Pythagorean-hodograph Cycloidal curves

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Abstract. In the paper, Pythagorean-hodograph cycloidal curves as an extension of PH cubics are introduced. Their properties are examined and a constructive geometric characterization is established. Further, PHC curves are applied in the Hermite interpolation, with closed form solutions been determined. The asymptotic approximation order analysis carried out indicates clearly which interpolatory curve solution should be selected in practice. This makes the curves introduced here a useful practical tool, in particular in algorithms that guide CNC machines.

Keywords. Pythagorean hodograph, C-curves, trigonometric functions, characterization, Hermite interpolation, asymptotic approximation order.

2010 Mathematics Subject Classification. 65D05, 65D17, 65D10.

1 Introduction

Polynomial Pythagorean-hodograph (PH) curves play significant role in the theory as well as in practical applications of polynomial curves. They are characterized by the property that the Euclidean norm of their hodograph, called the parametric speed, is also a polynomial. As a consequence, they have rational tangent, rational offsets, polynomial arc-length, etc. These properties are of a particular importance in developing algorithms that guide CNC (computerized numerical control) machines. As an example, suppose that a part of the machine should move at a given constant distance from the manufactured shape. A rational offset reduces this task to elementary numerical operations in algorithms. CNC machines in general are capable of linear and circular motions, and the programming is usually done in either G-code or M-code describing such motions [11, e.g.]. But the instruction set is usually extended by additional elementary operations such as sine or cosine evaluation (e.g., G107, G108).

This property makes the use of non-polynomial parametric curves attractive from the practical standpoint as well as from the more general mathematical one. In particular, it may be required that parametric curve class selected should reproduce exactly geometric objects that appear often in practice, such as segments, circular arcs, cycloids, helices, etc. Thus it makes sense to extend the PH property
from polynomial parametric curves to the more general ones. In the most general setting, we may consider a parametric curve, based upon a blending system of functions \((\phi_i)_{i=0}^n\) on \([\alpha, \beta]\):

\[
p : [\alpha, \beta] \to \mathbb{R}^d : t \mapsto \sum_{i=0}^n b_i \phi_i(t), \quad b_i \in \mathbb{R}^d.
\]

We assume that \((\phi_i)_{i=0}^n\) is an Extended Complete Chebyshev (ECC) system and that \(\phi_0 = 1, \phi_1 = t\) in order to reproduce linear segments exactly. In addition we assume that any

\[
f \in S := \text{span}\{ (\phi_i)_{i=0}^n \}
\]

could be efficiently evaluated. The curve \(p\) is a \emph{Pythagorean-hodograph} curve if its parametric speed \(\sigma\) is in the first reduced ECC-system:

\[
\sigma \in S_R := \text{span}\{ (\dot{\phi}_i)_{i=1}^n \} \subset S, \quad \sigma^2 = \dot{p}^T \dot{p}, \quad \text{on } [\alpha, \beta].
\]  

Here, and throughout the paper, \(x^T y\) denotes the scalar product of \(x, y \in \mathbb{R}^d\), \(\|x\| := \sqrt{x^T x}\) the Euclidean norm of \(x\), and the dot the derivation with respect to \(t\). If \(p\) is regular, (1) simplifies to \(\sigma = \sqrt{\dot{p}^T \dot{p}}\), and an evaluation of the parametric speed or the offset is not harder than computing a point at the original curve \(f\).

There has been a respectful amount of work devoted to the polynomial PH curves since their first appearance in [5]. For more information see e.g. [2], [3], [4], [10], and references therein. In this paper we show that the analysis could be extended to a practically important non-polynomial case, capable of reproducing line and circular segments exactly too.

The outline of the paper is as follows. In the second section, Pythagorean-hodograph C-curves (PHC curves in short) are introduced and their properties studied. In particular, Theorem 2.2 gives a geometric characterization of such curves, heavily exploited in the subsequent sections, and Theorem 2.5 proves that PHC curves distinct from a line segment are always regular and, in the planar case, convex too. In Section 3, Hermite interpolation with PHC curves is outlined. The analysis shows that there may be one, two or no solutions depending on the data supplied, similar to the PH cubic polynomial case. In section 4, the asymptotic error analysis of PHC Hermite problem solutions is carried out. The asymptotic approximation order turns out at most four as expected. The last section adds some numerical examples.
2 Pythagorean-hodograph C-curves

Let us consider the PH condition (1) applied to $n$-cycloidal curves (or $C_n$-curves) based upon the ECC system

$$1, t, \ldots, t^n, \cos t, \sin t, \quad t \in [0, \alpha], \quad \alpha \in (0, 2\pi), \quad n \in \mathbb{N}.$$  

We will call such curves Pythagorean-hodograph $C_n$-curves or shortly PHC$_n$ curves. The number of parameters that determine a PHC$_n$ curve is given by the following lemma.

**Lemma 2.1.** A PHC$_n$ curve in $\mathbb{R}^d$ has $(n + 3)d - 3n + 1$ degrees of freedom.

**Proof.** Let $t \in [0, \alpha], \quad \alpha \in (0, 2\pi), \quad n, k \in \mathbb{N}$, and

$$P_{n,k} := \text{span}\{1, t, \ldots, t^n, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt\},$$

$$I_n := \text{span}\{t \cos t, t \sin t, t^2 \cos t, t^2 \sin t, \ldots, t^n \cos t, t^n \sin t\}.$$  

Quite clearly, if $p$ is a $C_n$ curve, each component of $\dot{p}$ lies in $P_{n-1,1}$. Also,

$$\dot{p}^T \dot{p} \in P_{2n-2,2} \cup I_{n-1}.$$  

Since it is required $\sigma \in P_{n-1,1}$, and

$$\dim P_{n-1,1} = n + 2, \quad \dim(P_{2n-2,2} \cup I_{n-1}) = 4n + 1,$$

the PH condition (1) implies exactly $3n - 1$ equations. 

In this paper we will consider the first important case, namely PHC$_1$ curves that will simply be called Pythagorean-hodograph C-curves or shortly PHC curves. Note that C-curves have been studied quite thoroughly in [13] and [14]. It would be nice to have the possibility of reproducing a periodic curve exactly with respect to its natural parametric domain, i.e. $[0, 2\pi]$, without necessity to use splines. However, the above simple case cannot be extended to larger intervals, even from $(0, 2\pi)$ to $[0, 2\pi]$. The lowest order Chebyshev space on $[0, 2\pi]$ with this property is of order 6, see [1], making the subsequent analysis much more involved.

Let us recall that a C-curve can be expressed as

$$p = \sum_{i=0}^{3} b_i Z_i, \quad \text{on } [0, \alpha], \quad \alpha \in (0, 2\pi),$$  

(2)
where \( \mathbf{b}_i \in \mathbb{R}^d, d \geq 2 \), are the C-Bézier control points and \( Z_i := Z_{i,\alpha} \) are the C-Bézier basis functions ([13]) of

\[
S := S_\alpha := \text{span}\{1, t, \cos t, \sin t\}, \quad t \in [0, \alpha], \quad \alpha \in (0, 2\pi),
\]

defined as

\[
Z_{3,\alpha}(t) := \frac{t - \sin t}{\alpha - \sin \alpha}, \quad Z_{0,\alpha}(t) := Z_{3,\alpha}(\alpha - t),
\]

\[
Z_{2,\alpha}(t) := \frac{\sin \alpha}{\alpha - 2\nu(\alpha)} \left( \frac{1 - \cos t}{1 - \cos \alpha} - \frac{t - \sin t}{\alpha - \sin \alpha} \right), \quad Z_{1,\alpha}(t) := Z_{2,\alpha}(\alpha - t),
\]

with

\[
\nu(\alpha) := \frac{\alpha - \sin \alpha}{1 - \cos \alpha}.
\]

Note that functions \( Z_i \) form the partition of unity and that \( \nu(\alpha) > 0 \) for \( \alpha \in (0, 2\pi) \). The basis \( (Z_i)_{i=0}^3 \) is well defined also for \( \alpha = \pi \), in particular

\[
Z_{2,\pi}(t) = \lim_{\alpha \to \pi} Z_{2,\alpha}(t) = \frac{1}{2}(1 - \cos t) - \frac{1}{\pi}(t - \sin t).
\]

Let \( \Delta \mathbf{b}_i := \mathbf{b}_{i+1} - \mathbf{b}_i \) denote a forward difference of control points. From (2) and the partition of unity we observe that

\[
p = \sum_{i=0}^3 \mathbf{b}_i Z_i = \mathbf{b}_0 + \sum_{i=0}^2 \Delta \mathbf{b}_i \sum_{j=i+1}^3 Z_j,
\]

so the hodograph of \( p \) can be expressed as

\[
\dot{p} = \Delta \mathbf{b}_0 w_0 + \Delta \mathbf{b}_1 w_1 + \Delta \mathbf{b}_2 w_2,
\]

where

\[
w_i(t) := \sum_{j=i+1}^3 \dot{Z}_j(t), \quad i = 0, 1, 2,
\]

and further in the closed form

\[
w_2(t) = \frac{1 - \cos t}{\alpha - \sin \alpha}, \quad w_0(t) = w_2(\alpha - t),
\]

\[
w_1(t) = \frac{1 - \cos(t - \alpha) + \cos \alpha - \cos t}{\alpha - 2 \sin \alpha + \alpha \cos \alpha}.
\]
The functions $(w_i)^2_{i=0}$ are clearly a basis of $S_R$ with $S$ introduced in (3). Again, the basis is well defined for $\alpha = \pi$ too. Let us denote

$$w := \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}, \quad G := \begin{pmatrix} \|\Delta b_0\|^2 & \Delta b_0^T \Delta b_1 & \Delta b_0^T \Delta b_2 \\ \Delta b_1^T \Delta b_1 & \|\Delta b_1\|^2 & \Delta b_1^T \Delta b_2 \\ \Delta b_2^T \Delta b_2 & \Delta b_1^T \Delta b_2 & \|\Delta b_2\|^2 \end{pmatrix}. \quad (6)$$

If we introduce angles

$$\varphi_{ij} := \angle(\Delta b_i, \Delta b_j), \quad \varphi_{ij} \in [0, \pi], \quad (7)$$

we could express the elements of $G$ also as

$$\Delta b_i^T \Delta b_j = \|\Delta b_i\| \|\Delta b_j\| \cos \varphi_{ij}.$$ 

Note that the matrix introduced in (6) is the Gram matrix of differences of control points, thus symmetric and positive semidefinite. If $(\Delta b_i)^2_{i=0}$ are linearly independent, it is actually positive definite, and in this case $p$ is necessarily regular. Indeed, the norm $\|\dot{p}(t)\|$ could by (5) vanish only if $w(t^*) = 0$ at some $t^*$. Since the relation $w_2(t^*) = 0$ implies $t^* = 0$, and since $w_0(0) = w_2(\alpha) \neq 0$, this is clearly not possible. In this notation, the parametric speed $\sigma$ of a C-curve satisfies

$$\sigma^2(t) = w(t)^T G w(t), \quad t \in [0, \alpha]. \quad (8)$$

For cubic polynomial curves there exist simple necessary and sufficient conditions on Bézier control polygon for a curve to be a PH curve [3, 5, e.g.]. Based upon the form (8), the next theorem provides similar conditions for PHC curves.

**Theorem 2.2.** Suppose that a C-curve, distinct from a line segment, is given in the C-Bézier form (2). If all control points are pairwise distinct, i.e., $b_{i+1} \neq b_i, i = 0, 1, 2$, the necessary and sufficient conditions for $p$ to be a PHC curve are

$$\varphi_{01} = \varphi_{12} \quad \text{and} \quad \frac{\|\Delta b_0\| \|\Delta b_2\|}{\|\Delta b_1\|^2} = 2 \rho(\alpha) \left( \frac{1 - \cos^2 \varphi_{01}}{1 - \cos \varphi_{02}} \right), \quad (9)$$

where

$$\rho(\alpha) := \left( \frac{\alpha - \sin \alpha}{2 \alpha \cos \frac{\alpha}{2} - 4 \sin \frac{\alpha}{2}} \right)^2,$$

and the angles $\varphi_{ij}$ are introduced in (7). For such a PHC curve, the parametric speed $\sigma$ is given as

$$\sigma = \sum_{i=0}^2 \sigma_i w_i = \|\Delta b_0\| w_0 + \cos \varphi_{01} \|\Delta b_1\| w_1 + \|\Delta b_2\| w_2. \quad (10)$$

If at least one of the control point differences $\Delta b_i, i = 0, 1, 2$, vanishes, a PHC curve $p$ reduces to a line segment.
Proof. If \( p \) is a PHC curve with parametric speed \( \sigma(t) \in S_R, t \in [0, \alpha] \), the PH condition (1) could be by (8) written as

\[
\chi(t) := w^T Gw(t) - (\xi_0 w_0(t) + \xi_1 w_1(t) + \xi_2 w_2(t))^2 = 0, \quad t \in [0, \alpha],
\]

for some scalars \( \xi_i \) yet to be determined. The equality (11) should hold identically on \([0, \alpha]\), and an evaluation of \( \chi \) at \( t \in \{0, \alpha\} \),

\[
\chi(0) = \frac{1}{\nu(\alpha)^2} \left( \|\Delta b_0\|^2 - \xi_0^2 \right), \quad \chi(\alpha) = \frac{1}{\nu(\alpha)^2} \left( \|\Delta b_2\|^2 - \xi_2^2 \right),
\]

determines \( |\xi_0| \) and \( |\xi_2| \).

\[
|\xi_0| = \|\Delta b_0\|, \quad |\xi_2| = \|\Delta b_2\|. \tag{12}
\]

Further, derivatives of \( \chi \) evaluated at the same points imply relations

\[
\xi_1 \xi_0 - \Delta b_0^T \Delta b_1 = 0 \implies \xi_1 = \text{sign}(\xi_0) \cos \varphi_{01} \|\Delta b_1\|, \tag{13}
\]

and

\[
\xi_1 \xi_2 - \Delta b_1^T \Delta b_2 = 0 \implies \xi_1 = \text{sign}(\xi_2) \cos \varphi_{12} \|\Delta b_1\|. \tag{14}
\]

The expressions (12), (13), and (14) now simplify \( \chi \) to

\[
\chi(t) = \frac{(1 - \cos t + \cos \alpha - \cos(t - \alpha))^2}{(\alpha - \sin \alpha)^2(1 + \cos \alpha)} \cdot 2\rho(\alpha) \left(1 - \cos^2 \varphi_{01}\right) \|\Delta b_1\|^2 - \left(\text{sign}(\xi_0\xi_2) - \cos \varphi_{02}\right) \|\Delta b_0\| \|\Delta b_2\| \right).
\]

This expression vanishes for all \( t \) iff \( \text{sign}(\xi_0\xi_2) = 1 \) and the second equation in (9) holds. But we may choose \( \xi_0 \) and \( \xi_2 \) both as positive. Since \( \varphi_{ij} \in [0, \pi] \), (13) and (14) prove the first assertion in (9) and (10) too. The proof of the last assertion follows simply by rereading the part of the proof already completed, with an additional assumption. As an example, suppose that \( \Delta b_0 = 0 \). Then (12) implies \( \xi_0 = 0 \), which together with (14) simplifies \( \chi \) to

\[
\chi(t) = 2\rho(\alpha) \frac{(1 - \cos t + \cos \alpha - \cos(t - \alpha))^2}{(\alpha - \sin \alpha)^2(1 + \cos \alpha)} \left(1 - \cos^2 \varphi_{12}\right) \|\Delta b_1\|^2.
\]

But then \( \varphi_{12} \in \{0, \pi\} \), and \( p \) is a line segment. The other cases follow similarly, and the proof is completed. \( \square \)

The following corollary connects the C-curve case and the polynomial one, studied already in [6, 7, 9, e.g.].
Corollary 2.3. If $\alpha \to 0$, the PHC curve $p = \sum_{i=0}^{3} b_{i,\alpha} Z_{i,\alpha}$ reduces to a cubic PH polynomial curve $p^0$, determined as a limit

$$p^0(u) := \lim_{\alpha \to 0} p(\alpha u), \quad u \in [0, 1].$$

Proof. Let us reparameterize the curve $p$ to a fixed parameter interval $[0, 1]$ by $t \to \alpha u$. The basis functions $Z_{i,\alpha}$ in the limit turn out as cubic Bernstein polynomials,

$$\lim_{\alpha \to 0} Z_{i,\alpha}(\alpha u) = B_i(u) := \binom{3}{i} u^i (1 - u)^{3-i}, \quad i = 0, 1, 2, 3,$$

and $p^0$ is a cubic polynomial curve. Since $\lim_{\alpha \to 0} \rho(\alpha) = 1$, the conditions (9) reduce to conditions for PH polynomial curves ([9]).

For planar PHC curves the conditions (9) can even be simplified.

Corollary 2.4. The necessary and sufficient conditions for a planar C-curve, distinct from a line segment, which has pairwise distinct control points, to be a PHC curve are

$$\varphi_{01} = \varphi_{12}, \quad \frac{||\Delta b_0|| ||\Delta b_2||}{||\Delta b_1||^2} = \rho(\alpha), \quad (16)$$

and the control polygon of a curve is convex.

Proof. Suppose that the control polygon of a planar PHC curve is convex. It is straightforward to see that the relation $\varphi_{02} = \pi \pm (\varphi_{01} + \varphi_{12} - \pi)$ holds and therefore

$$\cos \varphi_{02} = \cos (\pi \pm (\varphi_{01} + \varphi_{12} - \pi)) = \cos \varphi_{01} \cos \varphi_{12} - \sin \varphi_{01} \sin \varphi_{12}. \quad (17)$$

The first relation in (9) and (17) imply

$$\cos \varphi_{02} = 2 \cos^2 \varphi_{01} - 1,$$

and the second relation in (9) simplifies to the second relation in (16). Suppose now that the control polygon is not convex. It follows $\varphi_{02} = \pm (\varphi_{01} - \varphi_{12})$, and thus

$$\cos \varphi_{02} = \cos (\pm (\varphi_{01} - \varphi_{12})) = \cos \varphi_{01} \cos \varphi_{12} + \sin \varphi_{01} \sin \varphi_{12}. \quad (18)$$

Now the first relation in (9) and (18) imply

$$\cos \varphi_{02} = 1.$$

Since $||\Delta b_1|| \neq 0$, the relation (15) implies $\cos \varphi_{01} = \cos \varphi_{12} = \pm 1$. Thus the control polygon is actually convex and a PHC curve is a line segment. □
Let us now discuss some significant geometric properties that PHC curves as a subset of C-curves possess.

**Theorem 2.5.** Suppose that \( p \) is a PHC curve on an interval \([0, \alpha]\), \( 0 < \alpha < 2\pi \), distinct from a line segment. Then \( p \) is a regular curve on this interval too,

\[
\|\dot{p}(t)\| > 0, \quad t \in [0, \alpha].
\]

If \( p \) is a planar curve then it must also be convex.

**Proof.** Let us recall that the true spatial PHC curves are always regular, and we have to consider planar PHC curves only. Since \( p \) is not a parameterized line segment by the assumption, Theorem 2.2 implies that none of the control point differences \( \Delta b_i \) could vanish. Let \( Q \) denote the intersection of lines \( b_0b_1 \) and \( b_2b_3 \) (Fig. 2), and let \( \mu_i \) be scalars defined by equations

\[
b_1 - Q = \mu_0 \Delta b_0, \quad Q - b_2 = \mu_3 \Delta b_2.
\]  \hspace{1cm} (19)

If \( Q \) doesn’t exist, we take \( \mu_i \) to be unbounded. Also, \( \mu_i = 0 \) reduces \( p \) to a line segment, so we assume \( \mu_i \neq 0 \). In [12, Theorem 2] shape properties of planar C-curves depending on \( \alpha \) and two parameters determined by \( Q \) were classified. A graphical representation of these properties [12, Fig. 4] in new variables \( \mu_i \) simplifies here to Fig. 1. A planar C-curve ([12]) has

- no inflection points in \( N := \{(\mu_0, \mu_3) ; \mu_0 < 0, \mu_3 < 0 \text{ or } \mu_0 > 0, \mu_3 > 0, \omega(\mu_0, \mu_3, \alpha) < 0\} \),

- one inflection point in \( S := \{(\mu_0, \mu_3) ; \mu_0 \mu_3 < 0\} \),

- two inflection points in \( D := \{(\mu_0, \mu_3) ; \mu_0 > 0, \mu_3 > 0, \omega(\mu_0, \mu_3, \alpha) > 0\} \),

- a cusp along the curve \( \omega(\mu_0, \mu_3, \alpha) = 0, \mu_0 > 0, \mu_3 > 0 \),

where

\[
\omega(\mu_0, \mu_3, \alpha) := \frac{1}{4\rho(\alpha)} - \mu_0 \mu_3.
\]

Note that \( 1/4 - \mu_0 \mu_3 < \omega(\mu_0, \mu_3, \alpha) < 1 - \mu_0 \mu_3 \), since \( 1/4 < \rho(\alpha) < 1 \). We proceed to show that the case \((\mu_0, \mu_3) \in N\) is the only possible for a PHC curve distinct from a line segment. Observe that Corollary 2.4 rules out the case \((\mu_0, \mu_3) \in S\), and we have to consider \( \mu_0, \mu_3 > 0 \) only (Fig. 2). The first relation in (16) and the equations (19) imply

\[
\angle (b_1b_2, b_2Q) = \angle (Qb_1, b_1b_2) = \pi - \varphi_{01}
\]
Figure 1. A simplified shape diagram of C-curves: N - no inflection points, S - a single inflection point, D - two inflection points, $\omega(\mu_0, \mu_3, \alpha) = 0$ - a cusp.

and

$$\|b_1 - Q\| = \mu_0 \|\Delta b_0\| = \|Q - b_2\| = \mu_3 \|\Delta b_2\|,$$

where $\varphi_{01}$ is defined in (7). From here, the law of sines gives

$$\frac{\|\Delta b_1\|}{\sin(2\varphi_{01} - \pi)} = \frac{\|b_1 - Q\|}{\sin(\pi - \varphi_{01})} = \mu_0 \frac{\|\Delta b_0\|}{\sin(\pi - \varphi_{01})} = \mu_3 \frac{\|\Delta b_2\|}{\sin(\pi - \varphi_{01})}.$$  \hfill (20)

From (20) and the second equation in (16) we compute

$$\mu_0 \mu_3 = \left( \frac{\sin(\pi - \varphi_{01})}{\sin(2\varphi_{01} - \pi)} \right)^2 \frac{\|\Delta b_1\|^2}{\|\Delta b_0\| \|\Delta b_2\|} = \frac{1}{4\rho(\alpha) \cos^2 \varphi_{01}} \geq \frac{1}{4\rho(\alpha)},$$

where the equality is reached iff $\varphi_{01} \in \{0, \pi\}$. In the latter case $p$ should be a line segment. This confirms $(\mu_0, \mu_3) \notin D$, and rules out a possible cusp. The proof is completed. □
Figure 2. Examples of a control polygon of a planar C-curve with $\mu_0, \mu_3 > 0$.

3 Hermite interpolation with PHC curves

Let us consider the following interpolation problem: suppose two points $P_0, P_1 \in \mathbb{R}^d$, $d \geq 2$, and two normalized tangent directions $d_0, d_1$ are prescribed. Find a PHC curve $p$ which interpolates these data in a geometric $G^1$ sense, i.e., the curve should satisfy the interpolation conditions

$$p(0) = P_0, \quad p(\alpha) = P_1, \quad \dot{p}(0) = \lambda_0 d_0, \quad \dot{p}(\alpha) = \lambda_1 d_1, \quad (21)$$

where $\lambda_0$ and $\lambda_1$ are unknown tangent lengths that should be positive. Conditions (21) give $4d$ equations for $4d + 2$ unknowns ($4d$ coefficients, $\lambda_0$ and $\lambda_1$). The additional two equations are given by PH condition simplified by Theorem 2.2 to (9).

Let us look for the interpolating curve $p$ in the Bézier form (2). The C-Bézier basis functions (4) allow us to incorporate the interpolation conditions (21) in the control points. Since

$$\dot{p}(0) = w_0(0) \Delta b_0 = \frac{1}{\nu(\alpha)} \Delta b_0, \quad \dot{p}(\alpha) = w_2(\alpha) \Delta b_2 = \frac{1}{\nu(\alpha)} \Delta b_2,$$

the control points are given by

$$b_0 = P_0, \quad b_1 = P_0 + \lambda_0 \nu(\alpha) d_0, \quad b_2 = P_1 - \lambda_1 \nu(\alpha) d_1, \quad b_3 = P_1. \quad (22)$$

This solves the linear part of the interpolation problem, and leaves us with two unknowns $\lambda_0$ and $\lambda_1$ that should be determined by the PH condition (9). For simplicity, let us define constants that depend entirely on the data

$$\delta := ||\Delta P_0||, \quad c_{ij} := \cos \theta_{ij}, \quad \lambda_0, \lambda_1 > 0,$$

where

$$\theta_{01} := \angle(d_0, \Delta P_0), \quad \theta_{12} := \angle(\Delta P_0, d_1), \quad \theta_{02} := \angle(d_0, d_1).$$
The cosines $c_{ij}$ are not independent, but must satisfy (see [6], e.g.)

$$(c_{02} - c_{01}c_{12})^2 \leq (1 - c_{01}^2)(1 - c_{12}^2).$$

(24)

The equality is reached only for the planar data.

By using Theorem 2.2 we can derive the equations for the unknowns $\lambda_0$ and $\lambda_1$. The relation between angles in (9) determines the function

$$e_1(\lambda_0, \lambda_1) := (c_{01} - c_{12})\delta + (c_{02} - 1)\nu(\alpha)(\lambda_0 - \lambda_1),$$

and from the other condition in (9) we obtain

$$e_2(\lambda_0, \lambda_1) := \lambda_0\lambda_1\nu(\alpha)^2(1 - c_{02}) - 2\rho(\alpha)\left(\|\Delta b_1\|^2 - (d_0^T\Delta b_1)^2\right)$$

$$= 2\left(c_{01}^2 - 1\right)\delta^2\rho(\alpha) + 4(c_{12} - c_{01}c_{02})\delta\nu(\alpha)\rho(\alpha)\lambda_1$$

$$+ 2(c_{02}^2 - 1)\nu(\alpha)^2\rho(\alpha)\lambda_1^2 + (1 - c_{02})\nu(\alpha)^2\lambda_0\lambda_1.$$

Since the first function is linear and the second one quadratic, the non-linear system

$$e(\lambda_0, \lambda_1) := (e_1(\lambda_0, \lambda_1), e_2(\lambda_0, \lambda_1)) = (0, 0)$$

(25)

has two solution pairs. With the help of the functions

$$h(x, \alpha) := 2\left(x^2 - 1\right)\rho(\alpha) - x + 1,$$

$$g_1(x, y, z, \alpha) := 4(xz - y)\rho(\alpha) + y - x,$$

$$g_2(x, y, z, \alpha) := -8\rho(\alpha)(x^2 - 1)h(z, \alpha) + g_1(x, y, z, \alpha)^2,$$

(26)

$$\zeta^\pm(x, y, z, \alpha) := \frac{4\delta\rho(\alpha)(x^2 - 1)}{\nu(\alpha)\left(g_1(x, y, z, \alpha) \mp \sqrt{g_2(x, y, z, \alpha)}\right)},$$

the two solutions $(\lambda_{0,i}, \lambda_{1,i}), i = 1, 2$, are simplified to

$$\lambda_{0,1} := \zeta^+(c_{12}, c_{01}, c_{02}, \alpha), \quad \lambda_{1,1} := \zeta^+(c_{01}, c_{12}, c_{02}, \alpha),$$

$$\lambda_{0,2} := \zeta^-(c_{12}, c_{01}, c_{02}, \alpha), \quad \lambda_{1,2} := \zeta^-(c_{01}, c_{12}, c_{02}, \alpha).$$

(27)

It is straightforward to check that $g_2(x, y, z, \alpha) = g_2(y, x, z, \alpha)$. Next theorems give sufficient and necessary conditions on the data for a PHC curve to exist, together with the exact number of solutions (see also Fig. 3).
Theorem 3.1. Suppose that data $d_0$, $P_0$, $P_1$, $(P_1 \neq P_0)$, and $d_1$ are prescribed and let $|c_{01}| < 1$, $|c_{12}| < 1$. Then there is precisely one interpolant (determined by $(\lambda_{0,1}, \lambda_{1,1})$ in (27)). iff

$$-1 \leq c_{02} < \vartheta(\alpha) := -1 + \frac{1}{2 \rho(\alpha)} \quad \text{or} \quad c_{02} = \vartheta(\alpha), \; c_{01} + c_{12} > 0.$$  

If $\vartheta(\alpha) < c_{02} < 1$, then the interpolation problem has two solutions (given by the pairs $(\lambda_{0,1}, \lambda_{1,1})$, and $(\lambda_{0,2}, \lambda_{1,2})$), iff

$$c_{01} + c_{12} > 0, \quad g_2(c_{01}, c_{12}, c_{02}, \alpha) \geq 0. \quad (28)$$

Otherwise, there are no solutions. The two solution pairs coincide iff in the last relation of (28) the equality is reached.

Proof. Note that $h(c_{02}, \alpha) = 0$ for $c_{02} = 1$ or $c_{02} = \vartheta(\alpha)$. Suppose first that $-1 \leq c_{02} < \vartheta(\alpha)$. Then $h(c_{02}, \alpha) > 0$ and

$$g_2(x, y, c_{02}, \alpha) > g_1(x, y, c_{02}, \alpha)^2, \; |x| < 1.$$  

Clearly the first pair $(\lambda_{0,1}, \lambda_{1,1})$ is admissible for this $c_{02}$ range, but the second one is not. Secondly, let $c_{02} = \vartheta(\alpha)$. Then $h(c_{02}, \alpha) = 0$ and

$$g_2(x, y, c_{02}, \alpha) = g_1(x, y, c_{02}, \alpha)^2.$$  

Furthermore,

$$g_1(c_{01}, c_{12}, \vartheta(\alpha), \alpha) = g_1(c_{12}, c_{01}, \vartheta(\alpha), \alpha) = (1 - 4 \rho(\alpha))(c_{01} + c_{12}).$$  

Since $1 - 4 \rho(\alpha) < 0$ for $\alpha \in (0, 2\pi)$, the function $g_1(c_{01}, c_{12}, \vartheta(\alpha), \alpha)$ is negative for $c_{01} + c_{12} > 0$ and positive for $c_{01} + c_{12} < 0$. Thus for $c_{01} + c_{12} > 0$ only the first solution $(\lambda_{0,1}, \lambda_{1,1})$ is admissible, since the second solution goes to infinity. For $c_{01} + c_{12} \leq 0$ no admissible solution exists. Namely, the first solution $(\lambda_{0,1}, \lambda_{1,1})$ goes to infinity, but the second solution pair is negative and approaches infinity as $c_{01} + c_{12} \uparrow 0$. It remains to analyse the case $\vartheta(\alpha) < c_{02} < 1$. Now $h(c_{02}, \alpha) < 0$ and

$$g_2(x, y, c_{02}, \alpha) < g_1(x, y, c_{02}, \alpha)^2, \; |x| < 1.$$  

Both solution pairs $(\lambda_{0,1}, \lambda_{1,1})$ and $(\lambda_{0,2}, \lambda_{1,2})$ are admissible or not at the same time. They are both real and positive iff

$$g_2(c_{01}, c_{12}, c_{02}, \alpha) = g_2(c_{12}, c_{01}, c_{02}, \alpha) \geq 0 \quad \text{and} \quad (29)$$

$$g_1(c_{01}, c_{12}, c_{02}, \alpha) < 0, \quad g_1(c_{12}, c_{01}, c_{02}, \alpha) < 0. \quad (30)$$
Using cylindrical decomposition one can prove that conditions (29) and (30) are equivalent to
\[ g_2(c_{01}, c_{12}, c_{02}, \alpha) \geq 0, \quad c_{01} + c_{12} > 0. \]
Finally let us show that for \( c_{02} = 1 \) none of the solutions is admissible. In this case the data are planar and \( c_{01} = c_{12} \). Moreover, \( g_1(x, x, 1, \alpha) = g_2(x, x, 1, \alpha) = 0 \), which implies \( \zeta^\pm(x, x, 1, \alpha) \to \infty \). The proof is completed.

![Figure 3](image_url)

Figure 3. The admissible choice of parameters \( c_{01}, c_{12} \) and \( c_{02} \) for \( \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{8} \) and \( \frac{6\pi}{4} \), respectively. In the black region there exists precisely one PHC interpolant, in the dark gray region there are two PHC interpolants, while in the bright gray region there are no PHC interpolants, although the relation (24) is satisfied.

We are left to consider the particular planar cases which were omitted in Theorem 3.1.
Theorem 3.2. Suppose that $\delta > 0$ and $|c_{01} c_{12}| < 1$, and let

$$(\Lambda_0, \Lambda_1) := \frac{\delta}{\nu(\alpha)} \left( \frac{1+c_{02}}{c_{02} \vartheta(\alpha) - 2\rho(\alpha)(c_{02} - \vartheta(\alpha))}, \frac{1}{c_{02} - \vartheta(\alpha)} \right).$$

(a) For $c_{01} = 1$ or $c_{12} = 1$ there exists (precisely one) admissible solution of the system (25), iff $\vartheta(\alpha) < c_{02} < 1$. The solution is given by the pair $(\Lambda_0, \Lambda_1)$ and $(\Lambda_1, \Lambda_0)$, respectively.

(b) For $c_{01} = -1$ or $c_{12} = -1$ there exists (precisely one) admissible solution of the system (25), iff $-1 < c_{02} < \vartheta(\alpha)$. The solution is given by the pair $(-\Lambda_0, -\Lambda_1)$ and $(-\Lambda_1, -\Lambda_0)$, respectively.

Proof. Suppose first that $c_{01} = 1$. The relation (24) implies $c_{12} = c_{02}$. Using these assumptions in the equations (25), we obtain the solutions

$$(\lambda_0, \lambda_1) = (\Lambda_0, \Lambda_1), \quad (\lambda_0, \lambda_1) = \left( \frac{\delta}{\nu(\alpha)}, 0 \right).$$

Clearly, the second solution is not admissible, and the first solution is positive iff $\vartheta(\alpha) < c_{02} < 1$. The proof for other cases follows similarly and will be omitted.

So far we have assumed $\delta > 0$. However, also the case $P_0 = P_1$ implies planar data.

Theorem 3.3. The planar problem $P_0 = P_1$, $d_1 \neq d_0$, has a solution only if $c_{02} = \vartheta(\alpha)$. In this case, any pair $\lambda_0 = \lambda_1 > 0$ determines a regular interpolant.

Proof. Since $c_{02} \neq 1$, equation $e_1(\lambda_0, \lambda_1) = 0$ implies $\lambda_0 = \lambda_1$. Considering this in equation $e_2(\lambda_1, \lambda_1) = 0$, we obtain

$$2(c_{02} - 1) \nu(\alpha)^2 \lambda_1^2 \rho(\alpha)(c_{02} - \vartheta(\alpha)) = 0,$$

and the assertion of the theorem follows.

Finally let us consider the cases $|c_{01} c_{12}| = 1$. It is straightforward to see that we obtain a regular line segment if $c_{01} = c_{12} = c_{02} = 1$ and that there are no regular solutions in all other cases.

The obtained results have a nice geometric interpretation. The condition (24) is equivalent to

$$\frac{(c_{01} + c_{12})^2}{2(1 + c_{02})} + \frac{(c_{01} - c_{12})^2}{2(1 - c_{02})} \leq 1, \quad c_{02} \in (-1, 1), \quad (31)$$
or
\[ c_{02} = \pm 1, \quad c_{01} = \pm c_{12}. \]

For every fixed \( c_{02}, \ |c_{02}| < 1 \), this represents the interior of the ellipse with axes \( c_{01} + c_{12} \) and \( c_{01} - c_{12} \). Similarly, the condition \( g_2(c_{01}, c_{12}, c_{02}, \alpha) \geq 0 \) is equivalent to
\[
\frac{(c_{01} + c_{12})^2}{a(c_{02}, \alpha)} + \frac{(c_{01} - c_{12})^2}{b(c_{02}, \alpha)} \geq 1, \tag{32}
\]
where
\[
a(c_{02}, \alpha) := -\frac{8 \rho(\alpha) (c_{02} - \vartheta(\alpha))}{1 - 4 \rho(\alpha)}, \quad b(c_{02}, \alpha) := -\frac{8 \rho(\alpha) (1 - c_{02})}{1 - 4 \rho(\alpha)}.
\]

Since \( a(c_{02}, \alpha) \) and \( b(c_{02}, \alpha) \) are both positive for \( c_{02} \in (\vartheta(\alpha), 1) \), condition (32) represents the exterior of an ellipse with axes \( c_{01} + c_{12} \) and \( c_{01} - c_{12} \). Therefore, the boundary of the admissible region (28) for a fixed \( c_{02} \in (\vartheta(\alpha), 1) \) is obtained through ellipses (31) and (32) as in Fig. 4.

Figure 4. The admissible parameter regions at \( c_{02} = \frac{1}{4} \) for \( \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \), respectively. The black boundary is determined by (31), the dashed one by (32), and the admissible area is coloured gray.

The next lemma shows how to compute the length of a PHC curve \( p \) and compares the lengths of the interpolants determined by (27).

**Lemma 3.4.** The length of a PHC curve \( p \) that satisfies (21) is computed as
\[
\ell(p) = \int_0^\alpha \|\dot{p}(t)\| \, dt = \delta c_{01} + (1 - c_{02}) \nu(\alpha) \lambda_1 = \delta c_{12} + (1 - c_{02}) \nu(\alpha) \lambda_0.
\]

If there are two admissible interpolants, then the one determined by \((\lambda_{0,1}, \lambda_{1,1})\) in (27) has a smaller length than the one determined by \((\lambda_{0,2}, \lambda_{1,2})\).

\[\]
Proof. From (10) and the equality $\|\dot{p}(t)\| = \sigma(t)$ it follows

$$\ell(p) = \sum_{i=0}^{2} \sigma_i = \|\Delta b_0\| + \cos \varphi_{01} \|\Delta b_1\| + \|\Delta b_2\|,$$

and from (22) one obtains (33). Let us denote by $p_i$ the interpolant determined by $(\lambda_{0,i}, \lambda_{1,i}), i = 1, 2$. It is straightforward to compute that

$$\ell(p_i) = \delta \left( c_{01} + c_{12} \left( 4 \rho(\alpha) - 1 \right) + (-1)^i \sqrt{g_2(c_{01}, c_{12}, c_{02}, \alpha)} \right) \frac{4 (1 + c_{02}) \rho(\alpha) - 2}{4 (1 + c_{02}) \rho(\alpha) - 2}.$$

By Theorem 3.1 the necessary condition for two admissible solutions is that $c_{02} > \vartheta(\alpha)$, which implies

$$4 (1 + c_{02}) \rho(\alpha) - 2 > 4 (1 + \vartheta(\alpha)) \rho(\alpha) - 2 = 0.$$

This clearly gives $\ell(p_1) \leq \ell(p_2)$, and the proof is completed. \qed

4 Asymptotic analysis

In this section we examine asymptotic approximation properties of the $G^1$ interpolation by PHC curves studied in Section 3. As usually, we assume that data are sampled from a smooth parametric curve, parameterized on an interval of length $2h$, and we analyse the error estimate behaviour as $h \to 0$. As an error measure, we take the parametric distance introduced in [8]. For parametric curves $f_1 : [\alpha_1, \beta_1] \to \mathbb{R}^d$ and $f_2 : [\alpha_2, \beta_2] \to \mathbb{R}^d$ it is defined as

$$\text{dist}_p(f_1, f_2) := \inf_{\phi} \max_{\alpha_1 \leq t \leq \beta_1} \|f_1(t) - f_2(\phi(t))\|,$$

where $\phi : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ is a differentiable regular reparameterization, i.e., a diffeomorphism between the intervals concerned.

Theorem 4.1. Suppose that the $G^1$ interpolation data (21) are sampled from a smooth parametric curve $f : [-h, h] \to \mathbb{R}^3 : s \mapsto f(s)$ with a non-vanishing curvature, parameterized by the arc-length,

$$P_0 = f(-h), \quad P_1 = f(h), \quad d_0 = f'(-h), \quad d_1 = f'(h), \quad \|f'\| = 1,$$

with $f' = \frac{df}{ds}$. Further, let

$$p_{i,h}^{\alpha} : [0, \alpha] \to \mathbb{R}^3, \quad i = 1, 2,$$
be two PHC curves that geometrically interpolate the data (34), which are determined by $\lambda_{0,i}$ and $\lambda_{1,i}$ in (27), respectively. For a fixed interval length $\alpha$, there exist constants $C_{1,\alpha}, C_{2,\alpha}, C_{2,\alpha} > 0$ and $h_\alpha > 0$ such that

$$\text{dist}_p(f, p^\alpha_{1,h}) \leq C_{1,\alpha} h^2, \quad C_{2,\alpha} h^2 \leq \text{dist}_p(f, p^\alpha_{2,h}) \leq C_{2,\alpha} h^2, \quad 0 < h \leq h_\alpha,$$

i.e., the asymptotic approximation order of both solutions is 2. If we additionally assume $\alpha = O(h)$, the approximation order of the first solution $p^\alpha_{1,h}$ rises to 4.

**Proof.** Without loss of generality, we may assume

$$f(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad f'(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f''(0) = \| f''(0) \| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (36)$$

Suppose that the curvature $\kappa$ and the torsion $\tau$ of the curve $f$ at $s = 0$ expand as

$$\kappa(s) = \kappa_0 + \frac{\kappa_1}{1!} s + \frac{\kappa_2}{2!} s^2 + \ldots, \quad \kappa_0 > 0,$$

$$\tau(s) = \tau_0 + \frac{\tau_1}{1!} s + \frac{\tau_2}{2!} s^2 + \ldots.$$

Then the Frenet-Serret formulae give an expansion of the curve, simplified by the assumptions (36) to

$$f(s) = \begin{pmatrix} s - \frac{1}{6}\kappa_0^2 s^3 - \frac{1}{8}\kappa_0\kappa_1 s^4 + O(s^5) \\ \frac{1}{6}\kappa_0 s^2 + \frac{1}{6}\kappa_1 s^3 + \frac{1}{12} (\kappa_2 - \kappa_0^3 - \kappa_0\tau_0^2) s^4 + O(s^5) \\ \frac{1}{6}\kappa_0\tau_0 s^3 + \frac{1}{12} (2\kappa_1 \tau_0 + \kappa_0\tau_1) s^4 + O(s^5) \end{pmatrix}. \quad (37)$$

From the expansion (37), it is straightforward to obtain expansions of the data (34) and the constants (23),

$$\delta = 2h - \frac{1}{3}\kappa_0^2 h^3 + O(h^5),$$

$$c_{01} = 1 - \frac{1}{2}\kappa_0^2 h^2 + \frac{1}{3}\kappa_0\kappa_1 h^3 + O(h^4),$$

$$c_{02} = 1 - 2\kappa_0^2 h^2 + O(h^4),$$

$$c_{12} = 1 - \frac{1}{2}\kappa_0^2 h^2 - \frac{1}{3}\kappa_0\kappa_1 h^3 + O(h^4).$$

From (26) and (27) we further obtain

$$\lambda_{j,1} = \frac{1}{\nu(\alpha)} \left( \frac{2\sqrt{\rho(\alpha)}}{2\sqrt{\rho(\alpha)} + 1} h + (-1)^j \frac{\kappa_1}{3\kappa_0} h^2 + O(h^4) \right), \quad j = 0, 1,$$
for the first solution, and
\[ \lambda_{j,2} = \frac{1}{\nu(\alpha)} \left( \frac{2\sqrt{\rho(\alpha)}}{2\sqrt{\rho(\alpha)} - 1} h + (-1)^j \frac{\kappa_1}{3\kappa_0} h^2 + O(h^4) \right), \quad j = 0, 1, \]
for the second one. Let us reparameterize the curve \( p_{i,h}^\alpha \) to a fixed parameter interval \([0, 1]\) by \( t \to \alpha u \), and let \( \phi_i : [0, 1] \to [-h, h] \) be a reparameterization of the data curve to the same interval,
\[ \phi_i := -h Z_0 + (-h + \lambda_{0,i} \nu(\alpha)) Z_1 + (h - \lambda_{1,i} \nu(\alpha)) Z_2 + h Z_3, \quad i = 1, 2. \]

With such a reparameterization \( f \) and \( p_{i,h}^\alpha \) agree twice at \( u = 0 \) and \( u = 1 \),
\[ p_{i,h}^\alpha(\alpha u) = f \circ \phi_i(\alpha u), \quad \frac{d}{du} (p_{i,h}^\alpha(\alpha u)) = \frac{d}{du} (f \circ \phi_i(\alpha u)), \quad u \in \{0, 1\}. \]

Since
\[ \phi_i(\alpha u) = \frac{\alpha}{2} (2u - 1) + (-1)^{i-1} \sin \left( \frac{\alpha}{2} (2u - 1) \right) \frac{\alpha}{2} + (-1)^{i-1} \sin \frac{\alpha}{2} h + O(h^2), \]
the reparameterization \( \phi_i \) is regular, at least for \( h \) small enough. This provides bounds
\[ \text{dist}_P \left( f, p_{i,h}^\alpha \right) \leq \max_{u \in [0,1]} \left\| f \circ \phi_i(\alpha u) - p_{i,h}^\alpha(\alpha u) \right\|, \quad i = 1, 2. \]

Further, the norm expansions yield
\[ \left\| f \circ \phi_i(\alpha u) - p_{i,h}^\alpha(\alpha u) \right\| = M_i(u, \alpha) h^2 + O(\alpha h^3) + O(h^4), \quad i = 1, 2, \quad (38) \]
where \( M_i \) are rather lengthy expressions. However, it is straightforward to conclude that
\[ \max_{u \in [0,1]} M_1(u, \alpha) = M_1 \left( \frac{1}{2}, \alpha \right) = \frac{\kappa_0}{2} \frac{\alpha}{2} - \sin \frac{\alpha}{2} =: M_{1,\alpha}, \]
\[ \max_{u \in [0,1]} M_2(u, \alpha) = M_2 \left( \frac{1}{2}, \alpha \right) = \frac{\kappa_0}{2} \frac{4 \tan \frac{\alpha}{4} - \frac{\alpha}{2} - \sin \frac{\alpha}{2}}{\frac{\alpha}{2} - \sin \frac{\alpha}{2}} =: M_{2,\alpha}. \]
So we can take \( \overline{C}_{i,\alpha} = M_{i,\alpha} + O(\alpha h_\alpha) + O(\alpha h^2_\alpha) \), with \( h_\alpha \) small enough. Note that \( M_{i,\alpha} \) are increasing functions of \( \alpha \in [0, 2\pi], \) and
\[ 0 = M_{1,0} \leq M_{1,\alpha} = \frac{1}{96} \kappa_0 \alpha^2 + O(\alpha^4), \quad \kappa_0 = M_{2,0} \leq M_{2,\alpha} = \kappa_0 + O(\alpha^2). \]
So a choice $\alpha = \mathcal{O}(h)$ decreases the bound (38) for the first solution only. This completes the upper bound part of the proof. As to the lower bound for the second solution $p_{2,h}^\alpha$, recall that the parametric distance is an upper estimate of the Hausdorff distance $\text{dist}_H\left(f, p_{2,h}^\alpha\right)$, and quite clearly,

$$\text{dist}_H\left(f, p_{2,h}^\alpha\right) \geq \min_{0 \leq u \leq 1} \|f(0) - p_{2,h}^\alpha(\alpha u)\| = \min_{0 \leq u \leq 1} \|p_{2,h}^\alpha(\alpha u)\| .$$

Further, the norm expansion reveals

$$\|p_{2,h}^\alpha(\alpha u)\| = \left[\frac{\alpha}{2}(2u - 1) - \sin\left(\frac{\alpha}{2}(2u - 1)\right)\right] h + \mathcal{O}(h^2),$$

and the upper bound (35) already confirmed implies that $u$ which minimizes (39) should be of the form $u = \tilde{u} := \frac{1}{2} + \text{const} h + \mathcal{O}(h^2)$. For such $u$ one gets

$$\|p_{2,h}^\alpha(\alpha \tilde{u})\| = h^2 \sqrt{M_{2,\alpha}^2 + M_{3,\alpha}^2} + \mathcal{O}(h^3),$$

with

$$M_{3,\alpha} := \frac{2\kappa_1 \sin \frac{\alpha}{2} \left(1 - \cos \frac{\alpha}{2}\right)}{3\kappa_0 (\alpha - \sin \alpha)} = \frac{\kappa_1}{4\kappa_0} - \frac{\kappa_1}{320\kappa_0} \alpha^2 + \mathcal{O}(\alpha^4),$$

and the lower bound can’t be improved over $\mathcal{O}(h^2)$. The proof is completed.

There are some remarks here to be added. First of all, it is straightforward to verify that $\alpha = \text{const} h$, $\text{const} \to 0$, gives the asymptotic approximation orders $\mathcal{O}(h^4)$ and $\mathcal{O}(h^2)$ of the PH polynomial solutions $p_{1,h}^0$ and $p_{2,h}^0$, respectively, obtained as a limit case of PHC interpolating curves as introduced in Corollary 2.3. Also, the solution $p_{1,h}^0$ should be the preferable algorithmic choice in general. Last, but not least: from the approximation point of view it is sensible to choose the parameter $\alpha$ closely connected to the distance between interpolated points, in particular if one is building a $C^1$ spline out of PHC pieces.

## 5 Examples

In this section some numerical examples are provided. As a first example let us consider a family of planar Hermite interpolation data,

$$P_0 = (0,0)^T, \ P_1 = (1,0)^T, \ d_0 = \left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right)^T, \ d_1 = (\cos \gamma \pi, \sin \gamma \pi)^T,$$  

(40)
where the tangent direction at \( P_1 \) changes with \( \gamma \),

\[
\gamma \in \{0.15, \ 0.05, \ 0.03, \ -0.04, \ -0.16, \ -0.35, \ -0.9, \ -1.01, \ -1.4\}. \quad (41)
\]

In Fig. 5, there is a corresponding subfigure to each choice of \( \gamma \) that shows all the planar PHC interpolants for different values of parameter \( \alpha \), \( \alpha = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4} \). As expected from Theorem 3.1, the number of interpolants (zero, one or two) clearly varies with \( \alpha \). Note also that all curves in Fig. 5 indeed interpolate given data, although in some cases this is difficult to recognize.

As another example let us consider the \( G^1 \) PHC spline interpolation. The data points are taken from a curve \( g : \mathbb{R} \to \mathbb{R}^3 \),

\[
g(t) = \left(\ln (t + 3) \cos 2t, \sin 2t, \frac{t^2}{2}\right)^T,
\]
as

\[ P_i := g \left( \frac{i}{2} \right), \quad d_i := \frac{g'(\frac{i}{2})}{\left\| g'(\frac{i}{2}) \right\|}, \quad i = 1, 2, \ldots, 8. \]

In order to construct a \( G^1 \) PHC Hermite spline, which interpolates the given points \( P_i \) and tangents \( d_i \), we have to determine three free parameters \( \lambda^k_0, \lambda^k_1, \alpha_k, k = 1, 2, \ldots, 7 \), for each cycloidal segment. Once parameters \( \alpha_k \) are determined, the tangent lengths \( \lambda^k_0, \lambda^k_1 \) are computed by (27). Two different cases are considered. Firstly, let us follow the asymptotic analysis suggestion and choose \( \alpha_k = \| P_{k+1} - P_k \|, \quad k = 1, 2, \ldots, 7. \) (42)

Since the given data ensure two admissible PHC interpolants for each segment, we can combine them in several different ways. In Fig. 6 (black curves) two \( G^1 \) PHC splines are shown, where in the left (right) figure the first (second) solution in (27) is taken for each segment. As a second case, we can choose \( \alpha \) to be the same for each segment. In Fig. 6 the gray curves present \( G^1 \) PHC splines with \( \alpha_k = \frac{3\pi}{2}, \quad k = 1, 2, \ldots, 7. \) There is not much difference between both splines generated by the first solutions, but a significant one for the second choice of \( \lambda^k_1. \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{\( G^1 \) PHC Hermite splines with parameters \( \alpha_k = \| P_{k+1} - P_k \| \) (black curves) and \( \alpha_k = \frac{3\pi}{2} \) (gray curves), \( k = 1, 2, \ldots, 7. \) In the left figure the curve segments are determined by \( (\lambda_{0,1}, \lambda_{1,1}) \) in (27) and in the right figure by \( (\lambda_{0,2}, \lambda_{1,2}) \).}
\end{figure}

However, if a particular nature of data to be interpolated is known in advance, the choice (42) may be improved. As an example, take a family of spirals with the first two components been periodic while the third one monotonously grows, and let us choose the data from the representative curve

\[ \tilde{g}(t) = \left( \cos 3t, \sin 3t, \sqrt{3 + t^2} \right)^T. \] (43)
Table 1. Parameters \( \alpha_k^{(i)}, k = 1, 2, \ldots, 7 \), and the Hausdorff distances between the PHC segments \( p_k^{(i)} \) and the curve segments \( \tilde{g}_k \) for the data based upon (43).

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</tbody>
</table>

Let us consider the interpolation by the first solution spline only. A PHC interpolant could reproduce a circle segment only if the parameter \( \alpha \) is equal to the corresponding arc length. This circle reproduction property suggests an iterative improvement. Let us determine \( p^{(0)} \) with \( \alpha_k^{(0)}, k = 1, 2, \ldots, 7 \), as proposed in (42). As a next guess, let us compute parameters \( \alpha_k^{(1)}, k = 1, 2, \ldots, 7 \), as lengths of the projections of PHC segments \( p_k^{(0)} \) to the first two components. This gives the new PHC spline interpolant \( p^{(1)} \), etc. In Table 1, the parameters \( \alpha_k^{(i)} \) and the Hausdorff distances between the PHC segments \( p_k^{(i)} \) and the curve segments \( \tilde{g}_k := [p_k, p_{k+1}] \) are shown for the first three steps. No observable error reduction has been achieved after the first correction.

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Received ???.

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