C}_i$, podanimi v (1). Potem velja*

$$\boldsymbol{B}_{n-k,k,0} = (\tau_k, 1 - \tau_k, 0)^T, \quad k = 0, 1, \ldots, n,$$

pri čemer je

$$\tau_k := \tau_k(\xi_0) := \frac{\alpha^n - \alpha^k}{\alpha^n - \alpha^k + (\alpha^k - 1)\,\xi_0}, \quad k = 0, 1, \ldots, n, \qquad \alpha := \sqrt[n]{\xi_0 \xi_1 \xi_2}.$$

Dokaz temelji na tako imenovani "cik-cak" konstrukciji, reševanju posebne rekurzivne enačbe, ter enačbi z rešitvami, ki so sorazmerne korenom enote.

S pomočjo Pappusovega izreka lahko nato dokažemo naslednji izrek.

IZREK 10. *Naj bodo centri $\boldsymbol{C}_i$ mreže treh šopov določeni s parametri ξ_i, kot v (1), in naj bo $\alpha = \sqrt[n]{\xi_0\xi_1\xi_2}$. Nadalje naj bodo*

$$v_i := \alpha^i, \quad w_i := \sum_{j=0}^{i-1} v_j, \qquad i = 0, 1, \ldots, n.$$

Baricentrične koordinate točk mreže treh šopov so enake

$$\boldsymbol{B}_{n-k-j,k,j} = \begin{pmatrix} \dfrac{v_{k+j}w_{n-k-j}}{v_{k+j}w_{n-k-j} + (v_j w_k + w_j\xi_1)\,\xi_0} \\ \dfrac{v_{n-k}w_k}{v_{n-k}w_k + (v_{n-k-j}w_j + w_{n-k-j}\xi_2)\,\xi_1} \\ \dfrac{v_{n-j}w_j}{v_{n-j}w_j + (v_k w_{n-k-j} + w_k\xi_0)\,\xi_2} \end{pmatrix}.$$

Posplošimo sedaj to analizo na mreže $d+1$ šopov na simpleksu v $\mathbb{R}^d$. Točke mreže so ponovno določene z $d+1$ kontrolnimi točkami. Kontrolne točke lahko v baricentričnih koordinatah sedaj zapišemo kot

$$\begin{aligned} \boldsymbol{P}_i &= \Bigg(\underbrace{0,0,\ldots,0}_{i}, \frac{1}{1-\xi_i}, -\frac{\xi_i}{1-\xi_i}, \underbrace{0,0,\ldots,0}_{d-1-i}\Bigg)^T, \quad i = 0,1,\ldots,d-1, \\ \boldsymbol{P}_d &= \Bigg(-\frac{\xi_d}{1-\xi_d}, \underbrace{0,0,\ldots,0}_{d-1}, \frac{1}{1-\xi_d}\Bigg)^T, \end{aligned}$$

kjer so $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ prosti parametri. Kot vidimo, gre za direktno posplošitev ravninskega primera. V nadaljevanju bomo potrebovali indeksne množice

$$\mathcal{I}_n^d := \left\{ \boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T \in \mathbb{N}_0^{d+1}, \quad |\boldsymbol{\gamma}| = \sum_{i=0}^{d} \gamma_i = n \right\}.$$

Naslednji izrek nam poda baricentrične koordinate točk mreže $d+1$ šopov.

IZREK 11. *Baricentrične koordinate točk mreže $d+1$ šopov reda n na simpleksu $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$, glede na oglišča $\triangle$, so določene z $d+1$ pozitivnimi parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ kot*

$$(\boldsymbol{B}_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_n^d} := (\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}))_{\boldsymbol{\gamma}\in\mathcal{I}_n^d},$$

pri čemer je

$$\boldsymbol{B}_{\boldsymbol{\gamma}} = \frac{1}{D_{\boldsymbol{\gamma},\boldsymbol{\xi}}} \left(\alpha^{n-\gamma_0}\,[\gamma_0]_\alpha\,, \xi_0\,\alpha^{n-\gamma_0-\gamma_1}\,[\gamma_1]_\alpha\,, \xi_0\xi_1\,\alpha^{n-\sum_{i=0}^{2}\gamma_i}\,[\gamma_2]_\alpha\,, \ldots, \xi_0\cdots\xi_{d-1}\,\alpha^0\,[\gamma_d]_\alpha\right)^T \tag{2}$$

in

$$D_{\boldsymbol{\gamma},\boldsymbol{\xi}} = \alpha^{n-\gamma_0} \left[\gamma_0\right]_\alpha + \xi_0 \, \alpha^{n-\gamma_0-\gamma_1} \left[\gamma_1\right]_\alpha + \ldots + \xi_0 \xi_1 \cdots \xi_{d-1} \, \alpha^0 \left[\gamma_d\right]_\alpha , \quad \alpha^n = \prod_{i=0}^{d} \xi_i .$$

Iz dejstva, da mreže $d+1$ šopov zadoščajo GC pogoju, sledi, da so Lagrangeevi bazni polinomi produkti linearnih faktorjev. Naslednji izrek podaja Lagrangeeve bazne polinome v baricentrični obliki.

IZREK 12. *Naj bo mreža $d+1$ šopov reda n na simpleksu $\triangle$ podana v baricentrični obliki s parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$, kot v izreku 11, in naj bodo*

$$f_{\boldsymbol{\gamma}} \in \mathbb{R}, \quad \boldsymbol{\gamma} \in \mathcal{I}_n^d,$$

dani podatki. Polinom $p_n \in \Pi_n^d$, ki interpolira podatke $(f_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma} \in \mathcal{I}_n^d}$ nad točkami $(\boldsymbol{B}_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma} \in \mathcal{I}_n^d}$, je oblike

$$p_n(\boldsymbol{x}; \boldsymbol{\xi}) = \sum_{\boldsymbol{\gamma} \in \mathcal{I}_n^d} f_{\boldsymbol{\gamma}} \, \mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}; \boldsymbol{\xi}), \qquad \boldsymbol{x} \in \mathbb{R}^{d+1}, \ \sum_{i=0}^{d} x_i = 1.$$

Lagrangeevi bazni polinomi $\mathcal{L}_{\boldsymbol{\gamma}}$ so produkti linearnih členov,

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}; \boldsymbol{\xi}) = \prod_{i=0}^{d} \prod_{j=0}^{\gamma_i - 1} h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x}; \boldsymbol{\xi}), \tag{3}$$

pri čemer je

$$h_{i,j,\boldsymbol{\gamma}}\left(\boldsymbol{x}; \boldsymbol{\xi}\right) := \frac{c_{i,\boldsymbol{\gamma}}}{1 - \dfrac{\left[n-\gamma_i\right]_\alpha}{\left[n-j\right]_\alpha}} \left(x_i + \left(1 - \frac{\left[n\right]_\alpha}{\left[n-j\right]_\alpha}\right) q_i\left(\boldsymbol{x}; \boldsymbol{\xi}\right) \right), \tag{4}$$

in

$$q_i\left(\boldsymbol{x}; \boldsymbol{\xi}\right) := \sum_{t=i+1}^{i+d} \frac{1}{\prod\limits_{k=i}^{t-1} \xi_k} \, x_t, \quad c_{i,\boldsymbol{\gamma}} := \left(1 - \frac{\left[n-\gamma_i\right]_\alpha}{\left[n\right]_\alpha}\right) \frac{1}{\left(\boldsymbol{B}_{\boldsymbol{\gamma}}\right)_{i+1}}.$$

Linearni faktorji $h_{i,j,\boldsymbol{\gamma}}$ v (4) so odvisni le od $(i+1)$-ve komponente pripadajoče točke $\boldsymbol{B}_{\boldsymbol{\gamma}}$. Tega dejstva iz klasične predstavitve Lagrangeevih polinomov ni moč razbrati, je pa vsekakor pomembno za učinkovito implementacijo.

Mreže na simplicialnih particijah

V tem poglavju je obravnavana razširitev mreže $d+1$ šopov z enega simpleksa na končno, regularno simplicialno particijo v $\mathbb{R}^d$. Simplicialni particiji rečemo, da je *regularna*, če ima poljuben par sosednjih simpleksov skupno celotno r-lice za nek $r \in \{0, 1, \ldots, d-1\}$ (slika 3.1 (levo)). Zanimajo nas globalne mreže, kjer se točke na skupnih licih simpleksov ujemajo, saj takšne mreže zagotavljajo vsaj zveznost interpolanta nad to mrežo. Razširitev mreže temelji na baricentrični predstavitvi, ki smo jo izpeljali v prejšnjem poglavju.

Obravnavajmo najprej mreže na triangulacijah, torej ravninski primer. To nam bo služilo kot osnova za študij v višjih dimenzijah. Privzemimo, da imamo dva trikotnika in mreži na njima. Poglejmo si, kaj mora veljati, da se točke obeh mrež ujemajo na skupni stranici obeh trikotnikov.

IZREK 13. *Naj bosta* $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$ *in* $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2 \rangle$ *dana trikotnika in naj bosta pripadajoči mreži definirani s parametri* $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2)^T$ *in* $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \xi'_2)^T$. *Baricentrične koordinate točk mrež se ujemajo na stranicah* $\langle \boldsymbol{T}_0, \boldsymbol{T}_1 \rangle$ *in* $\langle \boldsymbol{T}'_0, \boldsymbol{T}'_1 \rangle$ *natanko tedaj, ko velja*

$$\xi_0 \xi'_1 \xi'_2 = \xi'_0 \xi_1 \xi_2,$$

v primeru $n = 2$, *in*

$$\xi_1 \xi_2 = \xi'_1 \xi'_2,\ \xi_0 = \xi'_0,\ (\alpha' = \alpha), \quad \text{ali} \quad \xi_0 \xi'_1 \xi'_2 = 1,\ \xi'_0 \xi_1 \xi_2 = 1,\ (\alpha' \alpha = 1)\,,$$

za $n \geq 3$.

Dokaz izreka temelji na Descartesovem pravilu o številu pozitivnih realnih ničel realnega polinoma. Nekatere primere ujemanja dveh mrež na skupni stranici trikotnikov vidimo na slikah 3.3 in 3.4.

Poenostavimo za nekaj časa notacijo in označimo oglišča trikotnika $\boldsymbol{T}_0$, $\boldsymbol{T}_1$ in $\boldsymbol{T}_2$ z $0, 1$ in 2. Izrek 13 nam da relacije, ki zagotavljajo ujemanje točk na skupni stranici za posebno številčenje oglišč trikotnikov. Ker želimo skonstruirati mrežo na celotni triangulaciji, bi morali podoben rezultat imeti za poljubna številčenja. To rešimo tako, da uporabimo rotacije in zrcaljenja iz simetrične grupe S_3 na ogliščih triotnikov. Zrcaljenja okrog simetral kotov trikotnika nam predstavljajo permutacije (0 1), (0 2) in (1 2), medtem ko nam rotacije predstavljajo permutacije (0 1 2), (0 2 1) in (0)(1)(2). Zanima nas seveda, kako te transformacije vplivajo na parametre ξ_i. Za rotacijo (0 1 2) po zvezah (1) velja

$$\xi_0 \to \xi_1, \quad \xi_1 \to \xi_2, \quad \xi_2 \to \xi_0, \quad \alpha \to \alpha,$$

medtem ko zrcaljenje (0 1) implicira

$$\xi_0 \to \xi_0^{-1}, \quad \xi_1 \to \xi_2^{-1}, \quad \xi_2 \to \xi_1^{-1}, \quad \alpha \to \alpha^{-1}.$$

Ker je grupa S_3 generirana s permutacijama (0 1 2) in (0 1), lahko vse ostale transformacije centrov dobimo s kompozicijami teh dveh.

Mrežo na dani regularni triangulaciji lahko sedaj skonstruiramo na naslednji način. Najprej izberemo poljuben trikotnik in uporabimo izrek 10. Sedaj ponavljamo naslednje korake, dokler nimamo mreže določene na celotni triangulaciji:

- izberi poljuben trikotnik, tako da je trenutna podtriangulacija enostavno povezana;
- uporabi transformacije iz grupe S_3 in izrek 13 ter skonstruiraj mrežo na novem trikotniku.

Mreža na prvem trikotniku je seveda določena s tremi parametri. Izrek 13 pove, da vsak naslednji dodan trikotnik doprinese en prost parameter, razen, če je mreža že določena na dveh stranicah trikotnika, kar se zgodi, ko dopolnimo neko celico, t.j. triangulacijo, ki ima natanko eno notranjo točko.

IZREK 14. *Naj bo $n > 2$. Globalno mrežo treh šopov na regularni, enostavno povezani triangulaciji z V vozlišči lahko skonstruiramo s pomočjo izreka 13 in transformacij iz grupe S_3. Pri tem je mreža določena z V prostimi parametri (glej sliko 3.7).*

Izrek nakazuje, da bi to morda lahko veljalo v splošnem, zato postavimo naslednjo domnevo.

DOMNEVA 15. *Naj bo $\mathcal{T}$ regularna, enostavno povezana simplicialna particija z V vozlišči v $\mathbb{R}^d$. Potem obstaja globalna mreža $d+1$ šopov na $\mathcal{T}$, ki ima V prostostnih stopenj.*

Obravnavajmo sedaj mreže štirih šopov v $\mathbb{R}^3$ ter poskusimo dokazati domnevo 15 za ta primer. Podobno kot v ravninskem primeru, odgovorimo najprej na vprašanje, kaj mora veljati za parametre dveh mrež, da se ujemata na skupni stranici obeh tetraedrov.

IZREK 16. *Naj bosta $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ in $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ dana tetraedra in naj bodo $(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}))_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ in $\left(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')\right)_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ baricentrične koordinate obeh mrež štirih šopov reda n, $n \geq 3$, na $\triangle$ in $\triangle'$. Mreži se ujemata na skupni stranici*

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle, \qquad 0 \leq i_0 < i_1 \leq 3, \quad 0 \leq i'_0 < i'_1 \leq 3,$$

natanko tedaj, ko velja

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j = \prod_{j=i'_0}^{i'_1-1} \xi'_j \quad in \ \ \alpha' = \alpha\right) \ \ ali \ \ \left(\prod_{j=i_0}^{i_1-1} \xi_j = \alpha^n \prod_{j=i'_0}^{i'_1-1} \xi'_j \quad in \ \ \alpha'\alpha = 1\right),$$

kjer sta $\alpha^n = \prod_{j=0}^{3} \xi_j$ in $\alpha'^n = \prod_{j=0}^{3} \xi'_j$.

Tetraedra imata poleg skupne stranice lahko tudi skupen trikotnik (slika 3.9).

POSLEDICA 17. *Naj bosta $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ in $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ dana tetraedra in naj bodo $(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}))_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ in $\left(\boldsymbol{B}_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')\right)_{\boldsymbol{\gamma}\in\mathcal{I}_n^d}$ baricentrične koordinate obeh mrež štirih šopov reda n, $n \geq 3$, na $\triangle$ in $\triangle'$. Naj bo še $\alpha^n = \prod_{j=0}^{3} \xi_j \neq 1$ in*

$$\widetilde{\triangle} := \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \boldsymbol{T}_{i_2} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1}, \boldsymbol{T}'_{i'_2} \rangle,$$

$0 \leq i_0 < i_1 < i_2 \leq 3$, $0 \leq i'_0 < i'_1 < i'_2 \leq 3$, *skupen trikotnik obeh tetraedrov. Potem se mreži ujemata na $\widetilde{\triangle}$ natanko tedaj, ko velja*

$$\prod_{j=i_k}^{i_{k+1}-1} \xi_j = \prod_{j=i'_k}^{i'_{k+1}-1} \xi'_j, \quad k = 0, 1, \quad in \quad \alpha' = \alpha.$$

Sedaj smo si pripravili orodja, da lahko mrežo razširimo z dveh sosednjih tetraedrov na tetraedrsko particijo. Torej, dokazati želimo domnevo 15 za $d = 3$. Dokažimo jo najprej za poseben primer, ko je tetraedrska particija kar celica v $\mathbb{R}^3$.

LEMA 18. *Naj bo $\mathcal{S}$ celica, sestavljena iz $V-1$ tetraedrov. Potem obstaja globalna mreža štirih šopov na $\mathcal{S}$, ki je določena z V prostostnimi stopnjami.*

Lemo najprej dokažemo za minimalno celico v $\mathbb{R}^3$, to je celica, ki sestoji iz štirih tetraedrov (slika 3.11). Pokažemo, da lahko na njej skonstruiramo mrežo s petimi prostostnimi stopnjami. Nato dokažemo, da lahko poljubno celico dobimo iz minimalne celice s posebnim postopkom. Ta temelji na Eulerjevi formuli in dejstvu, da ima vsaka triangulacija na sferi vsaj eno vozlišče stopnje 3, 4 ali 5. Ta postopek zagotovi, da lema velja za poljubne celice. Brez večjih težav lahko sedaj lemo posplošimo na tetraedrske particije.

IZREK 19. *Naj bo $\mathcal{T}$ regularna, enostavno povezana tetraedrska particija z V vozlišči. Potem obstaja globalna mreža štirih šopov na $\mathcal{T}$ in je določena z V prostostnimi stopnjami.*

Dokaza domneve 15 za $d = 2$ in $d = 3$ sta zelo specifična in ju ni možno direktno posplošiti na poljubne dimenzije. Zato bomo za dokončen dokaz domneve 15 potrebovali še nekaj novih orodij in lastnosti mrež šopov. Kot prvo si poglejmo, kaj velja za zožitev mreže $d+1$ šopov na poljubno lice simpleksa (slika 3.15).

IZREK 20. *Naj bo mreža $d+1$ šopov na d-simpleksu $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ v baricentrični predstavitvi določena s parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$, kot v (2). Naj indeksi*

$$\boldsymbol{i} = (i_0, i_1, \ldots, i_r)^T, \ \ 0 \le i_j \le d, \quad kjer\,\text{je} \ \ i_k \ne i_j, \ \ če\,\text{je} \ k \ne j, \quad r \le d, \ \ w(\boldsymbol{i}) = 1,$$

določajo r-lice $\triangle' = \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_r} \rangle \subset \triangle$. Zožitev mreže na $\triangle'$ je mreža $r+1$ šopov na $\triangle'$ z baricentričnimi koordinatami, glede na $\triangle'$, določenimi s parametri $\boldsymbol{\xi}' = (\xi_0', \xi_1', \ldots, \xi_r')^T$, pri čemer je

$$\xi_j' = \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)}, \quad j = 0, 1, \ldots, r,$$

in $\boldsymbol{\ell} = (\ell_j)_{j=0}^{r+1} = u\Big((i_0, i_1, \ldots, i_r, i_0)^T \Big)$; (glej sliko 3.16).

Ta izrek lahko uporabimo seveda tudi za zožitev zgolj na eno stranico oziroma povezavo simpleksa. Mrežo, ki jo dobimo kot zožitev na eno stranico simpleksa, imenujemo mreža dveh šopov. Iz te zožitve lahko ugotovimo, ali je pripadajoči α enak 1 ali različen od 1. V primeru, da je različen, obstajata dva tipa mrež, ki imata isto zožitev na to povezavo. Za en tip velja, da je produkt vseh parametrov, ki določajo mrežo, enak α^n, za drugi tip pa je enak ${\alpha'}^n = 1/\alpha^n$. Naslednji izrek pove, da lahko mreži pripadajoči α enolično določimo šele, če poznamo zožitev mreže na nek cikel povezav oz. stranic simpleksa.

IZREK 21. *Naj bodo baricentrične koordinate mreže $d+1$ šopov reda n na d-simpleksu $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ podane s parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ in naj bo $\prod_{k=0}^{d} \xi_k \neq 1$. Zožitev mreže na cikel povezav*

$$\langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{i_{k+1}} \rangle, \quad k = 0, 1, \ldots, r, \quad i_{r+1} := i_0, \quad \boldsymbol{i} = (i_k)_{k=0}^{r},$$

določa pripadajoči $\alpha = \sqrt[n]{\prod_{k=0}^{d} \xi_k}$ enolično, natanko tedaj, ko velja $w(\boldsymbol{i}) \neq \frac{r+1}{2}$ (glej sliko 3.18).

Očitno je, da poznamo celotno mrežo, če poznamo njeno zožitev na vse stranice simpleksa. Izkaže se, da je dovolj poznati že zožitev na zgolj $d+1$ izbranih stranic. Z $\mathcal{G}(S)$ bomo označevali graf, induciran z vozlišči in povezavami simplicialnega kompleksa S. Spomnimo še, da graf $\mathcal{G}(S_1)$ napenja graf $\mathcal{G}(S)$, če se množici vozlišč obeh grafov ujemata.

IZREK 22. *Mreža $d+1$ šopov na simpleksu $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ s parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ je enolično določena z zožitvami na povezave $e_k = \langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{j_k} \rangle$, $k = 0, 1, \ldots, d$, natanko tedaj, ko graf $g := \mathcal{G}\left(\cup_{k=0}^{d} e_k\right)$ napenja graf $\mathcal{G}(\triangle)$ in velja*

(a) $\prod_{k=0}^{d} \xi_k = 1$ *ali*

(b) *g vsebuje cikel $e_{t_q} = \langle \boldsymbol{T}_{i_{t_q}}, \boldsymbol{T}_{j_{t_q}} \rangle$, $q = 0, 1, \ldots, r$, $i_{t_{q+1}} = j_{t_q}$, $q = 0, 1, \ldots, r-1$, $j_{t_r} = i_{t_0}$, tako da je*
$$w\left(\left(i_{t_q}\right)_{q=0}^{r} \right) \neq \frac{r+1}{2}.$$

Če produkt α^n poznamo vnaprej, potem velja naslednja posledica.

POSLEDICA 23. *Privzemimo, da je produkt $\alpha^n = \prod_{k=0}^{d} \xi_k$ znan vnaprej. Potem je mreža enolično določena z zožitvami na stranice simpleksa $e_k = \langle \boldsymbol{T}_{i_k}, \boldsymbol{T}_{j_k} \rangle$, $k = 1, 2, \ldots, d$, natanko tedaj, ko graf $\mathcal{G}\left(\cup_{k=1}^{d} e_k\right)$ napenja graf $\mathcal{G}(\triangle)$.*

Sedaj nas zanima, kakšne zveze morajo veljati med parametri mrež na dveh sosednjih simpleksih, ki imata neko skupno lice, da se mreži na tem skupnem licu ujemata.

IZREK 24. *Naj bosta $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ in $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \ldots, \boldsymbol{T}'_d \rangle$ dana simpleksa in naj bosta mreži na njima določeni s parametri $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)^T$ in $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \ldots, \xi'_d)^T$. Označimo z*

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \ldots, \boldsymbol{T}_{i_r} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1}, \ldots, \boldsymbol{T}'_{i'_r} \rangle, \quad 1 \leq r \leq d,$$

$0 \leq i_0 < i_1 < \cdots < i_r \leq d$, skupno r-lice obeh simpleksov, kjer velja $\boldsymbol{T}_{i_k} = \boldsymbol{T}'_{i'_k}$. Naj velja še $(\ell_0, \ldots, \ell_{r+1})^T = u\left((i_0, \ldots, i_r, i_0)^T\right)$ in $(\ell'_0, \ldots, \ell'_{r+1})^T = u\left((i'_0, \ldots, i'_r, i'_0)^T\right)$. Če je $\alpha^n = \prod_{i=0}^{d} \xi_i \neq 1$, potem se mreži ujemata na skupnem r-licu natanko tedaj, ko velja ena izmed naslednjih možnosti:

(a) $w(\boldsymbol{i}') = 1$ *in*

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \ldots, r\,; \tag{5}$$

(b) $w(\boldsymbol{i}') = r$ *in*

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \alpha^n \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \ldots, r.$$

Če bi privzeli, da je $\alpha = 1$, potem bi se mreži lahko ujemali tudi v kakšnem drugem primeru, vendar bi s tem izgubili eno prostostno stopnjo, česar pa ne želimo.

Sedaj smo prišli do koraka, ko lahko dokončno dokažemo domnevo 15.

IZREK 25. *Naj bo $\mathcal{T}$ regularna, enostavno povezana simplicialna particija v $\mathbb{R}^d$ z V vozlišči. Potem obstaja globalna mreža $d+1$ šopov na $\mathcal{T}$ z natanko V prostostnimi stopnjami.*

Glavna ideja dokaza je, da najprej globalno oštevilčimo vsa vozlišča simplicialne particije in nato oglišča vsakega posameznega simpleksa oštevilčimo v skladu z globalnim številčenjem (glej sliko 3.22). Posledica takšnega številčenja je, da imata poljubna simpleksa na skupnem licu ovojni števili enaki ena, zato lahko uporabimo relacije (5).

V domnevi 15 in v vseh izrekih, ki so jo dokazovali, smo potihoma privzeli, da je graf $\mathcal{G}(\mathcal{T})$, pri čemer je $\mathcal{T}$ naša simplicialna particija, vsaj 2-povezan. Grafu rečemo, da je ℓ-povezan, če je moč najmanjše množice vozlišč, katere odstranitev iz grafa povzroči, da graf postane nepovezan, vsaj ℓ.

OPOMBA 26. *Če je $\mathcal{T}$ regularna, enostavno povezana simplicialna particija v $\mathbb{R}^d$ z V vozlišči, takšna, da $\mathcal{G}(\mathcal{T})$ ni 2-povezan, potem je mreža na $\mathcal{T}$ določena z $V+m$ prostostnimi stopnjami, pri čemer je m moč množice vozlišč, za katera velja, da odstranitev poljubnega od njih naredi graf $\mathcal{G}(\mathcal{T})$ nepovezan (glej sliko 3.24).*

Simplicialne particije, s katerimi smo se ukvarjali do sedaj, so bile enostavno povezane. Ker imamo v praksi mnogokrat opravka tudi s particijami, ki niso enostavno povezane, je potrebno obravnavati tudi te. Naivno gledano bi seveda lahko povečali takšno particijo do enostavno povezane, skonstruirali mrežo nad to novo particijo in jo zožili na prvotno. Toda takšen pristop povsem zanemari dejstvo, da simplicialna particija ni enostavno povezana. Pokazali bomo, da je moč to dejstvo izkoristiti, saj nam dodatne prostostne stopnje lahko povečajo fleksibilnost mreže.

Da lahko posplošimo izrek 25, potrebujemo naslednjo lemo.

LEMA 27. *Privzemimo, da poznamo produkt $\alpha^n = \prod_{k=0}^{d} \xi_k$, ki ustreza baricentrični predstavitvi mreže $d+1$ šopov s parametri $\boldsymbol{\xi}$ na simpleksu $\triangle \subseteq \mathbb{R}^d$. Naj bodo mreže d šopov z istimi α^n dane na r $(d-1)$-licih $\{f_i\}_{i=1}^r$ simpleksa $\triangle$, $r \in \{2, 3, \ldots, d+1\}$, in naj sovpadajo na skupnih licih. Če je $d \geq 3$ ali ($d = 2$ in $r = 2$), potem obstaja enolično določena mreža $d+1$ šopov na $\triangle$, katere zožitev na $\{f_i\}_{i=1}^r$ sovpada z danimi mrežami d šopov.*

IZREK 28. *Naj bo $\mathcal{T}$ regularna, povezana simplicialna particija v $\mathbb{R}^d$ z V vozlišči in H notranjimi luknjami, homeomorfnimi simplicialni krogli. Poleg tega naj bo $\mathcal{G}(\mathcal{T})$ 2-povezan. Potem obstaja globalna mreža $d+1$ šopov na $\mathcal{T}$, ki ima natanko $V + \delta_{d,2}H$ prostostnih stopenj.*

Privzemimo, da smo mrežo na simplicialni particiji $\mathcal{T}$, ki ni enostavno povezana, že uporabili za interpolacijo. Naknadno se lahko na nekaterih mestih, katerih lokacij vnaprej ne poznamo, topologija particije spremeni. Kot modelni problem lahko vzamemo difuzijski proces nad particijo z veliko luknjami. Tekom procesa se lahko zgodi, da tekočina vdre v določene luknje. Ker seveda želimo obstoječo mrežo ohraniti čim bolj nespremenjeno, še posebej, če je izračun funkcijskih vrednosti zelo drag, interpolirati pa moramo tudi podatke, ki smo jih na novo pridobili nad nekaterimi luknjami, nam to da motivacijo za obravnavo problema, kako razširiti mrežo na simplicialni particiji, ki ni enostavno povezana, čez določene luknje. Seveda pri tem želimo, da nova mreža še vedno porodi zveznost interpolanta nad njo. Torej, da se točke razširjene mreže še vedno ujemajo na vseh skupnih licih simpleksov.

Preden mrežo razširimo čez luknjo, moramo luknjo seveda nekako omejiti (če ni omejena) in jo razdeliti na simplekse, tako da postane simplicialna particija, ki jo bomo označevali s $\mathcal{H}$. V splošnem je problem takšen, da imamo mrežo vnaprej določeno na nekaterih delih roba luknje $\mathcal{H}$, določiti pa moramo preostanek mreže na celotni particiji $\mathcal{H}$. Za nadaljno obravnavo potrebujemo naslednji definiciji:

- *Notranje* $(d-1)$*-lice* simplicialne particije $\mathcal{T}$ v $\mathbb{R}^d$ je $(d-1)$-simpleks, ki je $(d-1)$-lice dveh simpleksov v $\mathcal{T}$. Sicer ga imenujemo *robno* $(d-1)$*-lice* particije $\mathcal{T}$. Množico vseh robnih $(d-1)$-lic particije $\mathcal{T}$ označimo z $\mathcal{B}(\mathcal{T})$.

- *Notranje vozlišče* simplicialne particije $\mathcal{T}$ je vozlišče, za katerega velja, da so vsi $(d-1)$-simpleksi, ki ga vsebujejo kot oglišče, notranja $(d-1)$-lica particije $\mathcal{T}$. V nasprotnem ga imenujemo *robno vozlišče* particije $\mathcal{T}$.

Obravnavajmo najprej ravninski primer. V tem primeru so vse luknje povezane triangulacije homeomorfne simplicialni krogli, zato lahko vse obravnavamo hkrati. Lema 27 nam nakaže, da se mreže ne bo dalo razširiti čez luknje, brez da bi jo na določenih mestih popravili. Bistvo ravninskega primera je zajeto v naslednjem izreku.

IZREK 29. *Naj bo $\mathcal{T}$ regularna, povezana triangulacija v $\mathbb{R}^2$ z notranjo trikotno luknjo $\triangle_{\mathcal{H}}$ in naj bo $\mathcal{L}$ globalna mreža $d+1$ šopov na $\mathcal{T}$. Potem obstaja globalna mreža $d+1$ šopov $\mathcal{L}'$ na $\mathcal{T}' := \mathcal{T} \cup \{\triangle_{\mathcal{H}}\}$, tako da se $\mathcal{L}$ in $\mathcal{L}'$ razlikujeta zgolj na nekem traku trikotnikov (slika 3.28): $\triangle_{\mathcal{H}}, \triangle_2, \triangle_3, \ldots, \triangle_m$, kjer je $\triangle_m$ poljuben robni trikotnik particije $\mathcal{T}'$ (sliki 3.29 in 3.30).*

V višjih dimenzijah postane problem, kako razširiti mrežo čez luknjo, precej bolj zapleten. Za dokaz naslednjih izrekov potrebujemo tudi nekaj orodij iz topologije.

IZREK 30. *Naj bo $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 3$, simplicialna particija, homeomorfna simplicialni krogli, z V_I notranjimi vozlišči. Nadalje naj bodo mreže d šopov dane na celotnem $\mathcal{B}(\mathcal{T})$ in naj se ujemajo na skupnih licih. Naj bo produkt parametrov lokalnih mrež enak isti konstanti α^n za vse simplekse v $\mathcal{B}(\mathcal{T})$. Potem obstaja globalna mreža na $\mathcal{T}$, katere zožitev sovpada z danimi mrežami na $\mathcal{B}(\mathcal{T})$ in je določena z V_I prostostnimi stopnjami.*

V višjih dimenzijah velja, da lahko ima $\mathcal{H}$ mrežo določeno le na nekaterih robnih licih. Primer particije s takšno luknjo je simplicialni torus. Za obravnavo teh tipov lukenj, potrebujemo naslednjo lemo.

LEMA 31. *Naj bo $\mathcal{T}$ triangulacija v $\mathbb{R}^2$, homeomorfna simplicialni krogli, in naj bo $\mathcal{L}$ mreža na $\mathcal{T}$. Nadalje naj ima $\mathcal{T}$ natanko tri robne povezave $\{e_i\}_{i=1}^3$ in naj bo $\triangle_B$ trikotnik s stranicami $\{e_i\}_{i=1}^3$. Naj bo produkt parametrov lokalnih mrež enak isti konstanti α^n za vse simplekse v $\mathcal{T}$. Potem obstaja mreža treh šopov $\mathcal{L}_B$ na $\triangle_B$, ki sovpada z $\mathcal{L}$ na $\{e_i\}_{i=1}^3$ (glej sliko 3.33).*

S pomočjo te leme lahko dokažemo naslednji izrek.

IZREK 32. *Naj bo $\mathcal{T} \subseteq \mathbb{R}^d$, $d \geq 3$, simplicialna particija, homeomorfna simplicialni krogli. Nadalje naj bodo mreže d šopov dane na $\mathcal{T}' \subset \mathcal{B}(\mathcal{T})$, tako da se ujemajo na vseh skupnih licih (glej sliko 3.36). Naj bo produkt parametrov lokalnih mrež enak isti konstanti α^n za vse simplekse v $\mathcal{B}(\mathcal{T})$. Potem obstaja mreža na $\mathcal{B}(\mathcal{T})$, ki se ujema z danimi mrežami na $\mathcal{T}'$. Za razširitev mreže imamo na voljo V_B prostih parametrov, kjer je V_B število robnih vozlišč, ki niso vozlišča particije $\mathcal{T}'$.*

Sedaj lahko dokončno odgovorimo na vprašanje, kako razširiti mrežo čez luknjo $\mathcal{H}$ simplicialne particije $\mathcal{T}$. Najprej je potrebno luknjo razdeliti na simplekse in jo po potrebi omejiti, da postane omejena simplicialna particija. Nato moramo particijo $\mathcal{H}$ razširiti, da postane homeomorfna simplicialni krogli. Sedaj lahko rečemo, da imamo mrežo dano na nekaterih robnih licih particije $\mathcal{H}$. Če je $d \geq 3$, lahko to mrežo razširimo na celotno $\mathcal{H}$ po izrekih 30 in 32. Za $d = 2$ pa moramo po izreku 28 nekoliko popraviti obstoječo mrežo na $\mathcal{T} \backslash \mathcal{H}$.

Newton-Cotesova kubaturna pravila

V tem poglavju obravnavam Newton-Cotesova kubaturna pravila nad mrežami $d+1$ šopov na simpleksu in na simplicialni particiji. Kubaturna pravila na simpleksu $\triangle \subset \mathbb{R}^d$, oblike

$$\sum_{\boldsymbol{\gamma}} \omega_{\boldsymbol{\gamma}} f(\boldsymbol{X}_{\boldsymbol{\gamma}}), \quad \boldsymbol{X}_{\boldsymbol{\gamma}} \in \triangle,$$

kjer so $f(\boldsymbol{X}_{\boldsymbol{\gamma}})$ funkcijske vrednosti nad točkami $\boldsymbol{X}_{\boldsymbol{\gamma}}$, $\omega_{\boldsymbol{\gamma}}$ uteži, $\boldsymbol{\gamma}$ pa multiindeksi, so najobičajnejši način, kako aproksimirati večdimenzionalen integral nad kompaktnim območjem (razdeljenim na simplekse) v $\mathbb{R}^d$.

Newton-Cotesova kubaturna pravila so v disertaciji posplošena z osnovnih mrež na mreže $d+1$ šopov. Posplošitev temelji na preprosti obliki Lagrangeevih baznih polinomov v baricentrični obliki (glej izrek 12). Označimo s $S_\triangle(f)$ integral skalarnega polja $f : \triangle \to \mathbb{R}$ nad simpleksom $\triangle$ in z

$$Q^{(n)}(f; \boldsymbol{\xi}) := \sum_{\boldsymbol{\gamma} \in \mathcal{I}_n^d} \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) f_{\boldsymbol{\gamma}}, \tag{6}$$

kubaturno pravilo stopnje n v baricentrični obliki nad standardnim simpleksom $\triangle_{d+1}^d \subseteq \mathbb{R}^{d+1}$. Ker so Newton-Cotesova kubaturna pravila interpolacijska pravila, lahko uteži $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ izračunamo kot

$$\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = S_{\triangle_{d+1}^d}(\mathcal{L}_{\boldsymbol{\gamma}}), \quad \boldsymbol{\gamma} \in \mathcal{I}_n^d.$$

Pri nadaljnji izpeljavi je primerneje pisati hiperravnine $h_{i,j,\boldsymbol{\gamma}}$ kot

$$h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}) := \frac{h_{i,j}(\boldsymbol{x};\boldsymbol{\xi})}{h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi})},$$

kjer je

$$h_{i,j}(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{t=i}^{i+d} a_{t,j}(\boldsymbol{\xi})\, x_t, \quad a_{t,j}(\boldsymbol{\xi}) = \begin{cases} [n-j]_\alpha, & t = i, \\ \left([n-j]_\alpha - [n]_\alpha\right)\left(\prod_{k=i}^{t-1} \xi_k\right)^{-1}, & t > i. \end{cases}$$

Pri tem indeksov pri $a_{t,j}(\boldsymbol{\xi})$ ne računamo po modulu $d+1$. Sedaj lahko (3) zapišemo kot

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x};\boldsymbol{\xi}) = \prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1} \frac{h_{i,j}(\boldsymbol{x};\boldsymbol{\xi})}{h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi})}, \qquad \boldsymbol{x} := (x_i)_{i=0}^d \in \mathbb{R}^{d+1}, \ \sum_{i=0}^d x_i = 1.$$

Uvedimo še indeksne množice $\Lambda_{\boldsymbol{\gamma}}^i$, $i = 0, 1, \ldots, d$, in $\Lambda_{\boldsymbol{\gamma}}$ (slika 4.1). Če je $\gamma_i \neq 0$, potem naj velja

$$\Lambda_{\boldsymbol{\gamma}}^i := \left\{ \boldsymbol{\lambda}^i := \left(\lambda_0^i, \lambda_1^i, \ldots, \lambda_{\gamma_i-1}^i\right)^T, \ \lambda_0^i = i, \ \lambda_j^i \in \{i, i+1, \ldots, i+d\}, \ 0 < j \leq \gamma_i - 1 \right\},$$

sicer naj bo $\Lambda_{\boldsymbol{\gamma}}^i := \emptyset$. Nadalje naj velja

$$\Lambda_{\boldsymbol{\gamma}} := \left\{ \boldsymbol{\lambda} := \left(\lambda_0^0, \lambda_1^0, \ldots, \lambda_{\gamma_0-1}^0, \ldots, \lambda_0^d, \lambda_1^d, \ldots, \lambda_{\gamma_d-1}^d\right)^T \in \mathbb{N}_0^n, \quad \left(\lambda_0^i, \ldots, \lambda_{\gamma_i-1}^i\right)^T \in \Lambda_{\boldsymbol{\gamma}}^i \right\}.$$

Naslednji izrek nam da formulo za uteži kubaturnega pravila nad mrežami $d+1$ šopov v zaključeni obliki.

IZREK 33. *Uteži $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ kubaturnega pravila* (6) *so enake*

$$\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = K(\boldsymbol{\xi}) \cdot \sum_{\boldsymbol{\lambda} \in \Lambda_{\boldsymbol{\gamma}}} \left(\prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1} a_{\lambda_j^i, j}(\boldsymbol{\xi}) \right) \frac{k_{\boldsymbol{\lambda}}!}{(n+d)!}, \tag{7}$$

pri čemer je

$$K(\boldsymbol{\xi}) := \left(\prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1} h_{i,j}(\boldsymbol{B}_{\boldsymbol{\gamma}};\boldsymbol{\xi}) \right)^{-1}, \quad k_{\boldsymbol{\lambda}} := (k_{\boldsymbol{\lambda},0}, k_{\boldsymbol{\lambda},1}, \ldots, k_{\boldsymbol{\lambda},d})^T$$

in nam $k_{\boldsymbol{\lambda},i}$ označuje frekvenco indeksa i v vektorju $\boldsymbol{\lambda}$.

Vzemimo sedaj simpleks $\triangle = \langle \boldsymbol{V}_0, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_d \rangle \subset \mathbb{R}^d$ in naj bodo $\boldsymbol{X}_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \mathcal{I}_n^d$, kartezične koordinate točk mreže.

POSLEDICA 34. *Newton-Cotesovo kubaturno pravilo reda n nad simpleksom $\triangle \subset \mathbb{R}^d$ je oblike*

$$Q_{\triangle}^{(n)}(f;\boldsymbol{\xi}) := \sum_{\boldsymbol{\gamma}\in\mathcal{I}_n^d} \omega_{\boldsymbol{\gamma},\triangle}(\boldsymbol{\xi})\, f(\boldsymbol{X}_{\boldsymbol{\gamma}}) = d!\,\mathrm{vol}\,(\triangle) \sum_{\boldsymbol{\gamma}\in\mathcal{I}_n^d} \omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})\, f(\boldsymbol{X}_{\boldsymbol{\gamma}}),$$

kjer so $\omega_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ uteži dane v (7) *in* $\mathrm{vol}\,(\triangle)$ *označuje prostornino simpleksa $\triangle$.*

Razširimo sedaj kubaturna pravila nad globalne mreže $d+1$ šopov na simplicialnih particijah. Ker za dovolj majhne stopnje večina točk mreže leži na licih simpleksov, nam dobljene globalne mreže iz prejšnjega poglavja omogočajo, da izračunamo funkcijske vrednosti nad temi točkami le enkrat in tako dobimo učinkovita kubaturna pravila nad simplicialnimi particijami. Pravila Newton-Cotesovega tipa dobijo pravo vrednost šele, ko jih uporabimo v adaptivnem smislu. Pomembno vlogo pri globalnih adaptivnih pravilih igra način zgostitve mreže, zato si poglejmo, kako lahko zgostimo mrežo na nekem izbranem simpleksu v našem primeru (glej sliko 4.2).

- Izberi simpleks $\triangle \in \mathcal{T}$, kjer je potrebna zgostitev lokalne mreže.
- Dodaj novo vozlišče $\boldsymbol{T}$ v notranjost simpleksa $\triangle$, da simplicialna particija ostane regularna.
- Dodaj $d+1$ povezav od točke $\boldsymbol{T}$ do oglišč simpleksa $\triangle$. Te povezave razdelijo simpleks $\triangle$ na $d+1$ novih simpleksov.
- Skonstruiraj nove lokalne mreže na teh simpleksih tako, da se za poljubna sosednja simpleksa točke mrež ujemajo na skupnem licu.

Naslednji izrek natačneje opiše zadnji korak algoritma.

IZREK 35. *Naj bo mreža $d+1$ šopov dana na simpleksu $\triangle$ in naj bo $\boldsymbol{T}$ točka v notranjosti $\triangle$, ki razdeli $\triangle$ na $d+1$ novih simpleksov $\{\triangle_i\}_{i=1}^{d+1}$. Potem obstajajo mreže $d+1$ šopov na $\{\triangle_i\}_{i=1}^{d+1}$, ki sovpadajo na skupnih licih simpleksov $\{\triangle_i\}_{i=1}^{d+1}$ in se ujemajo z začetno mrežo na $\mathcal{B}(\triangle)$. Za konstrukcijo teh mrež imamo na voljo en prost parameter (glej sliko 4.3).*

Sedaj lahko sestavimo globalen adaptiven algoritem, kjer vsak nivo sestoji iz naslednjih korakov:

- Iz trenutne simplicialne particije $\mathcal{T}'$ (na začetku je $\mathcal{T}' = \mathcal{T}$) izberi simplekse, kjer kubaturno pravilo ne da zadovoljive aproksimacije.
- Razdeli izbrane simplekse na $d+1$ novih simpleksov in določi mreže na teh simpleksih.
- Nadgradi simplicialno particijo $\mathcal{T}'$ z novimi simpleksi, uporabi lokalno kubaturno pravilo na vsakem novem simpleksu, pri čemer se izogni nepotrebnim večkratnim izračunom funkcijskih vrednosti nad istimi točkami, ter nadgradi globalno aproksimacijo za integral.

V disertaciji se posebej posvetim drugemu koraku tega algoritma ter primerjam dva postopka. Pri prvem imamo ves čas opravka le z osnovnimi mrežami, novo točko pri delitvi simpleksa na $d+1$ novih simpleksov pa vedno izberemo kot središče simpleksa. Pri drugem postopku nam poseben algoritem, na osnovi iskanja minimuma funkcije več spremenljivk, določi, kje naj leži nova točka in kakšen naj bo prost parameter iz izreka 35 pri zgostitvi mreže na določenem simpleksu. Na številnih primerih pokažem, da lahko z uporabo drugega postopka precej zmanjšamo število potrebnih izračunov funkcijskih vrednosti, kar je v praksi lahko zelo pomembno (glej tabelo 4.1 in sliko 4.6).

Izjava

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Vito Vitrih