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Parametric curves with Pythagorean binormal

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Abstract In this paper, a class of rational spatial curves that have a rational binormal is introduced. Such curves (called PB curves) play an important role in the derivation of rational rotation-minimizing conformal frames. The PB curve construction proposed is based upon the dual curve representation and the Euler-Rodrigues frame obtained from quaternion polynomials. The construction significantly simplifies if the curve is a polynomial one. Further, polynomial PB curves of the degree ≥ 7 and rational PB curves of the degree ≥ 6 that possess rational rotation-minimizing conformal frames are derived, and it is shown that no lower degree curves with such a property exist.

Keywords Pythagorean-hodograph · Pythagorean-binormal · rational curve · dual coordinates · rotation-minimizing frame · conformal frame

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1 Introduction

Orthonormal frames of spatial parametric curves find their use in quite a few practical applications, such as the computer animation, the motion planning, the swept surface construction, *etc*. One of such frames is the well known Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$, where $\boldsymbol{t}$ is the unit length tangent, $\boldsymbol{n}$ the principal normal, and $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$ the binormal. Every orthonormal

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frame $(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ is accompanied by the angular velocity vector field $\boldsymbol{\omega}$, such that

$$\boldsymbol{f}_i' = \boldsymbol{\omega} \times \boldsymbol{f}_i, \quad i = 1, 2, 3.$$

If $\boldsymbol{\omega}$ additionally satisfies $\boldsymbol{\omega} \cdot \boldsymbol{f}_\ell = 0$ for some $\ell \in \{1, 2, 3\}$, then the frame is called *rotation-minimizing* with respect to $\boldsymbol{f}_\ell$. In case of the Frenet frame, the angular velocity vector field is equal to $\boldsymbol{\omega} = \kappa \boldsymbol{b} + \tau \boldsymbol{t}$, where κ and τ are the curvature and the torsion of the curve. Since $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$, there is no instantaneous rotation of $\boldsymbol{b}$ and $\boldsymbol{t}$ about $\boldsymbol{n}$, so the Frenet frame is rotation-minimizing with respect to $\boldsymbol{n}$. Orthonormal frames of a spatial curve $\boldsymbol{r}$ with

$$\boldsymbol{f}_1 = \boldsymbol{t} = \frac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$$

are called *adapted* frames. In most but not all applications it is preferable to work with adapted frames which are rotation-minimizing with respect to the tangent, *i.e.*, *rotation-minimizing adapted* (RMA) frames. However, in practical computer aided design applications it is an imperative to deal with rationally represented objects only. So in the past years, a lot of the research has been devoted to the construction of *rational* RMA frames (RRMA frames) (see *e.g.*, [5], [6], [7], [11], [13], [14], [15], [22], [23]). For such frames it is necessary that a curve $\boldsymbol{r}$ is a *Pythagorean-hodograph* (PH) curve. Pythagorean-hodograph curves are characterized by the property that their unit length tangent is rational. They were first introduced in [12], and since then very well investigated (see [2], [3], [4], [17], [18], [19], [20], [24], [25], and the references therein). But clearly, the PH property alone does not ensure the existence of a RRMA frame and a construction of curves which possess such a frame is a difficult task since nonlinear constraints are involved.

Recently, a new class of orthonormal frames called the *rotation-minimizing conformal* (RMC) frames has been introduced in [8]. Such frames are needed in aerodynamics to construct "yaw-free" rigid body motions along a curved path, *i.e.*, motions that have no instantaneous rotation about the binormal $\boldsymbol{b}$. An orthonormal frame $(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ of a spatial curve $\boldsymbol{r}$ is conformal if

$$\boldsymbol{f}_3 = \boldsymbol{b} = \frac{\boldsymbol{r}' \times \boldsymbol{r}''}{\|\boldsymbol{r}' \times \boldsymbol{r}''\|},$$

and the remaining two vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ span the osculating plane. Furthermore, it is a RMC frame if additionally $\boldsymbol{\omega} \cdot \boldsymbol{b} = 0$. A necessary condition for a curve to have a *rational* rotation-minimizing conformal (RRMC) frame is that its binormal $\boldsymbol{b}$ is rational. This certainly is true if the curve has a rational Frenet frame. Such a curve is called a *double* PH (DPH) curve (see *e.g.*, [9], [10]). In [8] the existence of polynomial curves possessing RRMC frame has been studied, and DPH curves as first candidates have been considered. It was shown that no cubic or quintic DPH curves having RRMC frame exist. Moreover, at least two important open problems were exposed. The first one is the existence of RRMC frames on curves of degree 7 or higher. The second question is whether there exist curves which are not DPH curves, and possess RRMC frame. In this paper the positive answers to both of the questions are given. Namely, a new class of rational parametric curves, called PB curves, that have a rational binormal $\boldsymbol{b}$ is defined and the construction is provided based on the dual representation of spatial parametric curves (see [21], [26]). This approach enables us to avoid dealing with nonlinear constraints that characterize such curves. Furthermore, it is shown that polynomial PB curves can be obtained by the integration of a particular hodograph defined by the preimage quaternion curve. The same holds true also for PH curves, only that the preimage map is different. Therefore, most of the results known for PH curves and the corresponding rational adapted frames can be applied to rational conformal frames of PB curves. In particular, the Euler-Rodrigues frame (E-R frame) that has been introduced in [1] can be assigned as a conformal frame to the derived PB curve. Following the construction of a RRMA frame of a PH curve, we show that a RRMC frame can be obtained from the E-R frame by rotating the two vectors in the osculating plane. By using [5], the construction of polynomial PB curves of degree seven that possess RRMC frames is proposed. Furthermore it is shown that polynomial PB curves of degree < 7 derived from a quadratic quaternion polynomial exist, but it is proven that such curves can not possess a RRMC frame. This extends the result obtained in [8] for DPH curves since DPH curves are a subset of PB curves.

The paper is organized as follows. In the next section the dual construction of spatial rational curves is shortly reviewed and it is applied to PB curves in Section 3. The examples of rational PB curves derived from quaternion polynomials are given in Section 4. Polynomial PB curves are considered in Section 5 with the emphasis on low degree curves. Section 6 deals with conformal frames of PB curves and provides the construction of RRMC frames.

2 Dual representation of spatial rational curves

In [16] and [26] it was shown how spatial parametric curves can be constructed from a one parametric family of osculating planes based on geometric foundations. In [21] a completely algebraic construction was derived and a dual representation of spatial curves was incorporated. In this section the main results from [21] are reviewed.

Let $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ be a smooth regular parametric curve such that the Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$,

$$\boldsymbol{t} = \frac{1}{\|\boldsymbol{r}'\|}\boldsymbol{r}', \quad \boldsymbol{b} = \frac{1}{\|\boldsymbol{r}' \times \boldsymbol{r}''\|}\boldsymbol{r}' \times \boldsymbol{r}'', \quad \boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t},$$

is well defined on the parameter interval $[\alpha, \beta]$. The basis for dual construction is the fact that at every parameter value $t \in [\alpha, \beta]$ the point $\boldsymbol{r}(t)$ is the unique solution of a linear system

$$\boldsymbol{b}(t) \cdot (\boldsymbol{p} - \boldsymbol{r}(t)) = 0, \;\; \boldsymbol{n}(t) \cdot (\boldsymbol{p} - \boldsymbol{r}(t)) = 0, \;\; \boldsymbol{t}(t) \cdot (\boldsymbol{p} - \boldsymbol{r}(t)) = 0, \quad \boldsymbol{p} \in \mathbb{R}^3, \;\; t \in [\alpha, \beta], \tag{1}$$

which represents the intersection of the osculating, the rectifying and the normal plane at the value t. In [21] it was shown that under additional condition that the torsion τ does not vanish on $[\alpha, \beta]$, equations (1) are equivalent to

$$\boldsymbol{u}(t) \cdot \boldsymbol{p} - f(t) = 0, \;\; \boldsymbol{u}'(t) \cdot \boldsymbol{p} - f'(t) = 0, \;\; \boldsymbol{u}''(t) \cdot \boldsymbol{p} - f''(t) = 0, \quad \boldsymbol{p} \in \mathbb{R}^3, \;\; t \in [\alpha, \beta], \tag{2}$$

where

$$\boldsymbol{u} := \phi\, \boldsymbol{b}, \quad f := \phi\, \boldsymbol{b} \cdot \boldsymbol{r}, \tag{3}$$

and ϕ is any zero free function from $C^2([\alpha, \beta])$. The first equation in (2), which represents the osculating plane, determines the corresponding curve uniquely. Its coefficients are called *dual coordinates* of the parametric curve $\boldsymbol{r}$ and are denoted by

$$\boldsymbol{L} := (-f; \boldsymbol{u}) := \left(-f; (u_1, u_2, u_3)^T\right) := (-f, u_1, u_2, u_3)^T .$$

The dual coordinates $\boldsymbol{L}$ are clearly homogeneous, *i.e.*, $\boldsymbol{L} \sim \zeta \boldsymbol{L}$ for any smooth function ζ that does not vanish on $[\alpha, \beta]$. If the original curve $\boldsymbol{r}$ is rewritten in a homogeneous form too,

$$\boldsymbol{P} := (P_0, P_1, P_2, P_3)^T \sim (1; \boldsymbol{r}), \quad \boldsymbol{r} = \frac{1}{P_0}(P_1, P_2, P_3)^T, \quad P_0 \neq 0, \tag{4}$$

then both representations are connected as

$$\boldsymbol{P} \sim \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'', \quad \boldsymbol{L} \sim \boldsymbol{P} \wedge \boldsymbol{P}' \wedge \boldsymbol{P}'', \tag{5}$$

where $\cdot \wedge \cdot \wedge \cdot$ denotes the Grassmann wedge product between vectors in $\mathbb{R}^4$, defined as

$$\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3 := \left((-1)^i \det V^{[i]}\right)_{i=1}^4, \quad \boldsymbol{v}_j \in \mathbb{R}^4,$$

where $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \in \mathbb{R}^{4\times3}$ and $V^{[i]} \in \mathbb{R}^{3\times3}$ is a submatrix of V with i-th row of the original matrix omitted (see [21, Thm.1]).

From (4) and (5) it is clear that a polynomial dual form $\boldsymbol{L}$ defines a rational curve $\boldsymbol{r}$ and vice versa. A degree of a polynomial homogeneous representation is defined as a maximal degree of polynomials involved, under the condition that the polynomials are relatively prime. A homogeneous form with relatively prime components is called a primitive form. Rational curves with polynomial dual form $\boldsymbol{L}$ of degree m are called *class m curves*. Quite clearly $\deg \boldsymbol{r} = \deg \boldsymbol{P}$ provided $\boldsymbol{P}$ is primitive, but a connection between the degree and the class of a curve $\boldsymbol{r}$ is not that straightforward and it is treated in [21]. In particular, it is shown that a dual form $\boldsymbol{L}$ of degree m imply a point representation $\boldsymbol{P} = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}''$ of degree $3m-6$ in general. Thus a switch from a known dual representation $\boldsymbol{L}$ to a point representation $\boldsymbol{P}$ might increase the complexity of the curve representation significantly. But, a Beziér dual representation that has been introduced in [26], allows one to do the computations in an efficient and stable way by applying the de Casteljau algorithm to dual coordinates.

Dual coordinates turned out to be particularly useful when dealing with rational PH curves. In this paper we will show how to apply dual construction to obtain another practically important class of curves, *i.e.*, curves that have rational unit binormals.

3 Rational curves with pythagorean binormal

Definition 1 Rational (polynomial) spatial curve $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ is a *Pythagorean-binormal* (PB) curve if

$$\left\|\boldsymbol{r}' \times \boldsymbol{r}''\right\| = \sigma \tag{6}$$

for some rational (polynomial) function σ.

The algebraic characterization of a PB curve is easy to formulate. Suppose that $\boldsymbol{r}$ is a polynomial curve with components being polynomials of the degree n. Then $\boldsymbol{r}' \times \boldsymbol{r}''$ is in general of the degree $2(n-2)$ (see [21, Thm. 3.]), so σ must be a polynomial of the degree $2(n-2)$ too. Condition $\left\|\boldsymbol{r}' \times \boldsymbol{r}''\right\|^2 = \sigma^2$ gives $4(n-2)+1$ polynomial equations for $2(n-2)+1$ unknown coefficients of σ and $3(n+1)$ unknown coefficients of $\boldsymbol{r}$. The choice of the free σ coefficient is obvious, *i.e.*, $\sigma(0) = \left\|\boldsymbol{r}'(0) \times \boldsymbol{r}''(0)\right\|$, and it is a linear task to express the rest of the σ coefficients from

$$\frac{d^\ell}{dt^\ell}\sigma^2(t)\big|_{t=0} = \frac{d^\ell}{dt^\ell}\left(\left\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\right\|^2\right)\big|_{t=0}, \quad \ell = 1, 2, \ldots, 2(n-2),$$

by the coefficients of $\boldsymbol{r}$. So one is left with $2(n-2)$ equations that the coefficients of $\boldsymbol{r}$ should satisfy. Since $3(n+1)-2(n-2)=n+7$, the family of polynomial PB curves of the degree n is $(n+7)$-parametric. As an alternative, one may avoid the use of σ completely by simply applying a proper set of linear functionals that annihilates both sides of (6) ([20]).

A similar approach works for the rational case too. Let $\boldsymbol{r}=\frac{1}{q}\boldsymbol{p}$ be a rational curve of the degree n with the numerator $\boldsymbol{p}$ and with the denominator q. Then $\boldsymbol{r}$ depends on $3(n+1)+n-1=4n+2$ free coefficients. Further we obtain

$$\boldsymbol{r}'\times\boldsymbol{r}''=-\frac{q'}{q^3}\boldsymbol{p}\times\boldsymbol{p}''+\frac{q''}{q^3}\boldsymbol{p}\times\boldsymbol{p}'+\frac{1}{q^2}\boldsymbol{p}'\times\boldsymbol{p}''.$$

One needs to consider the PB property of the polynomial numerator part $q^3\,\boldsymbol{r}'\times\boldsymbol{r}''$ only. The degree of $q^3\,\boldsymbol{r}'\times\boldsymbol{r}''$ equals $3(n-2)$, so the corresponding σ should be of the same degree. Following the steps of the polynomial case we finally observe that the number of free parameters in the rational case equals

$$(3(n+1)+n-1)-((6(n-2)+1)-(3(n-2)+1))=n+8.$$

Even though the algebraic approach to PB curves outlined is precise, it may lack the practical functionality even for modest degrees n. But if we take a tiny step back from the generality, the PB curve construction turns out as a simple task. Indeed, if we express the curve by its dual form, described in Section 2, we only have to assure that the field of binormals is chosen as

$$\boldsymbol{b}=\frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u},$$

where the norm $\|\boldsymbol{u}\|$ is a polynomial. The following theorem gives the basic properties of rational PB curves obtained by the dual construction.

Theorem 1 *Suppose that $\boldsymbol{u}:\mathbb{R}\to\mathbb{R}^3$ is a polynomial curve, such that its norm $\|\boldsymbol{u}\|$ is polynomial too. For any polynomial f, the dual form $\boldsymbol{L}:=(-f;\boldsymbol{u})$ defines a rational PB curve $\boldsymbol{r}$ with the binormal $\boldsymbol{b}=\frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u}$ and the denominator of $\boldsymbol{r}$ being equal to*

$$\lambda:=\det\bigl(\boldsymbol{u},\boldsymbol{u}',\boldsymbol{u}''\bigr)\,. \tag{7}$$

Moreover, the hodograph of $\boldsymbol{r}$ is of the form

$$\boldsymbol{r}'=\psi\,\boldsymbol{u}\times\boldsymbol{u}', \tag{8}$$

where

$$\psi=\frac{\det\bigl(\boldsymbol{L},\boldsymbol{L}',\boldsymbol{L}'',\boldsymbol{L}'''\bigr)}{\lambda^2}, \tag{9}$$

and $\boldsymbol{r}'\times\boldsymbol{r}''=\psi^2\lambda\,\boldsymbol{u}$.

Proof Let $\boldsymbol{P}=(P_0,P_1,P_2,P_3)^T=\boldsymbol{L}\wedge\boldsymbol{L}'\wedge\boldsymbol{L}''$ and $\boldsymbol{r}=\frac{1}{P_0}(P_1,P_2,P_3)^T$. From the definition of the wedge product it is clear that the denominator P_0 is equal to λ. If we define

$$\boldsymbol{R}:=\frac{1}{\lambda}\boldsymbol{P}=(1;\boldsymbol{r})\sim\boldsymbol{P},$$

then

$$\boldsymbol{R}' = \left(0; \boldsymbol{r}'\right) = \frac{1}{\lambda}\boldsymbol{P}' - \frac{\lambda'}{\lambda^2}\boldsymbol{P} = \frac{1}{\lambda}\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}''' - \frac{\lambda'}{\lambda^2}\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'' = \\ = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \left(\frac{1}{\lambda}\boldsymbol{L}''\right)'.$$

Therefrom it follows that

$$\boldsymbol{R}' \cdot \boldsymbol{L} = 0 = \boldsymbol{r}' \cdot \boldsymbol{u}, \quad \boldsymbol{R}' \cdot \boldsymbol{L}' = 0 = \boldsymbol{r}' \cdot \boldsymbol{u}',$$

which shows that $\boldsymbol{r}'$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{u}'$. Thus

$$\boldsymbol{r}' = \psi \boldsymbol{u} \times \boldsymbol{u}' \tag{10}$$

for some function ψ. Since

$$\left(0; \boldsymbol{u} \times \boldsymbol{u}'\right) = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \left(1; (0,0,0)^T\right),$$

the equation (10) can be written also as $\boldsymbol{R}' = \psi \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \left(1; (0,0,0)^T\right)$. If we multiply it by $\boldsymbol{L}''$, we obtain

$$\left(\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \left(\frac{1}{\lambda}\boldsymbol{L}''\right)'\right) \cdot \boldsymbol{L}'' = \psi \left(\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \left(1; (0,0,0)^T\right)\right) \cdot \boldsymbol{L}''.$$

The left hand side is equal to

$$-\frac{1}{\lambda} \det\left(\boldsymbol{L}, \boldsymbol{L}', \boldsymbol{L}'', \boldsymbol{L}'''\right)$$

and the right hand side evaluates to $-\lambda\psi$, which confirms (9). With the help of the cross product identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c}) = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\, \boldsymbol{a}, \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3,$$

we derive

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \psi \left(\boldsymbol{u} \times \boldsymbol{u}'\right) \times \left(\psi' \left(\boldsymbol{u} \times \boldsymbol{u}'\right) + \psi \left(\boldsymbol{u} \times \boldsymbol{u}''\right)\right) = \\ = \psi^2 \left(\boldsymbol{u} \times \boldsymbol{u}'\right) \times \left(\boldsymbol{u} \times \boldsymbol{u}''\right) = \psi^2 \lambda \boldsymbol{u},$$

which completes the proof. □

The next theorem reveals when a rational PB curve reduces to a polynomial one.

Theorem 2 *Suppose that the assumptions of Theorem 1 hold. Then the PB curve $\boldsymbol{r}$ determined by the dual form $\boldsymbol{L} = (-f; \boldsymbol{u})$ is a polynomial of the form $\boldsymbol{r} = \int \boldsymbol{p}$ iff the polynomial f equals*

$$f = \boldsymbol{u} \cdot \int \boldsymbol{p}, \tag{11}$$

where $\boldsymbol{p}$ is a polynomial field, orthogonal to $\boldsymbol{u}$ and $\boldsymbol{u}'$.

Proof If $\boldsymbol{L}$ determines a polynomial PB curve $\boldsymbol{r} = \int \boldsymbol{p}$ then it follows from (8) that $\boldsymbol{p}$ must be orthogonal to $\boldsymbol{u}$ and $\boldsymbol{u}'$. Furthermore, by (3) f should be of the form (11).

To prove the theorem in the other direction let us assume that (11) holds where $\boldsymbol{p} = (p_1, p_2, p_3)$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{u}'$. Then

$$f' = \boldsymbol{u}' \cdot \int \boldsymbol{p}, \quad f'' = \boldsymbol{u}'' \cdot \int \boldsymbol{p},$$

and the homogeneous form evaluates to

$$\boldsymbol{P} = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'' = \left(\lambda, \lambda \int p_1, \lambda \int p_2, \lambda \int p_3\right)^T \sim \left(1; \int \boldsymbol{p}\right)$$

with λ defined in (7). Thus by (4), $\boldsymbol{r} = \int \boldsymbol{p}$ is clearly a polynomial curve, and it is the PB one by the construction. This concludes the proof. □

A rational unit vector field of binormals can be constructed by using the stereographic projection (*e.g.* [16]). The other standard way is the quaternion approach. The latter will be used in the paper.

Space of quaternions $\mathbb{H}$ is a 4–dimensional vector space with the standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

$$\mathbf{1} = \left(1, (0,0,0)^T\right), \ \mathbf{i} = \left(0, (1,0,0)^T\right), \ \mathbf{j} = \left(0, (0,1,0)^T\right), \ \mathbf{k} = \left(0, (0,0,1)^T\right).$$

If we write

$$\mathcal{A} := (a_0, \boldsymbol{a}) := \left(a_0, (a_1, a_2, a_3)^T\right), \quad \mathcal{B} := (b_0, \boldsymbol{b}) := \left(b_0, (b_1, b_2, b_3)^T\right),$$

then

$$\mathcal{A} + \mathcal{B} = (a_0 + b_0, \boldsymbol{a} + \boldsymbol{b}), \quad \mathcal{A}\mathcal{B} = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}, a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}).$$

Equipped with the quaternion operations sum and product $\mathbb{H}$ becomes an algebra. The first component a_0 of the quaternion $\mathcal{A}$ is called the scalar part, and the remaining three components form the vector part of the quaternion $\mathcal{A}$, $\mathrm{vec}(\mathcal{A}) := (a_1, a_2, a_3)^T$. For a given $\mathcal{A}$ the components will be denoted also as

$$(\mathcal{A})_{\mathbf{1}} := a_0, \quad (\mathcal{A})_{\mathbf{i}} := a_1, \quad (\mathcal{A})_{\mathbf{j}} := a_2, \quad (\mathcal{A})_{\mathbf{k}} := a_3.$$

Quaternions with a zero scalar part are called pure quaternions. As usually, they are identified with vectors in $\mathbb{R}^3$,

$$\mathcal{A} = (0, \boldsymbol{a}) \equiv \boldsymbol{a}.$$

By $\mathcal{A}^* := (a_0, -\boldsymbol{a})$ we denote the conjugate of $\mathcal{A} = (a_0, \boldsymbol{a})$, $\|\mathcal{A}\| = \sqrt{\mathcal{A}\mathcal{A}^*}$ is its norm, and $\mathbb{H}[t]$ is the ring of polynomials over $\mathbb{H}$. Any nontrivial quaternion polynomial $\mathcal{A} \in \mathbb{H}[t]$ induces three rational unit vector fields $\boldsymbol{e}_i = \boldsymbol{e}_i(\mathcal{A})$, $i = 1, 2, 3$, determined as pure quaternions

$$\boldsymbol{e}_1 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A}\,\mathbf{i}\,\mathcal{A}^*, \quad \boldsymbol{e}_2 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A}\,\mathbf{j}\,\mathcal{A}^*, \quad \boldsymbol{e}_3 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A}\,\mathbf{k}\,\mathcal{A}^*. \tag{12}$$

They form an orthonormal frame $(\boldsymbol{e}_i)_{i=1}^3$, called the Euler-Rodrigues frame (E-R frame). The angular velocity vector field $\boldsymbol{\omega}$, given by $\boldsymbol{e}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i$, is in the E-R frame coordinate system equal to

$$\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3,$$

where

$$\omega_1 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_2' = -\boldsymbol{e}_2 \cdot \boldsymbol{e}_3', \quad \omega_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3' = -\boldsymbol{e}_3 \cdot \boldsymbol{e}_1', \quad \omega_3 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' = -\boldsymbol{e}_1 \cdot \boldsymbol{e}_2'. \tag{13}$$

The speed of the E-R frame can be expressed as

$$\boldsymbol{e}_1' = \omega_3 \boldsymbol{e}_2 - \omega_2 \boldsymbol{e}_3, \quad \boldsymbol{e}_2' = -\omega_3 \boldsymbol{e}_1 + \omega_1 \boldsymbol{e}_3, \quad \boldsymbol{e}_3' = \omega_2 \boldsymbol{e}_1 - \omega_1 \boldsymbol{e}_2. \tag{14}$$

Furthermore, let us define

$$\rho := \|\mathcal{A}\|^2, \quad \boldsymbol{h}_i := \rho\, \boldsymbol{e}_i, \quad \boldsymbol{\nu} := (\nu_1, \nu_2, \nu_3)^T := \rho\, (\omega_1, \omega_2, \omega_3)^T. \tag{15}$$

Note that $\boldsymbol{h}_i$ and ν_i are the numerators of $\boldsymbol{e}_i$ and ω_i respectively. In terms of the quaternion polynomial $\mathcal{A}$ the coefficients $\boldsymbol{\nu}$ read

$$(\nu_1, \nu_2, \nu_3)^T = 2\, \mathrm{vec}\Big(\mathcal{A}^* \mathcal{A}'\Big). \tag{16}$$

To construct a PB curve, we choose the binormal $\boldsymbol{b}$ to be equal to $\boldsymbol{e}_1$. More precisely, if $\boldsymbol{u}$ in Theorem 1 is chosen as

$$\boldsymbol{u} = g\boldsymbol{h}, \quad \boldsymbol{h} := \boldsymbol{h}_1, \tag{17}$$

then the dual form

$$\boldsymbol{L} := (-f; g\boldsymbol{h}) \tag{18}$$

defines a rational PB curve $\boldsymbol{r}$ for any polynomials f and g. To obtain a primitive dual form $\boldsymbol{L}$ we must choose f and g such that $\gcd(f, g) = 1$. However, this does not assure that the components of $\boldsymbol{L}$ are relatively prime too. In [4] it was shown that there might exist a nonconstant factor ϑ, such that $\boldsymbol{h} = \vartheta \boldsymbol{h}_R$, $\gcd(\boldsymbol{h}_R) = 1$, even if the components of a quaternion polynomial $\mathcal{A}$ are relatively prime. In that case one replaces $\boldsymbol{L}$ by

$$(-f\vartheta; g\boldsymbol{h}) \sim (-f; g\boldsymbol{h}_R) =: \boldsymbol{L}_R,$$

and $\boldsymbol{L}_R$ is a primitive dual form for any relatively prime polynomials f and g. Note that by [4] such a common factor ϑ can't have real roots.

The hodograph of a PB curve derived from (18) follows from (8) as

$$\boldsymbol{r}' = \psi\, (g\boldsymbol{h}) \times (g\boldsymbol{h})' = \psi g^2 \boldsymbol{h} \times \boldsymbol{h}',$$

where ψ depends on the chosen functions f and g. Further, from (14) and (15) we conclude

$$\boldsymbol{h} \times \boldsymbol{h}' = \rho^2 \boldsymbol{e}_1 \times \boldsymbol{e}_1' = \rho^2 (\omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3) = \nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3,$$

what simplifies the hodograph to

$$\boldsymbol{r}' = \psi g^2 (\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3). \tag{19}$$

The irregular points clearly arise from the zeros of a function ψ. But, the hodograph can vanish also if the components of $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$ have a common factor. The conditions, that imply such a common factor in the case when the quaternion polynomial $\mathcal{A}$ is a linear or a quadratic one, are examined in [21].

4 Examples of rational PB curves

Let us first consider the PB curves generated by a linear quaternion polynomial

$$\mathcal{A}(t) = \mathcal{A}_0 B_0^1(t) + \mathcal{A}_1 B_1^1(t), \quad \mathcal{A}_i = \left(a_{i,0}, (a_{i,1}, a_{i,2}, a_{i,3})^T\right), \tag{20}$$

where

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-1}, \quad i = 0, 1, \ldots, n,$$

are the Bernstein basis polynomials of the degree n. As it is shown in [21],

$$\deg(\nu_i) = 0, \quad \deg(\boldsymbol{h}_i) \leq 2, \quad i = 1, 2, 3.$$

The dual form (18) is thus of the degree $m = \max\{2 + \deg(g), \deg(f)\}$. Further, (19) implies

$$\|\boldsymbol{r}'\|^2 = \boldsymbol{r}' \cdot \boldsymbol{r}' = \psi^2 g^4 \left(\nu_2^2 + \nu_3^2\right) \rho^2.$$

But $\nu_2^2 + \nu_3^2$ is a constant, so the curve $\boldsymbol{r}$ has a pythagorean hodograph, which proves the following assertion.

Proposition 1 *Rational PB curves derived from a linear quaternion polynomial* (20) *are rational DPH curves for any polynomials f and g.*

Since rational PH curves have already been thoroughly considered in [21], we skip the details here. The next case to be examined are PB curves generated by a quadratic quaternion polynomial

$$\mathcal{A}(t) = \mathcal{A}_0 B_0^2(t) + \mathcal{A}_1 B_2^1(t) + \mathcal{A}_2 B_2^2(t), \quad \mathcal{A}_i = \left(a_{i,0}, (a_{i,1}, a_{i,2}, a_{i,3})^T\right). \tag{21}$$

In this case

$$\deg(\nu_i) = 2, \quad \deg(\boldsymbol{h}_i) \leq 4, \quad i = 1, 2, 3,$$

and the dual form (18) is of the degree $m = \max\{4 + \deg(g), \deg(f)\}$. If we choose $g \equiv 1$ and $\deg(f) \leq 4$ then the obtained rational PB curve is of the degree ≤ 6. As an example let us choose

$$\mathcal{A}_0 = \left(1, (0,0,0)^T\right), \quad \mathcal{A}_1 = \left(0, (2,-1,1)^T\right), \quad \mathcal{A}_2 = \left(0, (3,3,0)^T\right),$$

and $f(t) = 2t^3 + t^4$. From (12), (15) and (17) we compute

$$\boldsymbol{h}(t) = \begin{pmatrix} -27t^4 + 16t^3 + 14t^2 - 4t + 1 \\ 2t\left(-7t^3 + 28t^2 - 14t + 2\right) \\ -2t\left(3t^3 - 2t^2 + t - 2\right) \end{pmatrix}.$$

The evaluation of the wedge product $\boldsymbol{P} = (P_0, P_1, P_2, P_3)^T = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}''$ gives the homogeneous coordinates of the curve $\boldsymbol{r} = \frac{1}{P_0}(P_1, P_2, P_3)^T$ (see Fig. 1) as

$$\begin{aligned}
P_0(t) &= 16\left(770t^6 - 2334t^5 + 1641t^4 - 712t^3 - 24t^2 + 78t - 13\right), \\
P_1(t) &= 16t^3\left(35t^3 + 102t^2 - 39t - 26\right), \\
P_2(t) &= 8t\left(91t^5 - 258t^4 - 12t^3 - 28t^2 + 3t + 6\right), \\
P_3(t) &= -16t\left(392t^5 - 231t^4 + 84t^3 - 14t^2 - 18t + 3\right).
\end{aligned} \tag{22}$$

The function ψ computed by (9) equals

$$\psi(t) = \frac{3\left(763t^4 - 70t^3 - 102t^2 + 26t + 13\right)}{4\left(-770t^6 + 2334t^5 - 1641t^4 + 712t^3 + 24t^2 - 78t + 13\right)^2}.$$

Since ψ has no real zeros and the components of $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$ do not have a common factor, the curve $\boldsymbol{r}$ is regular. The denominator of ψ vanishes at parameter values

$$t \in \{-0.299505,\ 2.27122,\ 0.241973 \pm 0.0839458\,\mathrm{i},\ 0.287754 \pm 0.543649\,\mathrm{i}\},$$

that corresponds to singular points of $\boldsymbol{r}$.

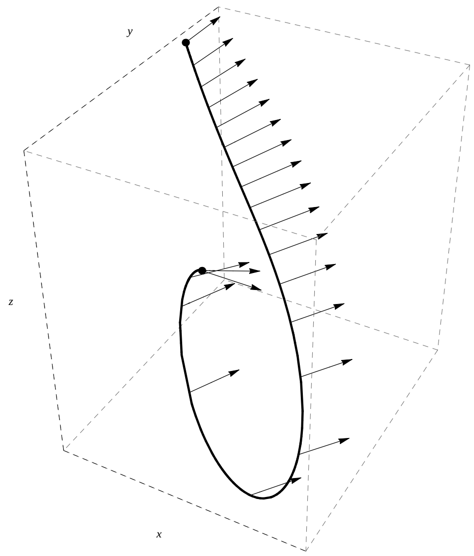

Fig. 1 The rational PB curve of degree 6 defined by (22) on the parameter domain $[0, 1]$.

As the example shows, a quadratic quaternion polynomial gives rational PB curves of degree 6, provided $\deg(g) = 0$ and $\deg(f) \leq 4$. In general, if polynomials g and f satisfy $\deg(g) = 0$ and $\deg(f) \leq 2k$, then a quaternion polynomial of degree k generates rational PB curves of the class $m = 2k$, *i.e.*, PB curves of the degree $\leq 3(2k) - 6 = 6(k-1)$. If the degree of f or g is increased, the class of a curve is increased too.

5 Polynomial PB curves

In the beginning of Section 3 it was observed that there is a small distinction between a rational PB curve and a polynomial one of the same degree as far as the number of degrees of freedom is concerned. So one is tempted to study the polynomial case in particular since rationals might not offer enough additional flexibility w.r.t. the polynomial ones. Theorem 2 provides a shortcut to polynomial PB curves, and together with the relation (19) implies the following observation.

Proposition 2 *Suppose that $\mathcal{A} \in \mathbb{H}[t]$ is a given quaternion polynomial which determines the E-R frame* (12)*, and the corresponding* (13) *and* (15)*. Then $\boldsymbol{r} = \int \boldsymbol{p}$ is a polynomial PB curve for any rational function τ, for which the field $\boldsymbol{p}$,*

$$\boldsymbol{p} = \tau\left(\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3\right) = 2\tau \left(\left(\mathcal{A}^* \mathcal{A}'\right)_{\mathbf{j}} \mathcal{A}\,\mathbf{j}\,\mathcal{A}^* + \left(\mathcal{A}^* \mathcal{A}'\right)_{\mathbf{k}} \mathcal{A}\,\mathbf{k}\,\mathcal{A}^* \right), \tag{23}$$

results polynomial.

The last equality in (23) follows from (16). The simplest function τ in (23) is a constant, and the degree of $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$ is $4k-2$ in general where k is the degree of $\mathcal{A}$. This yields degree $4k-1$ polynomial PB curves. If τ is a polynomial of the degree ≥ 1, its real zeroes contribute to the set of irregular points of the curve $\boldsymbol{r}$ what one avoids usually if possible. If the components of $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$ have common factors then τ could also be rational with the denominator part which consists of these common factors. Such a choice removes irregular points of the curve (except at infinity), and it decreases the degree of $\boldsymbol{r}$ too. Note that this degree may be lowered also by decreasing the degree of $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$. In [21, Sec. 6], a comprehensive analysis of conditions imposed on quadratic quaternion polynomials $\mathcal{A}$ that decrease the degree of $\boldsymbol{p}$ is outlined. Rather than to repeat them in the full generality, we apply the outcome provided there to generate examples of polynomial PB curves of the degrees $6, 5$, and 4. This answers a query posted in [8] too: there exist polynomial PB curves, even of small degrees ≥ 4.

An example of a PB curve of the degree 6. The quaternion polynomial

$$\mathcal{A} = \left(t^2 - t + 5, (t+2, t+2, t+2)^T\right)$$

generates the hodograph direction

$$\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3 = 4 \begin{pmatrix} 2(t-4)t \\ (1-t)t\left(t^3 + 2t^2 + 2t + 6\right) \\ t^5 + 4t^4 - 4t^3 - 10t^2 + 2t + 10 \end{pmatrix},$$

which is of the degree 5 only.The choice $\tau = \frac{1}{4}$ in (23) determines the polynomial PB curve of the degree 6,

$$\boldsymbol{r}_6(t) = \frac{1}{30} \begin{pmatrix} 20(t-6)t^2 \\ -t^2\left(5t^4 + 6t^3 + 40t - 90\right) \\ t\left(5t^5 + 24t^4 - 30t^3 - 100t^2 + 30t + 300\right) \end{pmatrix} + \boldsymbol{r}_6(0),$$

with the rational binormal

$$\boldsymbol{b}_6(t) = \frac{1}{t^4 + 2t^3 - 2t^2 - 8t + 10} \begin{pmatrix} -t^4 - 2t^3 + 2t^2 + 8t - 6 \\ 2\left(t^2 + 2t - 4\right) \\ 2t^2 \end{pmatrix}.$$

An example of a PB curve of the degree 5. Let the (linearly dependent) quaternion polynomial coefficients be given as

$$\mathcal{A}_0 = \left(5, (2,2,2)^T\right), \quad \mathcal{A}_1 = \frac{1}{2}\left(9, (5,5,5)^T\right), \quad \mathcal{A}_2 = \left(5, (3,3,3)^T\right).$$

Then

$$\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3 = -2\left(t^2+4t-7\right)\begin{pmatrix} 4(t+2)^2 \\ ((t-2)t+3)^2 \\ \left(t^2+7\right)^2 \end{pmatrix},$$

and hence the choice

$$\tau = -\frac{3}{40\left(t^2+4t-7\right)}$$

reduces the degree of the hodograph direction to 4. The curve reads

$$\boldsymbol{r}_5(t) = \frac{1}{100}\begin{pmatrix} 20t\left(t^2+6t+12\right) \\ t\left(3t^4-15t^3+50t^2-90t+135\right) \\ t\left(3t^4+70t^2+735\right) \end{pmatrix} + \boldsymbol{r}_5(0),$$

and its rational binormal is given as

$$\boldsymbol{b}_5(t) = \frac{1}{t^4-2t^3+14t^2+2t+37}\cdot\begin{pmatrix} \left(t^2+7\right)\left(t^2-2t+3\right) \\ 2(t+2)\left(t^2+7\right) \\ -2(t+2)\left(t^2-2t+3\right) \end{pmatrix}.$$

An example of a PB curve of the degree 4. The quaternion polynomial

$$\mathcal{A} = \left(t^2-5, (-6t, 8, 4)^T\right)$$

yields the degree 5 hodograph direction, but with a common quadratic divisor,

$$\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3 = 16\left(t^2+4\right)\begin{pmatrix} 120 \\ -2t^3-9t^2+24t-48 \\ -t^3+18t^2+12t+96 \end{pmatrix}.$$

So the choice

$$\tau = \frac{1}{300\left(t^2+4\right)}$$

gives

$$\boldsymbol{r}_4(t) = \frac{1}{75}\begin{pmatrix} 480t \\ -2t\left(t^3+6t^2-24t+96\right) \\ -t\left(t^3-24t^2-24t-384\right) \end{pmatrix} + \boldsymbol{r}_4(0),$$

and the rational binormal

$$\boldsymbol{b}_4(t) = \frac{1}{\left(t^2+4\right)\left(t^2+24\right)}\begin{pmatrix} t^4+28t^2-64 \\ 8\left(t^2-12t-4\right) \\ -16(t-1)(t+4) \end{pmatrix}.$$

All three example curves are shown in Fig. 2.

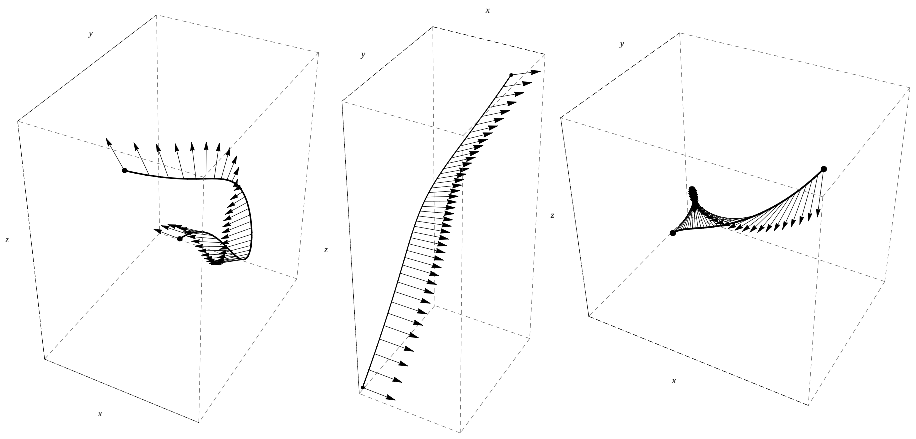

Fig. 2 The polynomial PB curves $\boldsymbol{r}_6$ (left), $\boldsymbol{r}_5$ (middle), and $\boldsymbol{r}_4$ (right) on $[-2, 2]$, with $\boldsymbol{r}_6(0) = \boldsymbol{r}_5(0) = \boldsymbol{r}_4(0) = (0, 0, 0)^T$, and the corresponding binormal fields $\boldsymbol{b}_6, \boldsymbol{b}_5$, and $\boldsymbol{b}_4$.

6 Rational rotation-minimizing conformal frames

A necessary condition for a spatial curve $\boldsymbol{r}$ to possess a rational conformal frame is that $\boldsymbol{r}$ is a PB curve. For any PB curve derived from a quaternion polynomial $\mathcal{A}$ by a dual construction the associated E-R frame has the property of being conformal and rational. Moreover, for any quaternion polynomial $\mathcal{Q} = \left(q_0, (q_1, 0, 0)^T\right)$ the E-R frame obtained from a product $\mathcal{A}\mathcal{Q}$ is also a conformal frame of a curve $\boldsymbol{r}$. Note that the same holds true for PH curves and the associated adapted frames. In both cases the rational rotation-minimizing frames could be constructed from the E-R frame, where $\boldsymbol{e}_1$ is taken as the unit binormal/tangent vector. Furthermore, conditions for a curve to possess a rational rotation-minimizing conformal/adapted frame depend only on a chosen quaternion polynomial $\mathcal{A}$ and not on the way the curve is derived from. Therefore all known results for RRMA frames on PH curves can be used also for RRMC frames on PB curves.

The analysis of RRMA frames is usually done by using a Hopf map representation of a PH curve, but all the results can be reformulated to the quaternion setting in a quite simple way. Some of the important results known for RRMA frames are summarized and applied to RRMC frames in the next two propositions. In the first one, sufficient and necessary conditions for the existence of a RRMC/RRMA frame are stated. The second proposition reveals the connection between the coefficients of a quadratic quaternion $\mathcal{A}$ that defines a curve which possesses a RRMC/RRMA frame. Proofs and more details can be found, *e.g.*, in [5], [13] and [15].

Proposition 3 *Let $\mathcal{A}$ be a quaternion polynomial which defines the binormal/tangent and the E-R frame of a PB/PH curve. The curve possesses a RRMC/RRMA frame iff there exists a quaternion polynomial $\mathcal{Q} := \left(q_0, (q_1, 0, 0)^T\right)$, where q_0 and q_1 are relatively prime, such that*

$$\left(\frac{\mathcal{A}^*\mathcal{A}'}{\|\mathcal{A}\|^2}\right)_{\mathbf{i}} = \left(\frac{\mathcal{Q}^*\mathcal{Q}'}{\|\mathcal{Q}\|^2}\right)_{\mathbf{i}}.$$

Proposition 4 *A PB/PH curve generated from a quadratic quaternion polynomial $\mathcal{A} = \mathcal{A}_0 B_0^2 + \mathcal{A}_1 B_1^2 + \mathcal{A}_2 B_2^2$ has a RRMC/RRMA frame iff the quaternion coefficients $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$*

satisfy

$$vec\Big(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*\Big) = \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^*. \tag{24}$$

Moreover, the RRMC/RRMA frame vectors can be determined as $\boldsymbol{e}_i(\mathcal{B})$, $i = 1, 2, 3$, *where*

$$\mathcal{B} := \mathcal{A}\mathcal{Q}^*, \tag{25}$$

with

$$\begin{aligned} q_0 &:= \|\mathcal{A}_0\|^2\, B_0^2 + \left(\mathcal{A}_0^*\mathcal{A}_1\right)_{\mathbf{1}}\, B_1^2 + \|\mathcal{A}_1\|^2\, B_2^2, \\ q_1 &:= \left(\mathcal{A}_0^*\left(\mathcal{A}_1 B_1^2 + \mathcal{A}_2 B_2^2\right)\right)_{\mathbf{i}}. \end{aligned} \tag{26}$$

Using Proposition 2 and Proposition 4 one can easily construct polynomial PB curves of degree seven that possess RRMC frames. As an example let the quaternion polynomial $\mathcal{A}$ be defined by (21) with

$$\mathcal{A}_0 = \Big(1, (0,0,0)^T\Big), \quad \mathcal{A}_1 = \Big(0, (2,-1,1)^T\Big), \quad \mathcal{A}_2 = \Big(2, (1,-4,-,4)^T\Big), \tag{27}$$

which generates the hodograph direction

$$\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3 = 4\begin{pmatrix} 16t^2\left(3t^4 - 45t^3 + 39t^2 - 11t + 3\right) \\ 548t^6 - 564t^5 + 270t^4 - 200t^3 + 72t^2 - 6t - 1 \\ 296t^6 - 612t^5 + 102t^4 - 48t^3 + 42t^2 - 18t + 1 \end{pmatrix}.$$

The choice $\tau = \frac{35}{4}$ in (23) determines the polynomial PB curve of the degree 7,

$$\boldsymbol{r}(t) = \begin{pmatrix} 4t^3\left(60t^4 - 1050t^3 + 1092t^2 - 385t + 140\right) \\ 5t\left(548t^6 - 658t^5 + 378t^4 - 350t^3 + 168t^2 - 21t - 7\right) \\ t\left(1480t^6 - 3570t^5 + 714t^4 - 420t^3 + 490t^2 - 315t + 35\right) \end{pmatrix} + \boldsymbol{r}(0), \tag{28}$$

shown in Fig. 3, with the rational binormal

$$\boldsymbol{b}(t) = \frac{1}{58t^4 - 52t^3 + 34t^2 - 4t + 1}\begin{pmatrix} 22t^4 + 20t^3 - 18t^2 + 4t - 1 \\ 4t\left(6t^3 - 8t^2 + 9t - 1\right) \\ 4t\left(12t^3 - 14t^2 + 3t + 1\right) \end{pmatrix}.$$

Since the coefficients (27) fulfil the condition (24), the obtained polynomial PB curve (28) possesses a RRMC frame. More precisely, the E-R frame $\boldsymbol{e}_i(\mathcal{B})$, $i = 1, 2, 3$, of the quaternion polynomial $\mathcal{B}$, determined by (25), (26) and (27) defines the degree eight RRMC frame of the curve (28). Let us illustrate the obtained results on a ruled surface, defined as

$$S(t,\xi) := \boldsymbol{r}(t) + \xi\boldsymbol{e}_3(\mathcal{B}(t)), \tag{29}$$

where $\boldsymbol{r}$ is given in (28). Tangent planes of the surface coincide with the osculating planes along the curve $\boldsymbol{r}$. Moreover, such a surface is in [8] called a rotation-minimizing ruled surface, since the pitch (a measure of the variation of the tangent plane along each ruling) is the smallest possible among all ruled surfaces determined by $\boldsymbol{r}$ and an osculating-plane vector. Note that the same holds true if $\boldsymbol{e}_2(\mathcal{B})$ or any vector of a fixed orientation relative to $\boldsymbol{e}_2(\mathcal{B})$ and $\boldsymbol{e}_3(\mathcal{B})$ is taken instead of $\boldsymbol{e}_3(\mathcal{B})$ in (29). Since $\boldsymbol{r}$ is a polynomial curve, S is a rational surface. The curve (28) together with the rational rotation-minimizing ruled surface S is shown in Fig. 3.

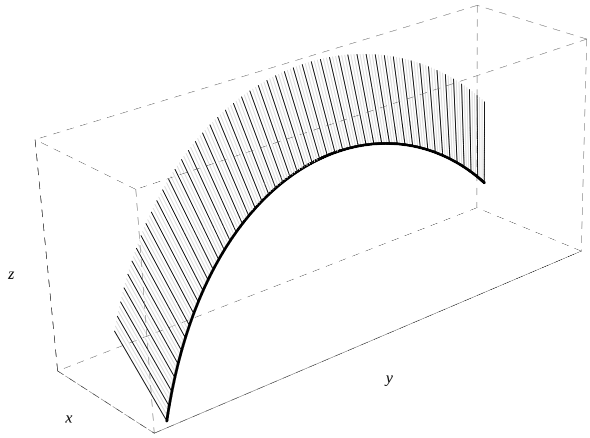

Fig. 3 The curve (28) and a rational rotation-minimizing ruled surface given in (29) for $t \in [0, 0.15]$ and $\xi \in [0, 1]$.

In a similar way rational PB curves of degree 6 that possess RRMC frames can be constructed. One only needs to assure that a quadratic quaternion polynomial that defines a dual form satisfies (24). In view of Section 5 one is therefore tempted to look also for polynomial PB curves of degrees ≤ 6 generated by a quadratic quaternion polynomial which possess a RRMC frame. Unfortunately, the following assertion wipes this possibility out.

Theorem 3 *There are no polynomial curves of the degree ≤ 6 generated by a quadratic quaternion polynomial which are accompanied by the corresponding RRMC frame.*

Proof The existence of the RRMC frame depends on the curve generating quaternion polynomial $\mathcal{A} \in \mathbb{H}[t]$ only. The qualifying relation (24) stays the same if $\mathcal{A}$ is multiplied by a constant nontrivial quaternion. So we may assume that the leading coefficient of $\mathcal{A} = \mathcal{A}_0 B_0^2 + \mathcal{A}_1 B_1^2 + \mathcal{A}_2 B_2^2$ is equal to $\mathbf{1}$, $\mathcal{A}_2 = \mathbf{1} - \mathcal{A}_0 + 2\mathcal{A}_1$. It is simpler to study $\mathcal{A}$ in new variables $\mathcal{C}_i := \left(c_{0,i}, (c_{1,i}, c_{2,i}, c_{3,i})^T\right)$, $i = 0, 1$, the coefficients of the quaternion polynomial in the standard basis. Note that $\mathcal{C}_0 = \mathcal{A}_0$, $\mathcal{C}_1 = 2(\mathcal{A}_1 - \mathcal{A}_0)$, and $\mathcal{A} = \mathcal{C}_0 + \mathcal{C}_1 t + \mathbf{1}t^2$. In variables $c_{i,j}$ the relation (24) turns out as

$$\begin{aligned} c_{0,0} - \frac{1}{4}\left(c_{0,1}^2 + c_{1,1}^2 - c_{2,1}^2 - c_{3,1}^2\right) &= 0, \\ c_{3,0} - \frac{1}{2}\left(c_{1,1}c_{2,1} + c_{0,1}c_{3,1}\right) &= 0, \\ -c_{2,0} + \frac{1}{2}\left(c_{0,1}c_{2,1} - c_{1,1}c_{3,1}\right) &= 0. \end{aligned} \tag{30}$$

Let us express $c_{0,0}$, $c_{3,0}$, and $c_{2,0}$ from (30). The leading part of the hodograph direction $\nu_2 \boldsymbol{h}_2 + \nu_3 \boldsymbol{h}_3$ in the remaining variables $c_{i,j}$ reads

$$\begin{pmatrix} -6c_{1,1}\left(c_{2,1}^2 + c_{3,1}^2\right)t^4 - 8\left(c_{1,0} + c_{0,1}c_{1,1}\right)\left(c_{2,1}^2 + c_{3,1}^2\right)t^3 + \dots \\ -2c_{2,1}t^6 + 6\left(c_{1,1}c_{3,1} - c_{0,1}c_{2,1}\right)t^5 + \dots \\ -2c_{3,1}t^6 - 6\left(c_{1,1}c_{2,1} + c_{0,1}c_{3,1}\right)t^5 + \dots \end{pmatrix}. \tag{31}$$

There is no way to lower the degree of $\nu_2\boldsymbol{h}_2 + \nu_3\boldsymbol{h}_3$ here. Namely, from (31) we observe that in this case one should have $c_{2,1} = c_{3,1} = 0$, and further from (30)

$$c_{0,0} = \frac{1}{4}\left(c_{0,1}^2 + c_{1,1}^2\right), \quad c_{2,0} = c_{3,0} = 0.$$

But then $\mathcal{A} = \left(\frac{1}{4}\left((c_{0,1}+2t)^2 + c_{1,1}^2\right), (c_{1,1}t + c_{1,0}, 0, 0)^T\right)$ and $\boldsymbol{e}_1 = (1,0,0)^T$, so $\nu_2\boldsymbol{h}_2 + \nu_3\boldsymbol{h}_3$ vanishes identically. Suppose now that $c_{2,1} \neq 0$, $c_{3,1} \neq 0$. The only way to lower the degree of the hodograph is to assure that the components (31) have a common polynomial divisor. But then the Gröbner basis of the ideal spanned by the components (31) should vanish at zeros of this polynomial too. The first polynomial of the Gröbner basis with respect to the variable t is constant, *i.e.*,

$$\left(c_{2,1}^2 + c_{3,1}^2\right)\left(4c_{1,0}^2 - 4c_{0,1}c_{1,1}c_{1,0} + c_{1,1}^2\left(c_{0,1}^2 + c_{1,1}^2 + c_{2,1}^2 + c_{3,1}^2\right)\right)^2 \cdot$$
$$\left(16c_{1,0}^2 - 16c_{0,1}c_{1,1}c_{1,0} + 4c_{0,1}^2c_{1,1}^2 + \left(2c_{1,1}^2 + c_{2,1}^2 + c_{3,1}^2\right)^2\right)^2.$$

Since $c_{2,1} \neq 0$, $c_{3,1} \neq 0$, it can vanish only if

$$c_{1,0} = \frac{1}{2}\left(c_{0,1}c_{1,1} \pm \sqrt{-c_{1,1}^2\left(c_{1,1}^2 + c_{2,1}^2 + c_{3,1}^2\right)}\right),$$

or

$$c_{1,0} = \frac{1}{2}c_{0,1}c_{1,1} \pm \frac{1}{4}\sqrt{-\left(2c_{1,1}^2 + c_{2,1}^2 + c_{3,1}^2\right)^2},$$

what it is clearly not possible since the coefficients $c_{i,j}$ are real unless $c_{1,1} = c_{1,0} = 0$. In this case,

$$\nu_2\boldsymbol{h}_2 + \nu_3\boldsymbol{h}_3 = -\frac{1}{32}\left(4t^2 + 4c_{0,1}t + c_{0,1}^2 + c_{2,1}^2 + c_{3,1}^2\right)^3\begin{pmatrix}0\\c_{2,1}\\c_{3,1}\end{pmatrix}$$

and the curve $\boldsymbol{r}$ is reparameterized line in the $y-z$ plane, with a trivial binormal $(0,0,0)^T$. The proof is completed. □

Similar negative result on the existence of RRMC frames on DPH curves of degree < 7 has already been proven in [8]. Theorem 3 extends this result to a wider class of PB curves.

7 Closure

Recently, the rotation-minimizing conformal frames of spatial curves have been introduced in [8] and it was shown that polynomial DPH curves of degree < 7 can not possess rational rotation-minimizing conformal frames. In this paper a new class of curves having a pythagorean binormal have been introduced which gives the answer to the first question given in [8]: are there curves that are not DPH and have a rational unit binormal. A proposed dual construction leads to rational PB curves of the degree 6 and polynomial PB curves of the degree 7 having the RRMC frames and gives a positive answer to another question exposed in [8]: do curves of degree ≥ 6 having a RRMC frame exist.

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