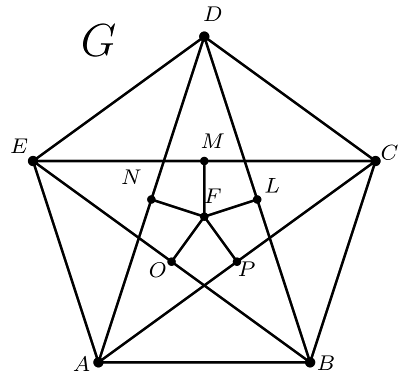
14 Graph Theory

83. Determine the number of edges in a graph with 6 vertices, where 2 vertices have degree 4 and 4 vertices have degree 2. Draw two such connected graphs.

84. The following graph G is given:



(i) Provide an example of a subgraph with 6 vertices that is not an induced subgraph.

(ii) Provide an example of an induced subgraph with 6 vertices.

(iii) Provide an example of a walk of length 7 that is not a path.

(iv) Provide an example of a graph H with vertex set $\{a, b, c, d, e, f, p, l, m, n, o\}$ that is isomorphic to the given graph G and is drawn in a different way.

(v) Is G planar? (Provide a detailed justification for your answer.)

(vi) Is $\chi(G) = 2$? (Provide a detailed justification for your answer.)

15 Graph

85. (i) Let $e_1, e_2, \ldots, e_n$ be the edges of graph G, and let x_{i-1} and x_i be the endpoints of edge e_i $(1 \leq i \leq n)$. Show that any closed walk $(e_1, e_2, \ldots, e_n)$ of length at least 3 with pairwise distinct vertices $x_1, x_2, \ldots, x_n$ is a cycle.

(ii) A graph that contains no cycles is called **acyclic**. A walk is acyclic if the subgraph consisting of the vertices and edges of the walk is acyclic. Prove: A walk has all distinct vertices if and only if it is an acyclic trail.

(iii) If u and v are distinct vertices of graph G, and there exists a walk in G from u to v, show that there exists an acyclic trail from u to v. *(Explain in detail why every shortest walk from u to v is in fact an acyclic trail.)*

86. Show that the sum of the degrees of all vertices in a graph is equal to twice the number of edges, i.e.,

$$\sum_{v \in V(G)} d(v) = 2 \cdot |E(G)|.$$

87. Let G be a simple graph with n vertices. Show that G must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

16 Tree

88. Given a tree with exactly five vertices of degree 2, three vertices of degree 3, four vertices of degree 4, and no vertices of degree greater than 4, determine how many vertices of degree 1 the tree has.

89. Let G be a simple graph with n vertices and $n-1$ edges. Is it true that G is a forest if and only if G is connected? Prove the statement or provide a counterexample.

90. Show that for any tree T, there exists exactly one path between every pair of vertices.

91. Show that if a graph G has exactly one path between every pair of vertices, then G is a tree.

92. Prove that every tree is a planar graph.

93. Let T be a tree. Show that if T contains a vertex of degree $k \geq 2$, then it also contains at least k vertices of degree 1.

94. Prove: If an edge e is the heaviest edge in a cycle C of a weighted graph G, then there exists a minimum spanning tree of G that does not include edge e.

All above math problems are taken from the following website:
https://osebje.famnit.upr.si/~penjic/teaching.html.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.