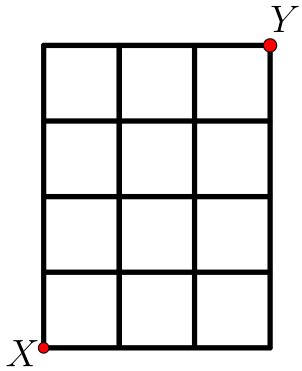
4 Ordered Selection with Repetition. Ordered Selection without Repetition.

17. A student wants to walk from point X to point Y along the streets as shown in the figure on the right. How many different shortest paths can they choose?



18. From a box containing balls numbered from 1 to 20, 6 balls are chosen. How many ways can this be done if

(a) the balls are selected one at a time and returned to the box?

(b) the balls are selected one at a time and not returned to the box?

(c) the balls are selected two at a time and not returned to the box?

(d) all 6 balls are selected at once?

Provide the result as an exact numerical value. For each part of the problem, give at least three examples of arrangements that can be made before calculating numerical values.

19. In a swimming competition with four swimmers, how many ways can the competition end if ties are possible?

20. A man has 12 relatives: 5 are men and 7 are women. His wife also has 12 relatives: 7 are men and 5 are women. They have no relatives in common. The couple decides to invite 12 relatives for a visit. In how many ways can they do this if

(a) each invites 6 relatives, and exactly 6 of the invited people are men?

(b) both invite the same number of women?

It is not necessary to express the answers with exact numerical values.

5 Permutations

21. In how many ways can 10 people be arranged into pairs? Before calculating the numerical value, provide at least three examples of possible arrangements.

22. In how many different ways can 4 different mathematics books, 3 different fiction books, and 5 different encyclopedias be arranged on a shelf if

(a) the books can be mixed freely?

(b) all mathematics books must be grouped together?

(c) books of the same type must be grouped together?

23. In how many ways can a group of 6 women and 5 men line up for a photograph if no two people of the same gender are standing next to each other?

24. First, list 12 distinct permutations of the following 10 letters: $a, a, a, b, b, n, n, e, s, t$. Then prove that the number of distinct permutations of n objects, where the first object appears k_1 times, the second object appears k_2 times, the third object appears k_3 times, and all remaining objects appear exactly once, is given by

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3!}.$$

Explain each step of the proof in detail.

25. (a) How many different "words" can be formed by rearranging the letters of the word *banana*?

(b) How many words from part (a) have the letter b immediately before the letter a?

(c) How many words from part (a) do not contain the sequence bnn?

(d) How many words from part (a) have the letter b appearing before all a's (not necessarily immediately before them)?

6 Principle of Double Counting

26. Prove the equality

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

by providing a combinatorial proof (using the principle of double counting) and addressing the following question: In how many different ways can we form a word of length n using any number of the letters A and B?

27. Prove the equality

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

by providing a combinatorial proof (using the principle of double counting).

28. Prove the equality

$$\sum_{j=0}^{t} \binom{a}{j} \binom{b}{t-j} = \binom{a+b}{t}$$

by providing a combinatorial proof (using the principle of double counting) and addressing the following question: In how many different ways can we form two words of lengths a and b using exactly t letters A (which must be used) and any number of letters B?

29. Provide a combinatorial proof (using the principle of double counting) of the following equality:

$$\binom{n+m}{k} = \sum_{\ell=0}^{k} \binom{n}{\ell} \cdot \binom{m}{k-\ell} \qquad (0 \le k \le m \le n).$$

30. Prove the equality

$$\sum_{k=0}^{m} k \cdot \binom{n}{k}\binom{m}{k} = n \cdot \binom{n+m-1}{n}$$

by providing a combinatorial proof (using the principle of double counting).

31. Prove the equality

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

(a) Provide an algebraic proof.

(b) Provide a combinatorial proof (using the principle of double counting), by addressing the following questions: How many different ways can we choose a pair of subsets (X, Y) such that $X \subseteq Y$ for a set of size n? How many different sequences of length n can we form using the digits 0, 1, and 2?

32. Prove the equality

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

by providing a combinatorial proof.

33. Prove the equality

$$\binom{n+1}{r+1} = \sum_{k=r+1}^{n+1} \binom{k-1}{r}$$

by providing a combinatorial proof.

34. (a) Provide a combinatorial proof of the equality

$$\sum_{k=2}^{n} \binom{n}{k}\binom{k}{2} = \binom{n}{2} 2^{n-2} \tag{1}$$

for $n \geq 2$.

(b) Write the identity (1) for $n = 5$ and interpret it using Pascal's triangle.

35. (a) Provide (algebraic or combinatorial) proof of the equality

$$\sum_{j=0}^{n} \binom{k+j}{j} = \binom{n+k+1}{n} \tag{2}$$

for any two numbers $n, k \in \mathbb{Z}^+$.

(b) Write the identity (2) for $k = 1$ and use the identity to compute the sum $\sum_{j=0}^{n} j$.

(c) Use the identity (2) to compute $\sum_{j=0}^{n} j^2$.

(d) Write the identity (2) for $n = 4$ and $k = 2$ and interpret it using Pascal's triangle.

All above math problems are taken from the following website:
https://osebje.famnit.upr.si/~penjic/teaching.html.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.