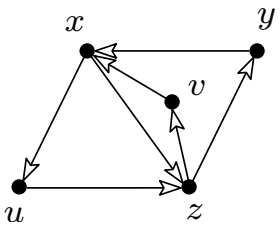
22 Digraph

110. Consider the following digraph D.

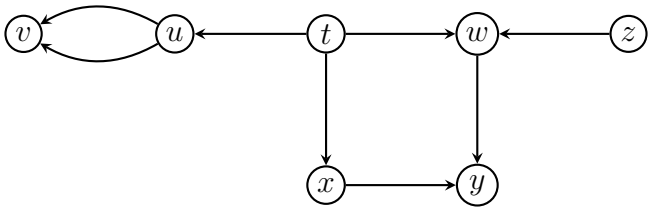


(i) Find the difference between the in-degree and out-degree for each vertex.

(ii) Does D contain an Eulerian circuit? Justify your answer.

111. Two special types of vertices are more interesting in digraphs than in undirected graphs. A vertex that is not the tail of any edge, i.e., with no "arrows" leading away from it, is called a **sink**. A **source** is a vertex that is not the head of any edge.

Consider the given digraph D.



(i) Is D weakly connected?

(ii) Is D strongly connected?

(iii) Does there exist a cycle in the digraph D?

(iv) Identify all sinks in the digraph D.

(v) Identify all sources in the digraph D.

112. (a) Provide three different examples of directed graphs with the vertex set $\{a, b, c, d, e, f, g, h\}$. Recall that a directed graph is a digraph whose underlying graph is simple.

(b) Determine the number of directed graphs with the vertex set $\{a, b, c, d, e, f, g, h\}$.

(c) Provide three different examples of digraphs with the vertex set $\{a, b, c, d, e, f, g, h\}$ that do not contain multiple edges or loops.

(d) Determine the number of digraphs with the vertex set $\{a, b, c, d, e, f, g, h\}$ that do not contain multiple edges or loops.

113. Prove that for every natural number $n \geq 2$, the edges of a complete graph on n vertices can be oriented such that all vertices of the resulting digraph have different out-degrees.

114. In class, we showed that the number of simple graphs with the vertex set $\{1, 2, \ldots, n\}$ is $2^{\binom{n}{2}}$.

(a) Determine the number of digraphs with the vertex set $\{1, 2, \ldots, n\}$ that do not contain multiple edges (i.e., more than one directed edge from the same start vertex to the same end vertex) or loops (i.e., directed edges that connect a vertex to itself).

(b) Determine the number of digraphs with the vertex set $\{1, 2, \ldots, n\}$ that do not contain multiple edges, but each vertex can have at most one loop.

(c) Determine the number of directed graphs with the vertex set $\{1, 2, \ldots, n\}$.

(d) A **tournament** is a directed graph whose underlying graph is complete. Determine the number of tournaments with the vertex set $\{1, 2, \ldots, n\}$.

115. Show that a digraph D in which all vertices have the same in-degree as out-degree is strongly connected if and only if it is weakly connected.

116. Two special types of vertices are more interesting in digraphs than in undirected graphs. A vertex that is not the tail of any edge, i.e., with no "arrows" leading away from it, is called a **sink**. A **source** is a vertex that is not the head of any edge.

Show that every acyclic digraph has at least one sink and at least one source.

117. If the vertices of a digraph with n vertices can be labeled from 1 to n such that $i > j$ whenever there is a path from vertex i to vertex j, then this labeling is called a **topological ordering**.

Show that every acyclic digraph has at least one topological ordering.

23 Tree and Digraph

118. Let T be a tree. Show that the edges of the tree T can be oriented so that in the resulting directed graph, the in-degree and out-degree of each vertex differ by at most one.

24 Matching

119. If all vertices of a graph G have the same degree k, the graph is called ***k*-regular**.
Let $k > 0$ and G be a bipartite k-regular graph with bipartition $\{A, B\}$. Show that the graph G has a matching that saturates A.

All above math problems are taken from the following website:
https://osebje.famnit.upr.si/~penjic/teaching.html.
The reader can find all solutions to the given problems on the same page.