# Hermite interpolation by rational $G^{k}$ motions of low degree 

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#### Abstract

Interpolation by rational spline motions is an important issue in robotics and related fields. In this paper a new approach to rational spline motion design is described by using techniques of geometric interpolation. This enables us to reduce the discrepancy in the number of degrees of freedom of the trajectory of the origin and of the rotational part of the motion. A general approach to geometric interpolation by rational spline motions is presented and two particularly important cases are analyzed, i.e., geometric continuous quartic rational motions and second order geometrically continuous rational spline motions of degree six. In both cases sufficient conditions on the given Hermite data are found which guarantee the uniqueness of the solution. If the given data do not fulfill the solvability conditions, a method to perturb them slightly is described. Numerical examples are presented which confirm the theoretical results and provide an evidence that the obtained motions have nice shapes.


Key words: motion design, geometric interpolation, rational spline motion, geometric continuity

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## 1 Introduction

Rational spline motions are motions of a rigid body with the property that each point travels along a trajectory which is a rational spline curve of a certain degree. The study of these motions can be traced back to classical texts in kinematical geometry [1]. For example, rational motions of degree two were analyzed thoroughly by G. Darboux in the $19^{\text {th }}$ century (see e.g. [1], [2]). More recently, rational spline motions have found numerous applications in robotics, computer graphics and related fields [3,4].

Given a sequence of positions of a rigid body, a rational spline motion that matches these data can be found by suitable interpolation algorithms. For instance, such algorithms can be derived by generalizing known techniques for curve design to the case of motions. Standard algorithms, such as $C^{1}$ Hermite interpolation [5], however, lead to rational motions of a relatively high polynomial degree. This is due to the discrepancy in the number of degrees of freedom that are present in the rotational and the translational part of a rational motion.

Rational motions of lower degree can be obtained by using geometric interpolation techniques. A first attempt was presented in [6], based on Bennett biarcs on Study's quadric, which give rational motions of degree 4. As an advantage of this method, collision detection between fixed and moving polyhedra can be performed by analyzing certain polynomials of degree 4 . As a disadvantage, however, this method generates motions of constant chirality only. Another method, which uses more general rational motions of degree 4 to overcome this limitation (but leading to slightly more involved collision tests) is described in [7] (see also Section 5).

More generally, geometric interpolation techniques have the potential to produce rational spline motions of the lowest possible degree needed to match certain data (e.g., Hermite-type boundary data). As an example, it was shown in [8] that a planar cubic can (under some reasonable restrictions) interpolate six geometric data, i.e., two boundary points together with two tangent directions and two curvature vectors. As a consequence, the approximation order is six, in comparison to the standard fourth order cubic approximation. Later, several other geometric interpolation schemes using polynomial curves also in higher dimensional spaces were developed (see [9], [10] and [11], e.g., and references therein). In addition, using geometric interpolation yields an automatically chosen parameterization. This is an important advantage in practice, since in classical interpolation methods, the parameterization should be chosen by an experienced designer and even this does not provide satisfactory motions in general.

In this paper we consider a generalization of the approach used in [7]. Geometric interpolation by parabolic splines was used there to construct $G^{1}$ quartic rational spline motions. Although the results are promising, the main drawback of this
method is the lowest possible degree of smoothness which might be insufficient in robotics and related fields. In order to obtain spline motions with continuous second order derivatives (after a suitable reparameterization) one has to consider $G^{2}$ continuity. We do the first obvious generalization by considering cubic geometric interpolation, which leads to $G^{2}$ rational spline motions of degree six. In addition, we derive a new $G^{1}$ quartic rational spline motion, for which examples show better shapes in comparison with the results of [7].

The paper is organized as follows. In the next section rational motions are presented. In Section 3, geometric continuity for motions is explained and a general approach to $G^{k}$ continuous Hermite spline motions is described. General interpolation problem by $G^{k}$ continuous Hermite spline motions is stated in Section 4. Section 5 deals with $G^{1}$ Hermite interpolation by rational quartic spline motions, and Section 6 considers the main problem of the paper, cubic $G^{2}$ Hermite interpolation by rational spline motions of degree six. A brief explanation how the translational part could be obtained is given in Section 7. In the next section some numerical examples are given and the paper is concluded with Section 9 that summarizes the main results of the paper and identifies possible future investigations.

## 2 Rational motions

A motion of a rigid body can be described by the trajectory $\boldsymbol{c}=\left(c_{1}, c_{2}, c_{3}\right)^{\top}$ of the origin of the moving system and by the $3 \times 3$ rotation matrix $\mathcal{R}$. By using quaternions $\boldsymbol{Q}=\left(q_{0}, q_{1}, q_{2}, q_{3}\right) \in \mathbb{H}$, the rotation matrix $\mathcal{R}$ can be represented as
$\mathcal{R}=\frac{1}{q_{0}^{2}+q_{1}^{2}+q_{2}^{2}+q_{3}^{2}}\left[\begin{array}{ccc}q_{0}^{2}+q_{1}^{2}-q_{2}^{2}-q_{3}^{2} & 2\left(q_{1} q_{2}-q_{0} q_{3}\right) & 2\left(q_{1} q_{3}+q_{0} q_{2}\right) \\ 2\left(q_{1} q_{2}+q_{0} q_{3}\right) & q_{0}^{2}-q_{1}^{2}+q_{2}^{2}-q_{3}^{2} & 2\left(q_{2} q_{3}-q_{0} q_{1}\right) \\ 2\left(q_{1} q_{3}-q_{0} q_{2}\right) & 2\left(q_{2} q_{3}+q_{0} q_{1}\right) & q_{0}^{2}-q_{1}^{2}-q_{2}^{2}+q_{3}^{2}\end{array}\right]$.
Note that all nonzero quaternions $\lambda \boldsymbol{Q}(\lambda \in \mathbb{R}, \lambda \neq 0)$ lying on the same line passing through the origin represent the same rotation. This equivalence relation defines a 3 -dimensional projective space, described by homogeneous quaternion coordinates. The bijective mapping between this space and the space of rotations is called the kinematic mapping (see [1]).

The trajectory of an arbitrary point $\widehat{\boldsymbol{p}}$ of the moving system is

$$
\begin{equation*}
\boldsymbol{p}(t)=\boldsymbol{c}(t)+\mathcal{R}(t) \widehat{\boldsymbol{p}} . \tag{1}
\end{equation*}
$$

Here $\widehat{\boldsymbol{p}}$ is expressed in a fixed local coordinate system of the original body position. In particular we are interested in rational spline motions which are obtained by choosing rational spline (i.e., piecewise rational) functions $q_{i}$ and $c_{i}$ representing
the coordinates of the quaternion and of the trajectory of the origin.
Rational motions can be classified by the degree of the curves involved, which is called the degree of the motion. In particular, by considering quadratic or cubic polynomial splines $q_{i}$, one obtains rational spherical spline motion of degree four or six, respectively. In order for the motion (1) to be of degree four or six, the functions $c_{i}$ should be chosen as

$$
\begin{equation*}
c_{i}=\frac{w_{i}}{r}, \quad \text { with } \quad r=q_{0}^{2}+q_{1}^{2}+q_{2}^{2}+q_{3}^{2}, \quad i=1,2,3, \tag{2}
\end{equation*}
$$

where $\boldsymbol{w}:=\left(w_{1}, w_{2}, w_{3}\right)$ is a parametric polynomial spline of degree $\leq 4$ or $\leq 6$, respectively.

## 3 Geometric continuity for motions

Spline motions (i.e., motions that are obtained by composing several pieces of rational motions) are useful for interpolation of a sequence of given positions. In order to obtain a globally smooth motion we need to study conditions that guarantee a smooth join between neighbouring segments. This problem leads to the concept of geometric continuity, which is well understood in curve design [12,13]. The generalization to motions is straightforward: a spline motion is said to be $G^{k}$ smooth, if all point trajectories generated by it are $G^{k}$ smooth spline curves. Here we present geometric continuity conditions for quaternion curves that imply geometric continuity of motions.

The trajectories

$$
\begin{aligned}
& \boldsymbol{p}:\left[t_{0}, t_{1}\right] \rightarrow \mathbb{R}^{3}, \quad \boldsymbol{p}(t)=\boldsymbol{c}(t)+\mathcal{R}(t) \widehat{\boldsymbol{p}}, \\
& \widetilde{\boldsymbol{p}}:\left[s_{0}, s_{1}\right] \rightarrow \mathbb{R}^{3}, \quad \widetilde{\boldsymbol{p}}(s)=\widetilde{\boldsymbol{c}}(s)+\widetilde{\mathcal{R}}(s) \widehat{\boldsymbol{p}},
\end{aligned}
$$

of an arbitrary point $\widehat{\boldsymbol{p}}$ join with a geometric continuity of order $k$ (or shortly with $G^{k}$ continuity) at the common point $\boldsymbol{p}(\tau)=\widetilde{\boldsymbol{p}}(\sigma)$ iff there exists a regular reparameterization $\varphi:\left[t_{0}, t_{1}\right] \rightarrow\left[s_{0}, s_{1}\right]$, such that

$$
\varphi^{\prime}>0, \quad \varphi(\tau)=\sigma
$$

and

$$
\left.\frac{d^{j} \boldsymbol{p}(t)}{d t^{j}}\right|_{t=\tau}=\left.\frac{d^{j}(\widetilde{\boldsymbol{p}} \circ \varphi)(t)}{d t^{j}}\right|_{t=\tau}, \quad j=0,1, \ldots, k,
$$

or equivalently

$$
\begin{align*}
\left.\frac{d^{j} \boldsymbol{c}(t)}{d t^{j}}\right|_{t=\tau} & =\left.\frac{d^{j}(\widetilde{\boldsymbol{c}} \circ \varphi)(t)}{d t^{j}}\right|_{t=\tau^{\prime}}  \tag{3}\\
\left.\frac{d^{j} \mathcal{R}(t)}{d t^{j}}\right|_{t=\tau} & =\left.\frac{d^{j}(\widetilde{\mathcal{R}} \circ \varphi)(t)}{d t^{j}}\right|_{t=\tau} \tag{4}
\end{align*}
$$

Suppose that the rotations are represented by quaternion curves $\boldsymbol{q}$ and $\widetilde{\boldsymbol{q}}$. Then the spherical motions given by $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ join with $G^{0}$ continuity at the common point iff

$$
\boldsymbol{q}(\tau)=\lambda(\tau) \widetilde{\boldsymbol{q}}(\varphi(\tau)),
$$

where $\lambda:\left[t_{0}, t_{1}\right] \rightarrow \mathbb{R}$ is a zero free scalar function, arising from the equivalence relation in the 3 -dimensional projective space. Thus, the geometric continuity conditions are the same as the ones for rational curves which are expressed in homogeneous coordinates.

Consequently, the $G^{k}$ continuity conditions (4) are equivalent to

$$
\begin{equation*}
\left.\frac{d^{j} \boldsymbol{q}(t)}{d t^{j}}\right|_{t=\tau}=\left.\frac{d^{j}}{d t^{j}}(\lambda(t) \widetilde{\boldsymbol{q}}(\varphi(t)))\right|_{t=\tau}, \quad j=1,2, \ldots, k . \tag{5}
\end{equation*}
$$

By using Faà di Bruno's formula, the conditions (5) can be written as

$$
\begin{align*}
& \left.\frac{d^{j} \boldsymbol{q}(t)}{d t^{j}}\right|_{t=\tau} \\
& =\quad \sum_{\ell=1}^{j}\binom{j}{\ell} \lambda^{(\ell-j)}(\tau) \sum_{i=1}^{\ell} \widetilde{\boldsymbol{q}}^{(i)}(\varphi(\tau)) \mathcal{B}_{\ell, i}\left(\varphi^{\prime}(\tau), \varphi^{\prime \prime}(\tau), \ldots, \varphi^{(\ell-i+1)}(\tau)\right), \tag{6}
\end{align*}
$$

where $\mathcal{B}_{\ell, i}$ are Bell polynomials ([14]),

$$
\begin{aligned}
& \mathcal{B}_{\ell, i}\left(x_{1}, x_{2}, \ldots, x_{\ell-i+1}\right) \\
& \quad=\sum \frac{\ell!}{j_{1}!j_{2}!\cdots j_{\ell-i+1}!}\left(\frac{x_{1}}{1!}\right)^{j_{1}}\left(\frac{x_{2}}{2!}\right)^{j_{2}} \cdots\left(\frac{x_{\ell-i+1}}{(\ell-i+1)!}\right)^{j_{\ell-i+1}},
\end{aligned}
$$

where the sum is taken over all sequences $j_{1}, j_{2}, \ldots, j_{\ell-i+1}$ of non-negative integers such that

$$
\sum_{k=1}^{\ell-i+1} j_{k}=i, \quad \sum_{k=1}^{\ell-i+1} k j_{k}=\ell .
$$

In practice, $G^{1}$ and $G^{2}$ continuity are most frequently used. The $G^{1}$ continuity condition at $t=\tau$ simplifies (6) to

$$
\boldsymbol{q}^{\prime}(\tau)=\lambda^{\prime}(\tau) \widetilde{\boldsymbol{q}}(\varphi(\tau))+\lambda(\tau) \varphi^{\prime}(\tau) \widetilde{\boldsymbol{q}}^{\prime}(\varphi(\tau))
$$

and $G^{2}$ additionally requires

$$
\begin{array}{r}
\boldsymbol{q}^{\prime \prime}(\tau)=\lambda^{\prime \prime}(\tau) \widetilde{\boldsymbol{q}}(\varphi(\tau))+2 \lambda^{\prime}(\tau) \varphi^{\prime}(\tau) \widetilde{\boldsymbol{q}}^{\prime}(\varphi(\tau))+ \\
\lambda(\tau) \varphi^{\prime}(\tau)^{2} \widetilde{\boldsymbol{q}}^{\prime \prime}(\varphi(\tau))+\lambda(\tau) \varphi^{\prime \prime}(\tau) \widetilde{\boldsymbol{q}}^{\prime}(\varphi(\tau)) .
\end{array}
$$

The reparameterization $\varphi$ and the scaling function $\lambda$ give the additional freedom that the geometric interpolation schemes have towards the standard parametric $(\mathcal{C})$ interpolation. The free parameters (derivatives of $\lambda$ and $\varphi$ ) will now be used to decrease the degree of a quaternion curve in the rotational part of the motion.

## 4 Interpolation by $G^{k}$ continuous Hermite spline motions

A standard interpolation problem in motion design is to find a rational spline motion that interpolates a sequence of given positions $\operatorname{Pos}_{i}, i=0,1, \ldots, N$, of a rigid body. Every position $\operatorname{Pos}_{i}$ is described by the position $\boldsymbol{C}_{i}$ of the center and by the associated rotation matrix $\mathcal{R}_{i}$. The rotations are represented by unit quaternions $\boldsymbol{Q}_{i} \in \mathbb{H},\left\|\boldsymbol{Q}_{i}\right\|=1$. This normalization still leaves two possible representatives for each rotation. In order to obtain good results the quaternions should be chosen in such a way that two neighbouring quaternions lie on the same hemisphere, i.e.,

$$
\left\langle\boldsymbol{Q}_{i}, \boldsymbol{Q}_{i+1}\right\rangle>0, \quad i=0,1, \ldots, N-1,
$$

where $\langle\cdot, \cdot\rangle$ is the standard inner product in $\mathbb{R}^{4}$. Every position $\operatorname{Pos}_{i}$ can thus be identified with the pair $\left\{\boldsymbol{C}_{i}, \boldsymbol{Q}_{i}\right\}$, which will be denoted as $\left\{\boldsymbol{C}_{i}, \boldsymbol{Q}_{i}\right\} \sim \operatorname{Pos}_{i}$.

The construction of the motion consists of two parts, the translational and the rotational one. The rotational part of the motion is obtained by applying the kinematical mapping to a polynomial (spline) quaternion curve of degree $n$, and the obtained motion is of degree $2 n$, provided that the translational part is of degree $\leq 2 n$. Since the degree of the motion is twice the degree of the corresponding quaternion curve, the degree of the latter should be as low as possible. This can be achieved by using geometric interpolation schemes.

The task is to construct a $G^{k}$ continuous rational spline motion $\boldsymbol{p}:[0, N] \rightarrow \mathbb{R}^{3}$ of degree $2 n$ with integer knots $[0,1, \ldots, N]$ that interpolates the positions $\operatorname{Pos}_{i}, i=$ $0,1, \ldots, N$. More precisely, the spline motion is composed of rational motions $\boldsymbol{p}_{i}$ : $[i, i+1] \rightarrow \mathbb{R}^{3}, i=0,1, \ldots, N-1$, of degree $2 n$ between two adjacent positions, such that $\boldsymbol{p}_{i}$ and $\boldsymbol{p}_{i+1}$ are $G^{k}$ continuous at the common knot $i+1$. Let $\boldsymbol{c}_{i}:[i, i+$ $1] \rightarrow \mathbb{R}^{3}$ denote the translational part of $\boldsymbol{p}_{i}$ and let $\boldsymbol{q}_{i}:[i, i+1] \rightarrow \mathbb{H}$ be the quaternion polynomial of degree $n$ that defines the rotational part. The interpolation conditions can be written as

$$
\begin{array}{ll}
\boldsymbol{c}_{i}(i)=\boldsymbol{C}_{i}, & \boldsymbol{c}_{i}(i+1)=\boldsymbol{C}_{i+1}, \\
\boldsymbol{q}_{i}(i)=\boldsymbol{Q}_{i}, & \boldsymbol{q}_{i}(i+1)=\boldsymbol{Q}_{i+1},
\end{array} \quad i=0,1, \ldots, N-1,
$$

where we have assumed that the quaternion curves are written in the standard form, i.e.,

$$
\begin{equation*}
\left\|\boldsymbol{q}_{i}(i)\right\|=\left\|\boldsymbol{q}_{i}(i+1)\right\|=1, \quad i=0,1, \ldots, N-1 . \tag{7}
\end{equation*}
$$

This assumption is similar to assuming the standard form of a Bézier rational curve, i.e., normalized weights at the first and the last control point which can always be obtained by a bilinear reparameterization (see [15], e.g.).

Clearly, only the positions $\operatorname{Pos}_{i}$ are not enough to determine the motion for $k>0$ and $n \geq 1$, and additional data are required. In order for the spline to be $G^{1}$ continuous we prescribe at each knot $i$ also a unit tangent vector $\boldsymbol{t}_{i}$ that determines
the derivative direction for the motion of the origin, and a unit quaternion $\boldsymbol{U}_{i}$ that corresponds to the Euler velocity quaternion for the rotational part. For $k \geq 2$ we assume that at each knot $i$ also curvature vectors $\boldsymbol{t}_{i}^{(j)} \in \mathbb{R}^{3}$ and curvature quaternions $\boldsymbol{U}_{i}^{(j)} \in \mathbb{H}, j=2,3, \ldots, k$, are prescribed. All these additional data may be specified by the user or they can be estimated from the positions $\operatorname{Pos}_{i}$ (see [5] and [16], e.g.).

Since the construction of $\boldsymbol{p}$ is local, it is enough to study only one segment of the spline. Thus let $N=1$ and let us consider the spherical motion first. The equations for geometric interpolation of tangent and curvature vectors at parameters $t=0$ and $t=1$ are derived from (5). Namely, the curve $\boldsymbol{q}$ must satisfy

$$
\begin{align*}
\boldsymbol{q}(i) & =\lambda_{i} \boldsymbol{Q}_{i}  \tag{8}\\
\boldsymbol{q}^{\prime}(i) & =\lambda_{i}^{(1)} \boldsymbol{Q}_{i}+\lambda_{i} \varphi_{i}^{(1)} \boldsymbol{U}_{i},  \tag{9}\\
\boldsymbol{q}^{\prime \prime}(i) & =\lambda_{i}^{(2)} \boldsymbol{Q}_{i}+2 \lambda_{i}^{(1)} \varphi_{i}^{(1)} \boldsymbol{U}_{i}+\lambda_{i} \varphi_{i}^{(2)} \boldsymbol{U}_{i}+\lambda_{i}\left(\varphi_{i}^{(1)}\right)^{2} \boldsymbol{U}_{i}^{(2)},  \tag{10}\\
\boldsymbol{q}^{(j)}(i) & =\sum_{\ell=1}^{j}\binom{j}{\ell} \lambda_{i}^{(\ell-j)} \sum_{r=1}^{\ell} \boldsymbol{U}_{i}^{(r)} \mathcal{B}_{\ell, i}\left(\varphi_{i}^{(1)}, \varphi_{i}^{(2)}, \ldots, \varphi_{i}^{(\ell-r+1)}\right), \quad j \leq k, \tag{11}
\end{align*}
$$

for $i=0,1$. The condition (7) implies

$$
\begin{equation*}
\lambda_{0}=\lambda_{1}=1, \tag{12}
\end{equation*}
$$

and the remaining free parameters $\left(\lambda_{i}^{(j)}\right)_{j=1}^{k}, i=0,1$, correspond to the derivatives of the scalar function $\lambda$ at $t=0$ and $t=1$. Similarly, $\left(\varphi_{i}^{(j)}\right)_{j=1}^{k}$ are free parameters that represent the derivatives of the reparameterization $\varphi$ at $t=0$ and $t=1$. In order for the reparameterization to be regular, the relations

$$
\varphi_{0}^{(1)}>0, \quad \varphi_{1}^{(1)}>0,
$$

must be satisfied. The additional $4 k$ parameters of freedom can be used to decrease the degree of the motion. These parameters together with $4(n+1)$ unknown coefficients of $\boldsymbol{q}$ are determined from $8(k+1)$ equations (8)-(11). The numbers of equations and unknowns are equal iff

$$
8(k+1)=4 k+4(n+1),
$$

which leads us to the following conjecture.
Conjecture 1 A spherical rational motion of degree $2 n=2 k+2(n>1)$ can geometrically interpolate the rotation, the velocity and $k-1$ curvature quaternions at each knot $i \in\{0,1\}$. The approximation order is $2 n$.

Note that the assumption $n>1$ is needed, since the conjecture is not true for quadratic quaternion curves as we shall see in the next section.

A similar conjecture was stated for geometric interpolation by parametric polynomial curves (see [17]) and it turned out as a difficult and still unsolved problem in general. As expected from the curve case, the equations involved in rational motion design are highly nonlinear, which makes the analysis difficult. As the first step, we will consider $G^{1}$ and $G^{2}$ rational motions generated by parabolic and cubic quaternion curves.

## $5 G^{1}$ Hermite interpolation by motions based on parabolic quaternion curves

With respect to Conjecture 1, the first case to be considered is $G^{1}$ parabolic interpolation. Let $\boldsymbol{Q}_{0}$ and $\boldsymbol{Q}_{1}$ be two given quaternions and $\boldsymbol{U}_{0}, \boldsymbol{U}_{1}$ given velocity quaternions at $\boldsymbol{Q}_{0}, \boldsymbol{Q}_{1}$, respectively. We would like to construct a parabolic quaternion interpolant $\boldsymbol{q}:[0,1] \rightarrow \mathbb{H}$, but if the data $\boldsymbol{U}_{0}, \boldsymbol{Q}_{1}-\boldsymbol{Q}_{0}$ and $\boldsymbol{U}_{1}$ are linearly independent, this can not be achieved, since parabolas are always planar curves. Perhaps the most appropriate remedy is to insert an additional quaternion $\boldsymbol{Q}_{A}$ and to try to construct two parabolic quaternion curves $\boldsymbol{q}_{0}:[0,1] \rightarrow \mathbb{H}$ and $\boldsymbol{q}_{1}:[0,1] \rightarrow \mathbb{H}$, such that $\boldsymbol{q}_{0}$ interpolates $\boldsymbol{Q}_{0}, \boldsymbol{Q}_{A}$ and $\boldsymbol{U}_{0}, \boldsymbol{q}_{1}$ interpolates $\boldsymbol{Q}_{A}, \boldsymbol{Q}_{1}$ and $\boldsymbol{U}_{1}$, and parabolic interpolants join with the $G^{1}$ continuity at the quaternion $\boldsymbol{Q}_{A}$.

Clearly, curves $\boldsymbol{q}_{0}$ and $\boldsymbol{q}_{1}$ can be written in the Bernstein-Bézier form as

$$
\begin{align*}
& \boldsymbol{q}_{0}(t)=\boldsymbol{Q}_{0} B_{0}^{2}(t)+\boldsymbol{B}_{0} B_{1}^{2}(t)+\boldsymbol{Q}_{A} B_{2}^{2}(t), \\
& \boldsymbol{q}_{1}(t)=\boldsymbol{Q}_{A} B_{0}^{2}(t)+\boldsymbol{B}_{1} B_{1}^{2}(t)+\boldsymbol{Q}_{1} B_{2}^{2}(t), \tag{13}
\end{align*}
$$

where $\boldsymbol{B}_{0}$ and $\boldsymbol{B}_{1}$ are two unknown control quaternions yet to be determined, and $B_{j}^{n}(t):=\binom{n}{j} t^{j}(1-t)^{n-j}$ are the Bernstein basis polynomials of degree $n$.

By (9) and (12), the $G^{1}$ interpolation conditions can be written as

$$
\boldsymbol{q}_{i}^{\prime}(i)=\lambda_{i}^{(1)} \boldsymbol{Q}_{i}+\varphi_{i}^{(1)} \boldsymbol{U}_{i}, i=0,1, \quad \boldsymbol{q}_{0}^{\prime}(1)=\lambda_{A}^{(1)} \boldsymbol{Q}_{A}+\varphi_{A}^{(1)} \boldsymbol{q}_{1}^{\prime}(0),
$$

where the parameters $\lambda_{0}^{(1)}, \lambda_{1}^{(1)}$ and $\lambda_{A}^{(1)}$ have to be positive. By using the basic properties of Bézier curves

$$
\begin{array}{ll}
\boldsymbol{q}_{0}^{\prime}(0)=2\left(\boldsymbol{B}_{0}-\boldsymbol{Q}_{0}\right), & \boldsymbol{q}_{0}^{\prime}(1)=2\left(\boldsymbol{Q}_{A}-\boldsymbol{B}_{1}\right), \\
\boldsymbol{q}_{1}^{\prime}(0)=2\left(\boldsymbol{B}_{1}-\boldsymbol{Q}_{A}\right), & \boldsymbol{q}_{1}^{\prime}(1)=2\left(\boldsymbol{Q}_{1}-\boldsymbol{B}_{1}\right),
\end{array}
$$

we obtain

$$
\begin{align*}
& \boldsymbol{B}_{0}=\boldsymbol{Q}_{0}+\frac{1}{2}\left(\lambda_{0}^{(1)} \boldsymbol{Q}_{0}+\varphi_{0}^{(1)} \boldsymbol{U}_{0}\right),  \tag{14}\\
& \boldsymbol{B}_{1}=\boldsymbol{Q}_{1}-\frac{1}{2}\left(\lambda_{1}^{(1)} \boldsymbol{Q}_{1}+\varphi_{1}^{(1)} \boldsymbol{U}_{1}\right),  \tag{15}\\
& 2\left(\boldsymbol{Q}_{A}-\boldsymbol{B}_{0}\right)=\lambda_{A}^{(1)} \boldsymbol{Q}_{A}+2 \varphi_{A}^{(1)}\left(\boldsymbol{B}_{1}-\boldsymbol{Q}_{A}\right) . \tag{16}
\end{align*}
$$

By inserting (14) and (15) into (16) we obtain a system of 4 scalar equations for 6 unknowns $\lambda_{0}^{(1)}, \lambda_{1}^{(1)}, \lambda_{A}^{(1)}, \varphi_{0}^{(1)}, \varphi_{1}^{(1)}$ and $\varphi_{A}^{(1)}$. It can be written in the matrix form $A \boldsymbol{x}=\boldsymbol{a}$, where

$$
A:=\left(\boldsymbol{Q}_{0} \boldsymbol{Q}_{A} \boldsymbol{Q}_{1} \boldsymbol{U}_{1}\right), \quad \boldsymbol{a}:=\varphi_{0}^{(1)} \boldsymbol{U}_{0}, \quad \text { and } \quad \boldsymbol{x}:=\left(\begin{array}{c}
-2-\lambda_{0}^{(1)} \\
2-\lambda_{A}^{(1)}+2 \varphi_{A}^{(1)} \\
\varphi_{A}^{(1)}\left(\lambda_{1}^{(1)}-2\right) \\
\varphi_{A}^{(1)} \varphi_{1}^{(1)}
\end{array}\right) .
$$

Let us denote

$$
D_{i}:=\frac{\operatorname{det} A^{(i)}\left(\boldsymbol{U}_{0}\right)}{\operatorname{det} A}, \quad i=1,2,3,4
$$

where $A^{(i)}(\boldsymbol{U})$ denotes the matrix $A$ with the $i$-th column replaced by the quaternion $\boldsymbol{U}$. By the Cramer's rule we can express unknowns $\lambda_{0}^{(1)}, \lambda_{1}^{(1)}, \lambda_{A}^{(1)}$ and $\varphi_{A}^{(1)}$ in terms of $\varphi_{0}^{(1)}$ and $\varphi_{1}^{(1)}$ as

$$
\begin{align*}
& \lambda_{0}^{(1)}=-2-\varphi_{0}^{(1)} D_{1}, \quad \lambda_{1}^{(1)}=2+\varphi_{1}^{(1)} \frac{D_{3}}{D_{4}},  \tag{17}\\
& \lambda_{A}^{(1)}=2+2 \frac{\varphi_{0}^{(1)}}{\varphi_{1}^{(1)}} D_{4}-\varphi_{0}^{(1)} D_{2}, \quad \varphi_{A}^{(1)}=\frac{\varphi_{0}^{(1)}}{\varphi_{1}^{(1)}} D_{4} .
\end{align*}
$$

By choosing any $\varphi_{0}^{(1)}>0$ and $\varphi_{1}^{(1)}>0$, the only solvability condition which has to be fulfilled is

$$
\begin{equation*}
D_{4}>0 . \tag{18}
\end{equation*}
$$

Let us summarize the discussion in the following theorem.
Theorem 2 Let $\boldsymbol{Q}_{i}, \boldsymbol{U}_{i}, i=0,1$, and $\boldsymbol{Q}_{A}$ be given data such that $A$ is nonsingular and $D_{4}>0$. Then there exists a two-parametric family of $G^{1}$ continuous pairs of parabolic quaternion curves $\boldsymbol{q}_{0}:=\boldsymbol{q}_{0}\left(t ; \varphi_{0}^{(1)}, \varphi_{1}^{(1)}\right)$ and $\boldsymbol{q}_{1}:=\boldsymbol{q}_{1}\left(t ; \varphi_{0}^{(1)}, \varphi_{1}^{(1)}\right)$, defined by (13), (14), (15) and (17).

Note that $\varphi_{i}^{(1)}$ affects only $\boldsymbol{q}_{i}, i=0,1$. In Fig. 1 the trajectories of a spherical part of a particular point for different choices of free parameters $\varphi_{0}^{(1)}$ and $\varphi_{1}^{(1)}$ are shown.

Remark 3 The proposed scheme generalizes the one presented in [7]. The analysis here is done directly in the quaternion space, while in [7] a projection of the data to a particular three-dimensional subspace has been applied and $\varphi_{0}^{(1)}, \varphi_{1}^{(1)}$ have been selected in advance and not left as degrees of freedom.

If $D_{4}<0$ in Theorem 2, we have to replace the inserted quaternion $\boldsymbol{Q}_{A}$, given by the user, by another quaternion. One possible way, which guarantees (18), would be to take

$$
\begin{equation*}
\boldsymbol{Q}_{A}:=\boldsymbol{r}\left(t^{*}\right) \tag{19}
\end{equation*}
$$



Fig. 1. The trajectories of a particular point for parameters $\varphi_{0}^{(1)}, \varphi_{1}^{(1)} \in\left\{\frac{1}{10}, \frac{1}{2}, 1,5,50\right\}$ (lighter curves correspond to higher parameter values).
where $t^{*}$ is any parameter from $(0,1)$ and $\boldsymbol{r}$ is the cubic quaternion curve, interpolating $\boldsymbol{Q}_{0}, \boldsymbol{Q}_{1}, \boldsymbol{U}_{0}$ and $\boldsymbol{U}_{1}$ in the $\mathcal{C}^{1}$ sense,
$\boldsymbol{r}:=\boldsymbol{r}\left(\cdot ; \boldsymbol{Q}_{0}, \boldsymbol{Q}_{1}, \boldsymbol{U}_{0}, \boldsymbol{U}_{1}\right)=\boldsymbol{Q}_{0} B_{0}^{3}+\left(\boldsymbol{Q}_{0}+\frac{1}{3} \boldsymbol{U}_{0}\right) B_{1}^{3}+\left(\boldsymbol{Q}_{1}-\frac{1}{3} \boldsymbol{U}_{1}\right) B_{2}^{3}+\boldsymbol{Q}_{1} B_{3}^{3}$.
Application of (19) and (20) gives

$$
\left.\left.D_{4}=\frac{-\operatorname{det}\left(\boldsymbol{Q}_{0} B_{2}^{3}\left(t^{*}\right) \boldsymbol{U}_{1}\right.}{} \boldsymbol{Q}_{1} \boldsymbol{U}_{0}\right)\right) \frac{t^{*}}{\operatorname{det}\left(\begin{array}{llll}
\boldsymbol{Q}_{0} & B_{1}^{3}\left(t^{*}\right) & \boldsymbol{U}_{0} & \boldsymbol{Q}_{1}
\end{array} \boldsymbol{U}_{1}\right)}=\frac{\left.1-t^{*}\right)}{(1-2}
$$

Remark 4 The shape parameter $t^{*}$ is usually chosen in such a way that $\| \boldsymbol{Q}_{A}-$ $\boldsymbol{r}\left(t^{*}\right)\left\|=\min _{t \in(0,1)}\right\| \boldsymbol{Q}_{A}-\boldsymbol{r}(t) \|$.

## $6 G^{2}$ Hermite interpolation by motions based on cubic quaternion curves

The second and perhaps the most important case is the cubic $G^{2}$ interpolation. Let $\boldsymbol{Q}_{j}$ be given quaternions and $\boldsymbol{U}_{j}, \boldsymbol{V}_{j}:=\boldsymbol{U}_{j}^{(2)}$ be given velocity and curvature quaternions at positions $\boldsymbol{Q}_{j}, j=0,1$. Our goal is to find a cubic quaternion interpolant $\boldsymbol{q}:[0,1] \rightarrow \mathbb{H}$,

$$
\begin{equation*}
\boldsymbol{q}(t)=\sum_{j=0}^{3} \boldsymbol{B}_{j} B_{j}^{3}(t) \tag{21}
\end{equation*}
$$

The interpolant $\boldsymbol{q}$ will be $G^{2}$ continuous if the relations (8), (9) and (10) together with (12) are satisfied. By using some basic properties of Bézier curves, one obtains

$$
\begin{align*}
& \boldsymbol{B}_{0}=\boldsymbol{Q}_{0}, \quad \boldsymbol{B}_{3}=\boldsymbol{Q}_{1}, \\
& 3 \Delta \boldsymbol{B}_{2 i}=\lambda_{i}^{(1)} \boldsymbol{Q}_{i}+\varphi_{i}^{(1)} \boldsymbol{U}_{i}, \quad i=0,1,  \tag{22}\\
& 6 \Delta^{2} \boldsymbol{B}_{i}=\lambda_{i}^{(2)} \boldsymbol{Q}_{i}+\left(2 \lambda_{i}^{(1)} \varphi_{i}^{(1)}+\varphi_{i}^{(2)}\right) \boldsymbol{U}_{i}+\left(\varphi_{i}^{(1)}\right)^{2} \boldsymbol{V}_{i}, \quad i=0,1,
\end{align*}
$$

where $\Delta(\cdot)_{i}:=(\cdot)_{i+1}-(\cdot)_{i}, \Delta^{2}(\cdot)_{i}:=\Delta\left(\Delta(\cdot)_{i}\right)$ are forward differences.

Equations (22) form a system of 24 nonlinear equations for the unknown control quaternions $\boldsymbol{B}_{j}, j=0,1,2,3$, and unknown scalar parameters

$$
\begin{equation*}
\varphi_{i}^{(1)}, \varphi_{i}^{(2)}, \lambda_{i}^{(1)}, \lambda_{i}^{(2)}, \quad i=0,1 . \tag{23}
\end{equation*}
$$

In addition, the unknowns $\varphi_{0}^{(1)}$ and $\varphi_{1}^{(1)}$ have to be positive. By a straightforward substitution, we reduce the system (22) to a system of 8 nonlinear equations

$$
\begin{align*}
& \left(\frac{2(-1)^{i}}{3} \lambda_{i}^{(1)}+\frac{1}{6} \lambda_{i}^{(2)}+1\right) \boldsymbol{Q}_{i}+\left(\frac{(-1)^{i}}{3} \lambda_{1-i}^{(1)}-1\right) \boldsymbol{Q}_{1-i}+ \\
& \left(\frac{2(-1)^{i}}{3} \varphi_{i}^{(1)}+\frac{1}{3} \lambda_{i}^{(1)} \varphi_{i}^{(1)}+\frac{1}{6} \varphi_{i}^{(2)}\right) \boldsymbol{U}_{i}+\frac{(-1)^{i}}{3} \varphi_{1-i}^{(1)} \boldsymbol{U}_{1-i}+\frac{1}{6}\left(\varphi_{i}^{(1)}\right)^{2} \boldsymbol{V}_{i}=\mathbf{0}, \tag{24}
\end{align*}
$$

where $i=0,1$, for parameters (23). Similarly as in the $G^{1}$ case, the system (24) can be written in the matrix form as

$$
\begin{equation*}
A_{i} \boldsymbol{x}_{i}=\boldsymbol{a}_{i}, \quad i=0,1, \tag{25}
\end{equation*}
$$

where

$$
A_{i}:=\left(\boldsymbol{Q}_{i} \boldsymbol{Q}_{1-i} \boldsymbol{U}_{i} \boldsymbol{V}_{i}\right), \quad \boldsymbol{a}_{i}:=\frac{(-1)^{1-i} \varphi_{1-i}^{(1)}}{3} \boldsymbol{U}_{1-i}
$$

and

$$
\boldsymbol{x}_{i}:=\left(\begin{array}{c}
\frac{2(-1)^{i}}{3} \lambda_{i}^{(1)}+\frac{1}{6} \lambda_{i}^{(2)}+1 \\
\frac{(-1)^{i}}{3} \lambda_{1-i}^{(1)}-1 \\
\frac{2(-1)^{i}}{3} \varphi_{i}^{(1)}+\frac{1}{3} \lambda_{i}^{(1)} \varphi_{i}^{(1)}+\frac{1}{6} \varphi_{i}^{(2)} \\
\frac{1}{6}\left(\varphi_{i}^{(1)}\right)^{2}
\end{array}\right)
$$

Here we assume that $A_{0}$ and $A_{1}$ are nonsingular matrices. Let us define

$$
D_{i, j}:=\frac{\operatorname{det} A_{i}^{(j)}\left(\boldsymbol{U}_{1-i}\right)}{\operatorname{det} A_{i}}, \quad j=1,2,3,4, i=0,1
$$

By applying the Cramer's rule, the system (25) simplifies to

$$
\begin{align*}
\frac{2(-1)^{i}}{3} \lambda_{i}^{(1)}+\frac{1}{6} \lambda_{i}^{(2)}+1 & =\frac{(-1)^{1-i}}{3} \varphi_{1-i}^{(1)} D_{i, 1}, \\
\frac{(-1)^{i}}{3} \lambda_{1-i}^{(1)}-1 & =\frac{(-1)^{1-i}}{3} \varphi_{1-i}^{(1)} D_{i, 2}, \\
\frac{2(-1)^{i}}{3} \varphi_{i}^{(1)}+\frac{1}{3} \lambda_{i}^{(1)} \varphi_{i}^{(1)}+\frac{1}{6} \varphi_{i}^{(2)} & =\frac{(-1)^{1-i}}{3} \varphi_{1-i}^{(1)} D_{i, 3},  \tag{26}\\
\frac{1}{6}\left(\varphi_{i}^{(1)}\right)^{2} & =\frac{(-1)^{1-i}}{3} \varphi_{1-i}^{(1)} D_{i, 4},
\end{align*}
$$

where $i=0,1$. The system (26) has 3 nontrivial solutions, but only one of them is
real:

$$
\begin{align*}
\varphi_{i}^{(1)}= & 2(-1)^{i} \sqrt[3]{D_{i, 4}^{2}\left|D_{1-i, 4}\right|} \operatorname{sign}\left(D_{1-i, 4}\right), \\
\varphi_{i}^{(2)}= & 4 \sqrt[3]{\left|D_{0,4} D_{1,4}\right|} \operatorname{sign}\left(D_{0,4} D_{1,4}\right) \\
& \cdot\left(\sqrt[3]{\left|D_{i, 4}\right|} \operatorname{sign}\left(D_{i, 4}\right)+\sqrt[3]{\left|D_{1-i, 4}\right|} \operatorname{sign}\left(D_{1-i, 4}\right)\left(D_{i, 3}+2 D_{i, 4} D_{1-i, 2}\right)\right),  \tag{27}\\
\lambda_{i}^{(1)}= & (-1)^{1-i}\left(3+2 D_{1-i, 2} \sqrt[3]{D_{i, 4}^{2}\left|D_{1-i, 4}\right|} \operatorname{sign}\left(D_{1-i, 4}\right)\right), \\
\lambda_{i}^{(2)}= & 2\left(3+4 D_{1-i, 2} \sqrt[3]{D_{i, 4}^{2}\left|D_{1-i, 4}\right|} \operatorname{sign}\left(D_{1-i, 4}\right)+2 D_{i, 1} \sqrt[3]{D_{1-i, 4}^{2}\left|D_{i, 4}\right|} \operatorname{sign}\left(D_{i, 4}\right)\right),
\end{align*}
$$

where $i=0,1$. Since $\varphi_{0}^{(1)}$ and $\varphi_{1}^{(1)}$ have to be positive, the only solvability conditions are $D_{0,4}<0$ and $D_{1,4}>0$. Let us summarize the obtained results.

Theorem 5 Let $\boldsymbol{Q}_{i}, \boldsymbol{U}_{i}$ and $\boldsymbol{V}_{i}, i=0,1$, be given data such that $A_{0}$ and $A_{1}$ are nonsingular and $D_{0,4}<0, D_{1,4}>0$. Then there exists a unique cubic interpolating quaternion curve $\boldsymbol{q}$, defined by (21) and (22), with

$$
\varphi_{i}^{(1)}=2(-1)^{i} \sqrt[3]{D_{i, 4}^{2} D_{1-i, 4}}, \quad \lambda_{i}^{(1)}=(-1)^{1-i}\left(3+2 D_{1-i, 2} \sqrt[3]{D_{i, 4}^{2} D_{1-i, 4}}\right),
$$

for $i=0,1$.
If $D_{0,4}>0$ or $D_{1,4}<0$, the given set of data have to be perturbed in order to guarantee the existence of the $G^{2}$ interpolating cubic. But the change in data affects non only one but two adjacent segments of the spline. Let $\boldsymbol{Q}_{i}, \boldsymbol{U}_{i}, \boldsymbol{V}_{i}:=\boldsymbol{U}_{i}^{(2)}, i=$ $0,1,2$, be given data on two neighbouring segments. The $G^{2}$ continuity condition at $\boldsymbol{Q}_{1}$ requires

$$
\begin{align*}
& D_{1,4}^{L}:=D_{1,4}^{L}\left(\boldsymbol{V}_{1}\right):=\frac{\operatorname{det}\left(\begin{array}{llll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{0} & \boldsymbol{U}_{1} & \boldsymbol{U}_{0}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{llll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{0} & \boldsymbol{U}_{1} & \boldsymbol{V}_{1}
\end{array}\right)}>0, \\
& D_{0,4}^{R}:=D_{0,4}^{R}\left(\boldsymbol{V}_{1}\right):=\frac{\operatorname{det}\left(\begin{array}{llll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{2} & \boldsymbol{U}_{1} & \boldsymbol{U}_{2}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{llll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{2} & \boldsymbol{U}_{1} & \boldsymbol{V}_{1}
\end{array}\right)}<0, \tag{28}
\end{align*}
$$

where the notation $(\cdot)^{L}$ and $(\cdot)^{R}$ refers to the left and right segment, respectively. It turns out that it is enough to modify $\boldsymbol{V}_{1}$ only if (28) is not satisfied. Since $\boldsymbol{V}_{1}$ is not involved in $D_{0,4}^{L}$ and $D_{1,4}^{R}$, this modification is local. If $D_{1,4}^{L}\left(\boldsymbol{V}_{1}\right)<0$ and $D_{0,4}^{R}\left(\boldsymbol{V}_{1}\right)>0$, changing $\boldsymbol{V}_{1}$ to $-\boldsymbol{V}_{1}$ will clearly satisfy (28). The other two cases are more involved. Suppose that $\boldsymbol{Q}_{0} \neq \boldsymbol{Q}_{2}$. Let

$$
\Pi_{1}:=\operatorname{det}\left(\begin{array}{lll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{0} & \boldsymbol{U}_{1} \boldsymbol{X}
\end{array}\right)=\mathbf{0}, \quad \Pi_{2}:=\operatorname{det}\left(\begin{array}{lll}
\boldsymbol{Q}_{1} & \boldsymbol{Q}_{2} & \boldsymbol{U}_{1} \boldsymbol{X}
\end{array}\right)=\mathbf{0}
$$

denote the hyperplanes in $\mathbb{R}^{4}$ passing through the common plane, determined by $\boldsymbol{Q}_{1}, \boldsymbol{U}_{1}, \mathbf{0}$, and $\boldsymbol{Q}_{0}, \boldsymbol{Q}_{2}$, respectively. In order to satisfy (28), $\boldsymbol{V}_{1}$ and $\boldsymbol{U}_{0}$ have to lie
on the same side of $\Pi_{1}$, while $\boldsymbol{V}_{1}$ and $\boldsymbol{U}_{2}$ have to be on the opposite sides of $\Pi_{2}$. Since $\Pi_{1}$ and $\Pi_{2}$ divide $\mathbb{R}^{4}$ into four subspaces and precisely one is the admissible for $\boldsymbol{V}_{1}$, an appropriate $\boldsymbol{V}_{1}$ always exists, provided that $\boldsymbol{Q}_{0} \neq \boldsymbol{Q}_{2}$. One possible way to determine it is the following. Recall (20) and let

$$
\begin{aligned}
\boldsymbol{V}_{i}^{L} & :=\boldsymbol{r}^{\prime \prime}\left(i ; \boldsymbol{Q}_{0}, \boldsymbol{Q}_{1}, \boldsymbol{U}_{0}, \boldsymbol{U}_{1}\right)=6\left(\boldsymbol{Q}_{1-i}-\boldsymbol{Q}_{i}\right)+2(-1)^{i+1}\left(\boldsymbol{U}_{1-i}+2 \boldsymbol{U}_{i}\right), \\
\boldsymbol{V}_{i}^{R} & :=\boldsymbol{r}^{\prime \prime}\left(i ; \boldsymbol{Q}_{1}, \boldsymbol{Q}_{2}, \boldsymbol{U}_{1}, \boldsymbol{U}_{2}\right)=6\left(\boldsymbol{Q}_{2-i}-\boldsymbol{Q}_{1+i}\right)+2(-1)^{i+1}\left(\boldsymbol{U}_{2-i}+2 \boldsymbol{U}_{1+i}\right),
\end{aligned}
$$

for $i=0,1$. Note that $D_{1,4}^{L}\left(\boldsymbol{V}_{1}^{L}\right)=\frac{1}{2}$ and $D_{0,4}^{R}\left(\boldsymbol{V}_{0}^{R}\right)=-\frac{1}{2}$. Suppose first that $D_{1,4}^{L}\left(\boldsymbol{V}_{1}\right)<0$ and $D_{0,4}^{R}\left(\boldsymbol{V}_{1}\right)<0$. If $\boldsymbol{D}_{0,4}^{R}\left(\boldsymbol{V}_{1}^{L}\right)<0$, then we can choose $\boldsymbol{V}_{1}^{L}$ for the new $\boldsymbol{V}_{1}$. Otherwise, we can connect given $\boldsymbol{V}_{1}$ and $\boldsymbol{V}_{1}^{L}$ by a line segment. Let us denote the intersections between the line segment and hyperplanes $\Pi_{1}, \Pi_{2}$ by $V_{\Pi_{1}}, V_{\Pi_{2}}$, respectively, and let $V_{\Pi}:=\frac{V_{\Pi_{1}}+\boldsymbol{V}_{\Pi_{2}}}{2}$. We have precisely two possibilities: $D_{1,4}^{L}\left(\boldsymbol{V}_{\Pi}\right)$ and $-D_{0,4}^{R}\left(\boldsymbol{V}_{\Pi}\right)$ are both positive or both negative. In the first (second) case, the new $\boldsymbol{V}_{1}$ is chosen as $\boldsymbol{V}_{\Pi}\left(-\boldsymbol{V}_{\Pi}\right)$, respectively. The symmetric case $D_{1,4}^{L}\left(\boldsymbol{V}_{1}\right)>0$ and $D_{0,4}^{R}\left(\boldsymbol{V}_{1}\right)>0$ follows similarly by using $\boldsymbol{V}_{0}^{R}$. Let us summarize the obtained observations in a short remark.

Remark 6 Let $\boldsymbol{Q}_{i}, \boldsymbol{U}_{i}, \boldsymbol{V}_{i}:=\boldsymbol{U}_{i}^{(2)}, i=0,1,2$, be given data on two neighbouring segments and suppose that $\boldsymbol{Q}_{0} \neq \boldsymbol{Q}_{2}$. Then we can always modify $\boldsymbol{V}_{1}$ such that $G^{2}$ continuity condition at $\boldsymbol{Q}_{1}$ is fulfilled.

## 7 Construction of the translational part

According to (1) we are left to construct a trajectory of the origin $\boldsymbol{c}$. By (2), polynomials $w_{i}, i=1,2,3$, of degree at most $2 n$, have to be determined. From interpolation conditions (3) polynomials $w_{i}$ of degree $\leq 4$ or $\leq 6$ for the $G^{1}$ or $G^{2}$ case are not uniquely determined. Therefore we will restrict the degrees to 3 and 5 .

The reparameterization $\varphi$, which has already been determined in the spherical part, must now by (3) and (4) be used in the translational part of the motion. In particular, for the $G^{2}$ interpolation, the polynomials $w_{i}$ are uniquely determined by the following conditions:

$$
\begin{aligned}
\boldsymbol{w}(i) & =r(i) \boldsymbol{C}_{i}, \\
\boldsymbol{w}^{\prime}(i) & =\varphi_{i}^{(1)} r(i) \boldsymbol{t}_{i}+r^{\prime}(i) \boldsymbol{C}_{i}, \\
\boldsymbol{w}^{\prime \prime}(i) & =\left(r(i) \varphi_{i}^{(2)}+2 \varphi_{i}^{(1)} r^{\prime}(i)\right) \boldsymbol{t}_{i}+\left(\left(\varphi_{i}^{(1)}\right)^{2} r(i)\right) \boldsymbol{f}_{i}+r^{\prime \prime}(i) \boldsymbol{C}_{i}, \quad i=0,1 .
\end{aligned}
$$

Note that parameters $\varphi_{i}^{(1)}$ and $\varphi_{i}^{(2)}, i=0,1$, are given by (27) and $\boldsymbol{f}_{i}:=\boldsymbol{t}_{i}^{(2)}$. Polynomials $\boldsymbol{w}$ can thus be computed by the standard Newton interpolation scheme componentwise, e.g.

## 8 Examples

Let us conclude the paper with some numerical examples. As the first one, let us sample the positions from a smooth motion defined by the quaternion curve $\widetilde{\boldsymbol{q}}$,

$$
\widetilde{\boldsymbol{q}}=\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}, \quad \boldsymbol{q}(t)=\left(t, t+\cos \left(\frac{\pi t}{4}\right), \sin \left(\frac{\pi t}{4}\right), \cos \left(\frac{\pi t}{10}\right)\right)^{\top},
$$

and by the trajectory of the center

$$
\widetilde{\boldsymbol{c}}(t)=(3 \log (t+1) \cos (t), 3 \log (t+1) \sin (t), 3(t+1))^{\top} .
$$

More precisely, let

$$
\begin{align*}
\boldsymbol{Q}_{i}=\widetilde{\boldsymbol{q}}\left(t_{i}\right), & \boldsymbol{U}_{i}=\widetilde{\boldsymbol{q}}^{\prime}\left(t_{i}\right) /\left\|\tilde{\boldsymbol{q}}^{\prime}\left(t_{i}\right)\right\|, & \boldsymbol{V}_{i}=\widetilde{\boldsymbol{q}}^{\prime \prime}\left(t_{i}\right),  \tag{29}\\
\boldsymbol{C}_{i}=\widetilde{\boldsymbol{c}}\left(t_{i}\right), & \boldsymbol{t}_{i}=\widetilde{\boldsymbol{c}}^{\prime}\left(t_{i}\right) /\left\|\widetilde{\boldsymbol{c}}^{\prime}\left(t_{i}\right)\right\|, & \boldsymbol{f}_{i}=\widetilde{\boldsymbol{c}}^{\prime \prime}\left(t_{i}\right),
\end{align*}
$$

where $t_{i}=i h, i=0,1, \ldots, N$.


Fig. 2. Nine positions of a cuboid interpolated by a $G^{1}$ continuous motion (left) and $G^{2}$ continuous motion (right).

Fig. 2 shows the $G^{1}$ spline motion (left) and $G^{2}$ spline motion (right) of a cuboid with $h=1$ and $N=8$. The interpolation positions are denoted by bold cuboids. The free parameters in the $G^{1}$ scheme are chosen as $\varphi_{0}^{(1)}=\varphi_{1}^{(1)}=1$ and every second quaternion $\boldsymbol{Q}_{2 i+1}$ is the additional one. This is perhaps the reason why the motions look quite similar, which can be observed also from Fig. 3, where the $G^{1}$ and $G^{2}$ continuous trajectories of a particular cuboid point $\hat{\boldsymbol{p}}$ are shown. In Fig. 4 the curvature and the torsion of a trajectory of the point $\widehat{\boldsymbol{p}}$ are shown for $G^{2}$ motion. Figures confirm that $G^{2}$ continuity implies the curvature continuity, but not


Fig. 3. The trajectory of a cuboid point of a $G^{1}$ motion (gray curve) and of a $G^{2}$ motion (black curve).


Fig. 4. A Curvature plot (left) and a torsion plot (right) of a $G^{2}$ trajectory of a cuboid point. the torsion continuity. Furthermore, the parametric distances ([18]) between trajectories of a point $\widehat{\boldsymbol{p}}$ of the original and the $G^{2}$ spline motions for different values $h$ are shown in Table 1. The last column numerically confirms that the approximation order is optimal, i.e. six.

Fig. 5 shows the spherical part of another $G^{2}$ motion of a cuboid. In this case the input data were only the unit quaternions $\boldsymbol{Q}_{i}$, given in (29), which correspond to the rotations. The remaining data $\boldsymbol{U}_{i}$ and $\boldsymbol{V}_{i}$ were estimated by using local quartic polynomials through five consecutive points. Quartic polynomials have been used since symmetry is preferred and parabolic arcs cause singularities.

As a last example, let us compare the spherical parts of the $G^{2}$ motion of degree 6

| h | Parametric distance | Decay exponent |
| :---: | :---: | :---: |
| 1 | $4.19949 \times 10^{-3}$ | $/$ |
| $\frac{1}{2}$ | $1.41924 \times 10^{-4}$ | 4.89 |
| $\frac{1}{4}$ | $3.51037 \times 10^{-6}$ | 5.34 |
| $\frac{1}{8}$ | $7.09768 \times 10^{-8}$ | 5.63 |
| $\frac{1}{16}$ | $1.27319 \times 10^{-9}$ | 5.80 |
| $\frac{1}{32}$ | $2.1373 \times 10^{-11}$ | 5.90 |

Table 1
The parametric distances between trajectories (of an arbitrary point $\widehat{\boldsymbol{p}}$ ) of the original and the $G^{2}$ spline motions for different values $h$.


Fig. 5. Spherical part of a $G^{2}$ rational motion of a cuboid with nine interpolated rotations where the velocity and curvature quaternions were estimated by using local quartic polynomials.
and the $C^{2}$ motion of degree 10 , which can be constructed using standard Hermite interpolation techniques. In order to recognize some difference between both motions we interpolate only every second data in (29). Fig. 6 shows that both motions are quite similar, but of course the degree of the geometrically continuous motion is much smaller.


Fig. 6. Spherical parts of the $C^{2}$ motion (left) and the $G^{2}$ motion (right) for every second data in (29).

## 9 Conclusion

In the paper we have studied the problem of interpolation by rational spline motions. Instead of classical approach, where usually $C^{k}$ continuous interpolants are constructed, geometric interpolation schemes were introduced in order to reduce the degree of the interpolating curves. As a consequence, $G^{k}$ continuous interpolating rational splines were obtained. A general theory of geometric interpolation by rational spline motions was presented. The analysis concentrated on two (practically important) cases, i.e., $G^{1}$ continuous quartic rational motions and $G^{2}$ continuous rational spline motions of degree six. A detailed study of the solvability conditions involving data quaternions was done. In some cases which do not guarantee a solution of the problem, some methods how to perturb given data in order to assure the solvability were proposed. Several numerical examples were given which confirm theoretical results.
The obtained interpolation schemes are of practical importance. They can be, e.g., used in robotics and related fields. The main advantage of $G^{k}$ interpolation schemes compared to classical $C^{k}$ schemes is the reduction of the degree of the resulting rational spline interpolants. In particular, for the interpolation of positions, velocity and curvature data (which is one of the classical problems in motion design) the degree reduces from 10 to 6 .
Although only Hermite case of interpolation has been studied, one could follow a general theory also for the Lagrange case (or combination of Hermite and Lagrange case). This would lead to new interpolation schemes, but usually also to more complicated (nonlinear) systems of equations to be solved, definitely interesting enough for some future work.

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