7 Properties of Binomial Coefficients

36. Prove that for $n > 0$ the following equality holds:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

37. Find a simpler expression for

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2z}{2}^k x^{n-k}.$$

The final expression should be the n-th power of a quadratic polynomial in variables x and z, written in simplified form (e.g., $ax^2 + bz^2 + cxz + dx + ez + f$, for some constants a, b, c, d, e, f).

38. Let X be a set of size $n > 0$.

(i) Let $\mathcal{S}$ be the set of all subsets of X with even cardinality, and let $\mathcal{L}$ be the set of all subsets of X with odd cardinality. Let x be any element of X, and define $f : \mathcal{S} \to \mathcal{L}$ as follows: for every $A \in \mathcal{S}$, let

$$f(A) = \begin{cases} A \setminus \{x\}, & \text{if } x \in A, \\ A \cup \{x\}, & \text{if } x \notin A. \end{cases}$$

Prove that the mapping f is bijective.

(ii) Using the result from part (i), provide a combinatorial proof of the identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

39. A thief is attempting to crack a door code. The code consists of 12 buttons labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, #, and $*$, and requires six digits followed by either # or $*$. Determine the number of possible codes:

(i) without any additional constraints.

(ii) if no digit is repeated.

The thief observes the buttons and notices the following:
- Buttons 4, 6, 7, 8, and $*$ are never used in the code.
- The button 1 appears at least twice.
- There are no guarantees for the usage of other buttons.

(iii) How many codes are possible given these new hints?

(iv) How many codes are possible if the thief also determines that 1 appears strictly more often than 9, and 9 appears strictly more often than 2?

8 Principle of Inclusion and Exclusion

40. A binary string of length eight is called a syllable. Find the number of syllables in which the second bit is 0 or the third bit is 1. Calculate the exact numerical value of your solution!

41. How many natural numbers ≤ 70 are coprime to 50 (i.e., they share no common divisors with 50 other than 1)?

42. How many ten-digit binary strings can be formed that either begin with 111 or end with 101 (or both), if:

(a) there are no additional restrictions?

(b) the string contains exactly 6 ones?

43. Out of 100 students, 46 play volleyball, 47 play handball, and 48 play football. Additionally, 12 students play any pair of these sports (but not the third), and 14 students play all three sports. How many students do not play any of these three sports?

44. Among the students in a dormitory:

- 12 play football (N), 20 play basketball (K), 20 play volleyball (O), and 8 play handball (R).
- 5 students play both N and K, 7 play both N and O, and 4 play both N and R.
- 16 students play both K and O, 4 play both K and R, and 3 play both O and R.
- 3 students play N, K, and O, 2 play N, K, and R, 2 play K, O, and R, and 3 play N, O, and R.
- 2 students play all four sports.
- Additionally, 71 students do not play any of the four sports.

Find the total number of students in the dormitory.

45. In how many ways can 20 identical candies be distributed among three children if we allow some children to receive no candies, and:

(a) there are no restrictions on the number of candies per child?

(b) at least one child receives at least 7 candies?

(c) at least one child receives at least 10 candies?

All above math problems are taken from the following website:
https://osebje.famnit.upr.si/~penjic/teaching.html.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.