Cross-polytopal Alexander Duality

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Abstract

This paper is inspired by classical and combinatorial variations of Alexander duality and motivated by applications to graphs with a perfect matching. We develop a notion with similar homological properties for complexes embedded into the cross-polytope. We relate to the work of Björner et al. in [3], and show that poset Bier spheres are vertex-decomposable. We further generalize their results to arbitrary triangulations of spheres. As a consequence, we also give an explicit method for constructing abstract n-spheres on $n+4$ vertices.

1 Introduction

Alexander duality stems from a result of J.W Alexander in 1915. It describes the homology of the complement of a subspace X in Euclidean space, a sphere, or another manifold. This theorem says:

Theorem 1.1 (Alexander duality theorem)**.** *If X is a compact, locally contractible, nonempty, proper subspace of the sphere $\mathbb{S}^n$, then:*

$$\forall i \ \tilde{H}_i(\mathbb{S}^n \setminus X) \approx \tilde{H}^{n-i-1}(X).$$

In 1983, Gil Kalai [7] first noticed that the *blocker* of a simplicial complex Δ on vertex set V, denoted as $\Delta^* = \{\sigma \in V : V \setminus \sigma \notin \Delta\}$, is homotopy equivalent to the complement of Δ in the simplex boundary. Thus:

Theorem 1.2. *Let Δ be a simplicial complex on the vertex set of size n, then for every i:*

$$\tilde{H}_i(\Delta) \approx \tilde{H}^{n-i-3}(\Delta^*).$$

Although this was an easy consequence of Theorem 1.1, a simple and self-contained proof was also presented by Björner et al. and Tancer [4]. We will review a variation of their proof later. Blocker has been a very useful tool, for example see [3], [5], [6].

In this paper, we will describe an analog to the blocker by working in the cross-polytope. This will give us a potentially useful tool to tackle problems regarding graphs with perfect matching, whose independence complexes correspond naturally to subcomplexes of the cross-polytope. We will then adapt the work of Björner et al. in [3] to our definitions and give a way of constructing many vertex-decomposable spheres.

This paper will be organized as follows: In the second section, we introduce some basic definitions of concepts related to simplicial complexes in general. In the third section, we introduce formally the notion of combinatorial Alexander duality as well as some results relating to it. In section four we introduce our new notion of Alexander duality on the cross-polytope and provide some basic calculations using it. We will then prove the main results of the paper.

2 Background

In this section, we will define the main mathematical object that we will use in this paper, a simplicial complex. There are a few ways to define a simplicial complex. We will begin with a purely combinatorial approach, by defining abstract simplicial complexes. Throughout the paper, we will see that some of the results and theorems about simplicial complexes sometimes come from combinatorial and algebraic arguments, while others are purely topological.

Definition 2.1. *An* abstract simplicial complex Δ *is a collection of sets that is closed under taking subsets.*

We call the elements of Δ *faces*, we will call a face with n elements a $(n-1)$-face. Maximal elements of Δ are called *facets*. Particularly 0-faces will be called vertices and 1-faces edges. In case all the facets have the same dimension we call the complex *pure*.
We can construct the topological space related to a complete abstract simplicial complex on n vertices (a *simplex*) by placing n linearly independent points $v_1, v_2, \dots, v_n$ into $\mathbb{R}^{n-1}$ and taking the convex hull.
Now for an arbitrary simplicial complex Δ we construct the corresponding topological space in two steps. First we identify the dimension of maximal facets, say m. We then place n points in $\mathbb{R}^{m-1}$ and repeat the above process for each facet of Δ on corresponding vertices. We obtain the corresponding space by taking the union over all facets of Δ. We call this topological space a *geometric realization* of Δ.

Example 2.1. *We will set a running example of*

$$\Delta = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_4\},$$

$$\{x_3, x_4, x_5\}\{x_1, x_3, x_5\}, \{x_1, x_2, x_6\}, \{x_2, x_4, x_6\}, \{x_1, x_5, x_6\}\}$$

We omit all the smaller faces and just list the facets in order to simplify writing. We will later see that Δ is actually the boundary of the octahedron, or 3-dimensional cross-polytope.

We can naturally define a face lattice $L(\Delta)$ of Δ by ordering faces with inclusion and adding an artificial top element $\hat{1}$.
Conversely, we can define the order complex $\Delta(P)$ of a bounded poset P with top $\hat{1}$ and bottom $\hat{0}$ as a complex on vertex set $P \setminus \{\hat{1}, \hat{0}\}$ with faces equal to chains in $P \setminus \{\hat{1}, \hat{0}\}$.

Example 2.2. *The Boolean lattice B_n is isomorphic to the face lattice of the boundary of the simplex on n vertices.*

Proposition 2.1 ([19] Page 6)**.** *Let K be a simplicial complex, then $\Delta(L(K))$ is isomorphic to the barycentric subdivision of K.*

Two important complexes related to a vertex v of a simplicial complex Δ are the link, the deletion and the star:

For a simplicial complex Δ on vertex set V and v we define the *deletion* of $v \in V$ as $\Delta \setminus v = \{\sigma \in \Delta : v \notin \sigma\}$, the *link* of v as $\operatorname{link}_\Delta v = \{\sigma \in \Delta \setminus \{v\} : \sigma \cup v \in \Delta\}$ and the *star* of v as $star_\Delta v = \{\sigma \in \Delta : \sigma \cup v \in \Delta\}$. See Figure 2
Counting information about a simplicial complex is stored in the corresponding f-vector:

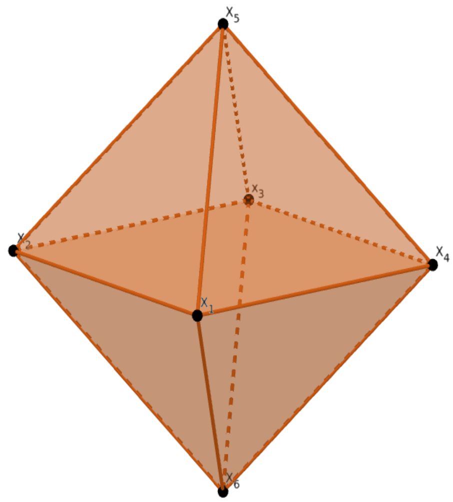

Figure 1: Geometric realization of Δ

Definition 2.2. *The* f-vector *of a simplicial complex* Δ *is the vector* $f_\Delta = (f_{-1}, f_0, f_1, \ldots, f_n)$ *where* f_n *is the number of* n*-dimensional faces. We set* $f_{-1} = 1$ *as we consider the empty set to be a* (-1)*-dimensional face.*

Sometimes it is useful and easier to talk about the h-vector (although it is not as clear what it counts):

Definition 2.3. *The* h*-vector of a simplicial complex* Δ *is the vector* $h_\Delta = (h_0, h_1, \ldots, h_n)$ *where* $\sum_{i=0}^{n} \frac{f_i t^i}{(1-t)^i} = \sum_{i=0}^{n} \frac{h_i t^i}{(1-t)^n}$.

There is no good combinatorial interpretation of the h-vector in general. However, it does exist for the class of partitionable complexes:

Definition 2.4. *We say that a simplicial complex* Δ *is* partitionable *if we can split* $L(\Delta)$ *into a disjoint union of intervals such that the top element of each interval is a facet.*

In the case Δ is pure and partitionable, we know that h_i counts the number of chains of length i in the face lattice of Δ [12].
We have already seen the bridge between posets and simplicial complexes. We will now take time to describe a relationship between commutative rings and simplicial complexes.
First, we note that in a polynomial ring $F[x_1, x_2, \ldots, x_n]$, a *square-free monomial* is any element of the form $\Pi_{i=0}^{d} x_i$. We call an ideal generated by such elements a *square-free monomial ideal.*
For every simplicial complex Δ we can naturally describe a square-free monomial ideal connected to it, and a corresponding factor ring. If Δ is a simplicial complex on a vertex set $V = \{x_1, x_2, \ldots, x_n\}$, and F is any field, then we will work in a ring $F[x_1, x_2, \ldots, x_n]$, where variables are in a natural correspondence with the vertices of the simplicial complex. We will also abuse notation and write $x_1x_2x_3$ for both the face and the corresponding polynomial in the ring. In this setting we can now define the following:

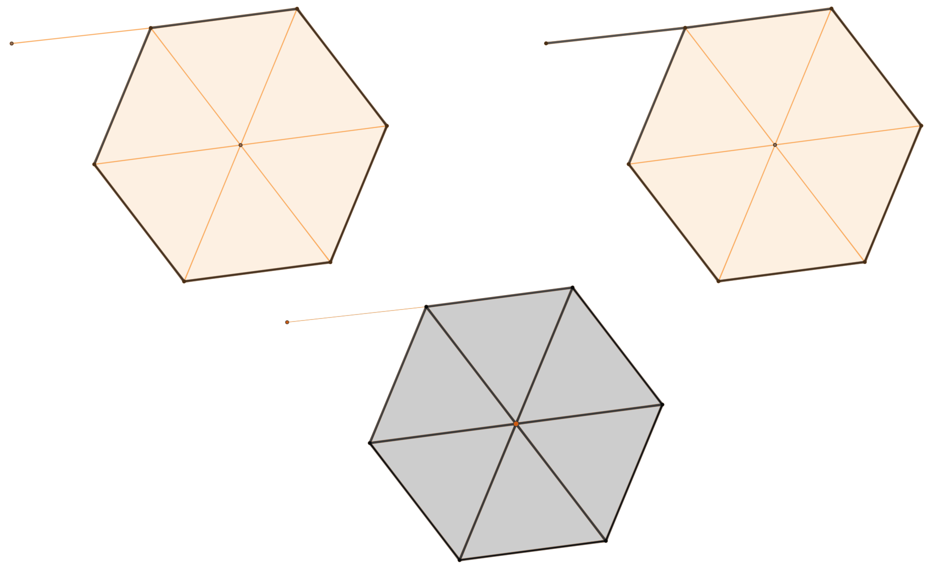

Figure 2: The link, deletion and star of the middle vertex

Definition 2.5. *For a simplicial complex* Δ*, the* Stanley-Reisner ideal *of* Δ *is* $I_\Delta = (\sigma \subset V : \sigma \notin \Delta)$.

We can see that I_Δ can be written with its minimal generators, i.e., the minimal non-faces of Δ. The quotient of the polynomial ring $F(x_1, x_2, \ldots, x_n)$ by $I(\Delta)$ is the *Stanley-Reisner ring of* Δ. This correspondance is reversible. If I is a square-free monomial ideal we can construct a corresponding simplicial complex as follows:

Definition 2.6. *For a square-free monomial ideal* $I \subset F[x_1, x_2, \ldots, x_n]$ *, the* Stanley-Reisner complex $\Delta(I)$ *of* I *is the simplicial complex consisting of the monomials not in* I*, i.e.,* $\Delta(I) = \{\sigma : \sigma \notin I\}$.

Example 2.3. *We go back to the octahedron:*

$$\Delta = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}\{x_1, x_3, x_5\},$$

$$\{x_1, x_2, x_6\}, \{x_2, x_4, x_6\}, \{x_1, x_5, x_6\}\}$$

Then we have: $I_\Delta = \{x_1x_4, x_2x_5, x_3x_6\}$.

We also state the following result, which is to be expected:
For any simplicial complex K, $\Delta(I(K)) = K$.
We now define two important homological properties of a simplicial complex, both of which are equivalent to notions in commutative algebra:

Definition 2.7. *The* depth *of simplicial complex* Δ *over a field* $\mathbb{F}$ *is the number*

$$\operatorname{depth}(\Delta) = \max\{d : \ \tilde{H}_i(\operatorname{link}_\Delta \sigma, \mathbb{F}) = 0 \ \forall \sigma \in \Delta; \ i < d - |\sigma|\}.$$

Definition 2.8. *The* Castelnuovo-Mumford regularity *of* Δ *over* $\mathbb{F}$ *is the number*

$$\operatorname{reg}(\Delta) = \max\{i : \tilde{H}_i(\Gamma, \mathbb{F}) \neq 0,\ \Gamma \textit{ is an induced subcomplex of } \Delta\},$$

or equivalently

$$\operatorname{reg}(\Delta) = \min\{i : \tilde{H}_i(\Gamma, \mathbb{F}) = 0, \quad \textit{for all induced subcomplexes } \Gamma \textit{ of } \Delta\}.$$

This allows us to define a few important classes of simplicial complexes and describe the relationships between them:

Definition 2.9. *If* $\operatorname{depth}(\Delta) = \dim(\Delta)$ *we say that* Δ *is* Cohen-Macaulay.

A closely related property is shellability:

Definition 2.10. *Let* Δ *be a simplicial complex. We say that* Δ *is* shellable *if its facets can be ordered linearly in such a way that the complex generated by the facet* σ_j*, for each* $j > 1$*, intersects in a pure* $(\dim \sigma_j - 1)$*-dimensional complex with the complex generated by the previous* $j - 1$ *facets in the ordering.*

The notion of shellability has proven to be useful. For example, it was used by McMullen [9] to prove the Upper Bound conjecture for the number of faces in convex polytopes.
Finally, we will discuss vertex-decomposable complexes: We will call a vertex v a *shedding vertex* if there exists another vertex w that satisfies:

$$\sigma \in \Delta, v \in \sigma \implies (\sigma \setminus v) \cup \{w\} \in \Delta.$$

Knowing this, we can give the following definition:

Definition 2.11. *We say that a simplicial complex* Δ *is* vertex decomposable *if it is a simplex or if it has a shedding vertex* v *such that* $\Delta \setminus v$ *and* $\operatorname{link}_\Delta(v)$ *are both vertex-decomposable.*

It is both easy and important to see that the definition gives us a way to recursively check whether a simplicial complex is vertex-decomposable.
The following list of implications is strict:

$$\text{Vertex-decomposable } + \text{pure} \implies \text{Shellable } + \text{pure} \implies \text{Cohen-Macaulay}.$$

3 Combinatorial Alexander Duality

In the introduction, we stated the Alexander duality theorem and presented shortly the concept of combinatorial Alexander duality arising from the blocker. We will now recall the definition and explore the material in more depth:

Definition 3.1. *Let* Δ *be a simplicial complex on a vertex set* V*. The* combinatorial Alexander dual *of* Δ *is the complex*

$$\Delta^* = \{\sigma : \bar{\sigma} \notin \Delta\}$$

The following statement is easy to see:

Proposition 3.1. *Let Δ be a simplicial complex, then $(\Delta^*)^* = \Delta$.*

Proof. The facets of $(\Delta^*)^*$ are $\{\sigma : \bar{\sigma} \notin \Delta^*\} = \{\sigma : \bar{\bar{\sigma}} \notin \Delta\} = \{\sigma : \sigma \in \Delta\}$, which are exactly the facets of Δ. □

Example 3.1. *If we look at the simplicial complex $\Delta = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$, whose geometric realization is a square, we see that it's minimal missing faces are $\{\{1,3\},\{2,4\}\}$. By taking complements, we get the facets of Δ^* to be $\{\{1,3\},\{2,4\}\}$.*

We now proceed to look at a connection between links and deletions of a vertex in Δ and Δ^*:

Lemma 3.1. *Let Δ be a simplicial complex on vertex set V and let $v \in V$. then:*

$$(\mathrm{link}_\Delta\, v)^* = \Delta^* \setminus v$$

$$(\Delta \setminus v)^* = \mathrm{link}_{\Delta^*}\, v.$$

Here we consider Δ^* to be a complex on vertex set $V \setminus \{v\}$.

Proof. 1. First let $\sigma \in (\mathrm{link}_\Delta\, v)^*$. This tells us that $\sigma^c \notin \mathrm{link}_\Delta\, v$ so either

$$\sigma^c \notin \Delta \implies \sigma \in \Delta^* \implies \sigma \in \Delta^* \setminus v$$

or alternatively:

$$\sigma^c \cup \{v\} \notin \Delta \implies \sigma \cap \{v\}^c = \sigma \in \Delta^* \implies \sigma \in \Delta^* \setminus v$$

so we have one inclusion. For the other inclusion let $\sigma \in \Delta^* \setminus v$, then $\sigma \in \Delta^*$ and $v \notin \sigma$, therefore:

$$v \in \sigma^c \implies \sigma^c \cup \{v\} = \sigma^c \notin \Delta \implies \sigma^c \notin \mathrm{link}_\Delta\, v \implies \sigma \in (\mathrm{link}_\Delta\, v)^*$$

2. Let $\sigma \in (\Delta \setminus v)^* \implies \sigma^c \notin \Delta \setminus v$ and as our vertex set doesn't include v this means that $\sigma^c \notin \Delta \implies \sigma \in \Delta^*$. Now assume that $\sigma \cup \{v\} \notin \Delta^*$, then $\sigma^c \cap \{v\}^c = \sigma \in \Delta$ giving us a contradiction and one inclusion. The other inclusion is trivial. □

We will now state the fundamental result regarding the Alexander dual. We will denote by $|\Delta|$ the topological space arising from Δ.

Theorem 3.1. *Let Δ be a nonempty, non-simplex complex on the vertex set V and let Σ be the boundary of the simplex on V. Then $|\Sigma| \setminus |\Delta|$ deformation retracts to $|\Delta^*|$.*

We will give multiple proofs of this. The first proof will use a strong result from Algebraic Topology, the Nerve Lemma. First, we define the nerve:

Definition 3.2. *If $J = U_1, U_2, \ldots, U_m$ form a cover of a topological space X, then the* nerve *of this cover is the simplicial complex with vertices $V = \{1,2,\ldots,m\}$, and faces consisting of subsets $\sigma \subset J$ such that $\bigcap_{i\in\sigma} U_i \neq \emptyset$.*

Example 3.2. *Consider the cover of the Octahedron by triangles (corresponding to the facets of the abstract simplicial complex). The nerve is a "thickened" cube. To see this we first have to notice that each vertex of the octahedron corresponds to the intersection of four triangles which will make up a tetrahedron in the nerve. Thus the nerve has as facets six tetrahedra. Collapsing an edge of each tetrahedron, we are left with 6 squares, each composed from two triangles. These make up a cube.*

The Nerve lemma gives us sufficient necessary conditions for this situation to occur:

Lemma 3.2 (Nerve lemma, [20] Theorem 1.7)**.** *If $U_1, U_2, \dots, U_n$ form a cover of a topological space X, and if for each $\sigma \in \{1, 2, \dots, n\}$ the intersection $\bigcap_{i\in\sigma} U_i$ is either empty or contractible, then the nerve of the cover is homotopy equivalent to X.*

From Lemma 3.2, together with the classical Alexander duality theorem, Theorem 3.1 follows as a corollary.

Proof of Theorem 3.1. We start by covering $|\Sigma|$ by small open neighborhoods of its facets, which automatically gives us a cover of $|\Sigma \setminus \Delta|$ by intersecting with $|\Sigma \setminus \Delta|$. Now we will call this cover A and look at $N(A)$. Before continuing we note that there is a one-to-one correspondence between facets of Σ and it's vertices, obtained by sending each facet to the unique vertex not contained in it. Now if we fix a facet σ and corresponding vertex v, intersecting any other facet with σ is equivalent to deleting v in $N(A)$. Thus, the faces of $N(A)$ are exactly the vertex subsets whose complement is not a face of Δ. Equivalently, they're faces of Δ^*. Applying the Nerve lemma, we get the desired result. $\square$

We will now move to the face lattice of a simplicial complex Δ. Then, we will see how we can use it to give a different description of Δ^*. We will reprove Theorem 3.1 to show that this description is worth considering. First, we need an additional definition and a lemma:

Definition 3.3. *For a poset $(P, \leq)$ we define the* dual poset *to be $P^{\updownarrow} = (P, \geq)$.*

Now in order to reprove Theorem 3.1, we need the following lemma:

Lemma 3.3. *If $P \subset Q$ are bounded posets, then $|\Delta(P)| \setminus |\Delta(Q)|$ deformation retracts to $|\Delta(Q) \setminus (P \cup \{\hat{1}, \hat{0}\})|$.*

For proof of 3.3 look at [2] [lemma 4.7.27]
With Lemma 3.3, we can reprove Theorem 3.1:

Proof. As Σ is the boundary of a simplex on the vertex set of our simplicial complex X, we have $L(\Sigma) = B_n$ where B_n is the boolean lattice. Now if we apply the lemma:

$$|\Delta(B_n)| \setminus |\Delta(L(X))| \simeq |\Delta(B_n \setminus L(X) \cup \{\hat{1}, \hat{(0)}\}|$$

If we call the poset on the right-hand side P, we can see that the faces of $\Delta(P)$ are chains of non-faces of Δ. We will now consider the map:

$$c : B_n \to B_n^{\updownarrow}$$

$$A \to \{1, 2, \dots, n\} \setminus A.$$

Thus is the complementation map. Thus, $c(P) \cong P^{\updownarrow}$ is the face lattice of X^*, which gives us the above-mentioned description and the desired result. $\square$

Before considering some more deep results that find usage for combinatorial Alexander duality, we will consider another way of describing it. This result was proven by Csorba in [5]. First, we will need two new definitions:

Definition 3.4. *For graph G we define G_n to be the graph G with every edge replaced with a path of length n.*

Definition 3.5. *For a graph G, we define* idependence complex *of G to be the simplicial complex whose faces are the independent sets of G. We will denote it* $\mathrm{Ind}(G)$*.*

Now we can state the main theorem from [5]:

Theorem 3.2. *For a graph G,* $\mathrm{Ind}(G)^*$ *is homotopy equivalent to* $\mathrm{Ind}(G_2)$*. Here we consider* $\mathrm{Ind}(G)$ *to be the complex on n+1 vertices with no face containing the additional vertex.*

Proof. We will make use of the nerve lemma, so we will first need to build our open cover. We do this in the following way:

1. Let $K_\emptyset$ be the subcomplex induced by $V(G_2) \setminus V(G)$
2. For each $v_i \; inV(G)$ we define $K_i = star_{\mathrm{Ind}(G_2)}(v_i)$.

Obviously, each K_i is open and a cone, hence contractible. Their union is $\mathrm{Ind}(G_2)$ and we just need to look at the intersection. First of all, $\bigcap K_i$ is going to be a cone at some vertex as $V(G)$ is an independent set in G_2 and thus it forms a facet in $\mathrm{Ind}(G_2)$ and all K_i will intersect as we wanted. Now we will let $k \leq n$ and consider: $K_\emptyset \cap K_1 \cap \cdots \cap K_k$, this intersection is empty if $V(G) \setminus v_1, \ldots, v_k$ forms an independent set. Otherwise, the intersection is a simplex where vertices correspond to the edges spanned by $V(G) \setminus v_1, \ldots, v_k$. By the nerve lemma $\mathrm{Ind}(G_2)$ is homotopy equivalent to the complex on $n+1$ vertices. Nonempty intersections correspond to non-independent sets, which are the non-faces of $\mathrm{Ind}(G)$, exactly as in Alexander dual. ☐

3.1 Eagon-Reiner theorem

Now we will discuss and prove the fundamental result proven by Eagon and Reiner in [6]. The result gives necessary and sufficient conditions for a simplicial complex Δ to be Cohen-Macaulay. To introduce the result, we need some algebraic notions. First of these is a resolution of a R-module for a ring R:

Definition 3.6. *For a ring R and R-module M, a resolution of M is the exact sequence of R-modules:*

$$\cdots \xrightarrow{d_{n+1}} E_n \cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} M \to 0.$$

Now as any ideal is also a free module we can also talk about resolutions of ideals over R. In particular, we say that resolution is free if all the modules E_i are free. If R is a polynomial ring then we say that an ideal I has a linear resolution if there exists a minimal free resolution where all modules in the sequence are of degree 1 in the standard grading of the polynomial ring R. Now we're set to state the theorem:

Theorem 3.3 (Eagon-Reiner theorem)**.** *Let Δ be a simplicial complex, it is Cohen-Macaulay if and only if I_{Δ^*} has a linear resolution.*

We will do the proof, as in [10], and for that we will need some strong results:

Theorem 3.4 (Reisner's criterion)**.** *Let Δ be a n-dimensional simplicial complex. Δ is Cohen-Macaulay if and only if, for every face σ, the link satisfies*

$$\tilde{H}^i(\mathrm{link}_\Delta(\sigma) = 0, \text{ whenever } i \neq \dim(\Delta) - |\sigma|.$$

Proof of Eagon-Reiner theorem. Assume that I_{Δ^*} is generated in degree d. By dual Hochester formula from [10, Corollary 1.40], the ideal has linear resolution if and only if for every $\sigma \in \Delta$

$$\tilde{H}_{i-1}(\mathrm{link}_\Delta(\sigma) = 0 \text{ whenever } i \neq |\bar{\sigma}| - d.$$

By our assumption Δ has dimension $n-d-1$, hence $\dim(\Delta) - |\bar{\sigma}| = |\sigma| - d - 1$. Therefore, Reisner's criterion holds for Δ. $\square$

We now consider a homological outlook on the Eagon-Reiner theorem, we first need the following lemma:

Lemma 3.4. *Let Δ be a complex on vertex set V, such that $|V| = n$, let σ be a face and F a field. Then*

$$\tilde{H}_i(\mathrm{link}_\Delta \sigma, F) = \tilde{H}_{n-|\sigma|-3-i}(\Delta^*, F).$$

Where Δ^* is considered as a complex on $V \setminus \sigma$.

The proof of this lemma boils down to two steps. First, we notice that $(\mathrm{link}_\Delta \sigma)^* = \Delta^*$ on the wanted vertex set. Second, we consider combinatorial Alexander dual on said vertex set and apply the homology/cohomology relation over the field.
Now we state the homological Eagon-Reiner theorem, as observed by Terai in [14]:

Theorem 3.5. *In the following calculations we will write $\Delta^*[V \setminus \sigma]$ to emphasize the vertex set we are working on.*

$$\begin{aligned}
\mathrm{depth}(\Delta) = \max\{d : \tilde{H}_i(\mathrm{link}_\sigma) = 0, \forall\sigma, i < d - |\sigma|\} = \\
\max\{d : \tilde{H}_{n-|\sigma|-3-i}(\Delta^*[V \setminus \sigma]), \forall\sigma, i < d - |\sigma|\} = \\
\max\{d : \tilde{H}_j(\Delta^*[V \setminus \sigma]), \forall\sigma, j > n - 3 - d\} = \\
n - 2 + \max\{d : \tilde{H}_j(\Delta^*[V \setminus \sigma]), \forall\sigma, j > -1 - d\} = \\
n - 2 + \min\{d : \tilde{H}_j(\Delta^*[V \setminus \sigma]), \forall\sigma, j > d - 1\}.
\end{aligned}$$

As $V \setminus \sigma$ is a non-face of Δ^* for each σ (in fact these are exactly the non-faces), and as each σ has 0-homology, the last line is $n - 2 - \mathrm{reg}(\Delta^*)$.

3.2 Alexander dual and f and h-vectors

In this section, we will describe the relationship between the f- and h-vectors of a simplicial complex Δ and those of Δ^*. As f-vectors are somewhat easier to deal with, we will first find a good way to transition between the $f-$ and h-vectors. This will be done using the following:

Definition 3.7. *For a n-dimensional simplicial complex Δ, we define the F-*polynomial *to be $F_\Delta(x) = \sum_{i=-1}^{n} f_i^\Delta x^{i+1}$. The H-*polynomial *is defined in a completely analogous way, $H_\Delta(x) = \sum_{i=0}^{n} h_i^\Delta x^i$.*

The result we are about to present was presented first in a highly circulated preprint by Terai [15] which was, unfortunately, never published. Parts of the preprint appeared in Terai's later works on Alexander duality [13, 14]. We compute as in [15] with $d^* = \dim(\Delta^*)$. Recall that

$$\frac{H_\Delta(x)}{(1-x)^d} = \sum_{i=0}^{d} \frac{f_i x^i}{(1-x)^i}$$

so that also

$$\frac{H_\Delta(1-x)}{x^d} = \sum_{i=0}^{d} \frac{f_i(1-x)^i}{x^i}.$$

Now we observe the following relationship between the H-polynomial of Δ and that Δ^*:

$$\frac{H_\Delta(x)}{(1-x)^d} = \sum_{i=0}^{d} \frac{f_i x^i}{(1-x)^i}$$

$$= \sum_{i=0}^{n} \frac{\binom{n}{i} x^i}{(1-x)^i} - \sum_{i=0}^{n} \frac{f^*_{n-i} x^i}{(1-x)^i}$$

$$\left(1+\frac{x}{1-x}\right)^n - \left(\frac{x}{1-x}\right)^n \sum_{i=0}^{n} \frac{f^*_{n-1}(1-x)^{n-i}}{x^{n-i}}$$

$$= \frac{1}{(1-x)^n} - \frac{x^n}{(1-x)^n} \sum_{j=0}^{d^*} \frac{f^*_j(1-x)^j}{x^j}$$

$$= \frac{1}{(1-x)^n} - \left(\frac{x}{1-x}\right)^n \frac{H_{\Delta^*}(1-x)}{x^{d^*}}$$

3.3 Bier spheres

In this section, we will look at the concept first presented by Bier in his unpublished notes[1], the so-called Bier spheres. Bier discovered a way to make a large number of simplicial spheres by gluing an abstract simplicial complex and its Alexander dual in a smart way. We will present this similarly as in [8].

Definition 3.8. *For a simplicial complex Δ, we define the* Bier sphere *of Δ as*

$$\text{Bier}(\Delta) = \{\sigma \uplus \delta | \sigma \in \Delta, \delta \in \Delta^*, \sigma \cap \delta\}.$$

Here, the symbol $\uplus$ means that we are taking faces from two copies of Δ and taking the simplex spanned by the union, i.e. $\sigma \uplus \delta = (\sigma \times \{1\}) \cup (\delta \times \{2\})$.
It is not obvious that this construction yields a sphere, except in extremely simple cases such as the boundary of a simplex or the empty complex. The following theorem tells us that this is in fact always the case.

Theorem 3.6. *For a simplicial complex Δ on a vertex set V with n vertices, the Bier sphere* $\text{Bier}(\Delta)$ *is a simplicial complex on at most $2n$ vertices, and has geometric realization homeomorphic to $\mathbb{S}^{n-2}$.*

In order to prove Theorem 3.6, we need the following definition and lemma:

Definition 3.9. *For simplicial complexes X and Y, we define their* join*, denoted, $X * Y$ to be the simplicial complex with faces:*

$$\{\delta \uplus \sigma : \delta \in X, \sigma \in Y\}.$$

Lemma 3.5. *For a simplicial complex Δ, the complex* $\mathrm{Bier}(\Delta)$ *has facets $\sigma \uplus \delta^c$ where $\sigma \subset \delta$ and $\sigma \in \Delta$, $\delta \notin \Delta$, $|\sigma \setminus \delta| = 1$.*

Proof. For any face $\sigma \uplus \delta^c$ it holds that $\sigma \subset \delta$ and there are σ' and δ' satisfying the conditions of the lemma, even further $\sigma \uplus \delta^c \subseteq \sigma' \uplus \delta'^c$ as faces of the Bier sphere. □

Now we can start proving the Theorem 3.6:

Proof. We will proceed by induction on k, the number of faces of a simplicial complex Δ on the vertex set V, such that $|V| = n$. If $k = 0$, then $\Delta = \{\emptyset\}$ and $\mathrm{Bier}(\Delta)$ has geometric realization homeomorphic to $\mathbb{S}^{n-2}$. We will now assume that $\mathrm{Bier}(\Delta)$ has the same property for arbitrary k and we will add a new face σ to Δ. We now have a simplicial complex with facets:

$$\Delta \setminus \{(\sigma \setminus \{x\}) \uplus \sigma^c : x \in \sigma\} \cup \{\sigma \uplus (\sigma \cup \{y\})^c : y \notin \sigma\}$$

We can now notice that the vertex set that is common for all added and removed faces is $\sigma \uplus \sigma^c$, and the subcomplexes induced by these sets in $\mathrm{Bier}(\Delta)$ and $\mathrm{Bier}(\Delta \cup \sigma)$ are two $n-2$ balls sharing a boundary. Also, if δ is a face with vertex outside of this set, it will not contain a face "inside" of these balls, only on the boundary. This means that adding σ to Δ is equivalent to taking out a ball and replacing it with a finer ball in $\mathrm{Bier}(\Delta)$. Thus, $\mathrm{Bier}(\Delta)$ and $\mathrm{Bier}(\Delta \cup \sigma)$ are topologically equivalent, proving our theorem. □

This gives us a way to construct simplicial spheres. Furthermore, we can use this to show that there are shellable triangulations of spheres which can't be realized as boundaries of convex polytopes. A proof of this fact can be found in [8]. Matoušek constructs $O(2^{(2^n/n)-2n^2})$ nonisomorphic spheres. This ends up being more than the number of convex polytopes on $2n$ vertices.

In later work, Björner et al. [3] showed that there is a beautiful generalization of Biers method. This is done by looking at posets and reinterpreting the construction in that setting. First, they generalized the construction to an arbitrary poset, their so-called *Bier posets*. They showed that the original construction of Bier comes out when considering ideals of the Boolean lattice. We will now take a look at their definition of a Bier poset and state a few important results from that paper. First, we recall the definition of a poset ideal:

Definition 3.10. *Let P be a poset and let $I \subset P$ be nonempty. We say that I is an* ideal *of P if: 1. whenever $x \in I$ and $y \leq x$, then $y \in I$ 2. whenever $x, y \in I$ and $y \leq x$, then there is a $z \in I$ such that $x, y \leq z$.*

Definition 3.11. *Let P be a bounded poset of finite length and I a proper ideal. Then the* Bier poset *of I in P is denoted* $\mathrm{Bier}(P, I)$*, and consists of the intervals of the form $[x, y]$ where $x \in I$ and $y \notin I$, ordered by reverse inclusion.*

In the original paper, it is shown that if P is the face lattice of a strongly regular piecewise linear CW-sphere, then so is $\mathrm{Bier}(P, I)$ for any ideal I. Here, strongly regular means that each attaching map is a homeomorphism, and the intersection of any two faces is again a face. It is also shown that if P is Cohen-Macaulay, then so is $\mathrm{Bier}(P, I)$. However, a main property of Bier spheres that they have proven is that Bier spheres are shellable ([3], Thm 4.1). This gives a construction of a large number of shellable spheres. In Subsection 4.1 we give an improvement to their shelling result.

3.4 Self-dual complexes

Now we will look at results by Timotijević [16] regarding the combinatorial structure of self-dual simplicial complexes. We first give a few definitions:

Definition 3.12. *Let Δ be a simplicial complex, then:*

1. *Δ is self-dual if it is isomorphic to Δ^*.*
2. *Δ is sub-dual if it is isomorphic to a subcomplex of Δ^*.*
3. *Δ is super-dual if it is isomorphic to a supercomplex of Δ^*.*

Self-dual complexes were extensively studied by other authors and have applications in Optimization Theory and Algebraic topology. For example, such complexes have been used by Matoušek in [8] as prime examples of nonembedable objects in $\mathbb{R}^{2k}$.
Now we will describe necessary and sufficient conditions needed for an arbitrary simplicial complex Δ to be self-dual:

Theorem 3.7. *Let Δ be a simplicial complex on the vertex set V, then:*

1. *Δ is sub-dual if and only if there is no simplex $\sigma \subseteq V$ such that both σ and $V \setminus \sigma$ are in Δ,*
2. *Δ is super-dual if and only if there is no simplex $\sigma \subseteq V$ such that both σ and $V \setminus \sigma$ are not in Δ, and*
3. *Δ is self-dual if and only if for any arbitrary $\sigma \subset V$, either $\sigma \in \Delta$ or $V \setminus \sigma \in V$.*

Proof. 1. First, we assume that Δ is sub-dual and that such σ exists. Then σ and $V \setminus \sigma$ are complements to non-faces of Δ and are therefore not in Δ, contradicting our assumption. Now we assume that there is no such σ. It follows that for arbitrary $\delta \in \Delta$, $V \setminus \delta \notin \Delta$ which means that $\delta \in \Delta^*$, finishing the proof.
2. This follows from the fact that Δ is super dual if and only if Δ^* is sub dual and applying the first part of the theorem.
3. In order for Δ to be self-dual it needs to be both super-dual and sub-dual. Thus by first two parts of the theorem, if $\sigma \in \Delta$, then $V \setminus \sigma$ is not in Δ as Δ is sub-dual, a similar argument works in case $\sigma \notin \Delta$.

□

We will also state the following facts regarding the structure of self-dual complexes without proof, all of which were proven by Timotijević in ([16] section 7).

Proposition 3.2. *Let Δ be a self-dual simplicial complex on the vertex set V. Then for any vertex v, $\operatorname{link}_\Delta(v)$ is a sub dual complex on vertex set $V \setminus \{v\}$.*

Proposition 3.3. *Let Δ be as before. Then for any vertex $v \in \Delta$ we have:*

$$\Delta = \operatorname{link}_\Delta(v)^c \cup C(\operatorname{link}_\Delta(v))$$

where $C(X)$ is the cone over X.

The next result is the main result of the original paper by Timotijević and gives us a pleasant tool for combinatorial classification of self-dual simplicial complexes.

Theorem 3.8. *Let X and Y be two self-dual simplicial complexes on the vertex set V. Then X and Y are isomorphic if and only if there exist vertices $x \in X$ and $y \in Y$ such that $\operatorname{link}_X(x)$ and $\operatorname{link}_Y(y)$ are isomorphic.*

4 Cross-polytopal Alexander duality

In this section, we will describe the main problem that we are trying to tackle. We will describe Alexander duality over subcomplexes of the cross-polytope, rather than a simplex. The *n-dimensional cross-polytope* is a convex hull of $2n$ vertices chosen as endpoints of unit vectors pointing along each co-ordinate axis. Since we will work on the boundary of the cross-polytope, we will say "cross-polytope" to mean the boundary. We describe this boundary as follows:

Definition 4.1. *The* n-dimensional cross-polytope *is the independence complex of the graph defined on vertex set* $\{x_1, y_1, x_2, y_2, \ldots x_n, y_n\}$ *with edges* (x_i, y_i) *for each* $1 \leq i \leq n$.

We use this definition to shine the light on why this concept of duality is interesting. We can easily construct a large number of complexes and their duals when working inside an octahedron. We do this by taking supergraphs of the graph described in the definition above, the "matching graph" and considering their independence complexes. Specifically, these graphs are interesting because they have a perfect matching.
To start our work on the cross-polytope we will first note that a 1-dimensional cross-polytope is the suspension of the empty set and in general, the n-dimensional cross-polytope is the suspension of the $(n-1)$-dimensional cross-polytope.
Now we will construct a crucial tool that we use in order to talk about our notion of duality, the *facet cover*. The facet cover of a simplicial complex Δ, embedded into n-dimensional cross polytope X is obtained by covering X with its facets and taking the intersection with $|\Delta|$.
With this, we are ready to define the first version of "cross dual":

Definition 4.2. *Let X be the n-dimensional cross-polytope and let Δ be a subcomplex of X. We define the* cross-dual *of Δ to be the nerve of the facet covering of $|\Delta|$ and denote it Δ^{cd}.*

Before we look at some examples, it is important to note that our definition has the same behavior as combinatorial and classical Alexander duality. This will be a consequence of the Nerve lemma we discussed earlier and the fact that homology/cohomology groups are homotopy type invariant.

Example 4.1. *Consider the* 2*-dimensional cross-polytope (square) S embedded inside the* 3*-dimensional cross-polytope (octahedron) X. We can see that $|X| \setminus |S|$ is two pyramids without the bottom boundary. Thus, the cross-dual of S is two tetrahedra. See Figure 3.*

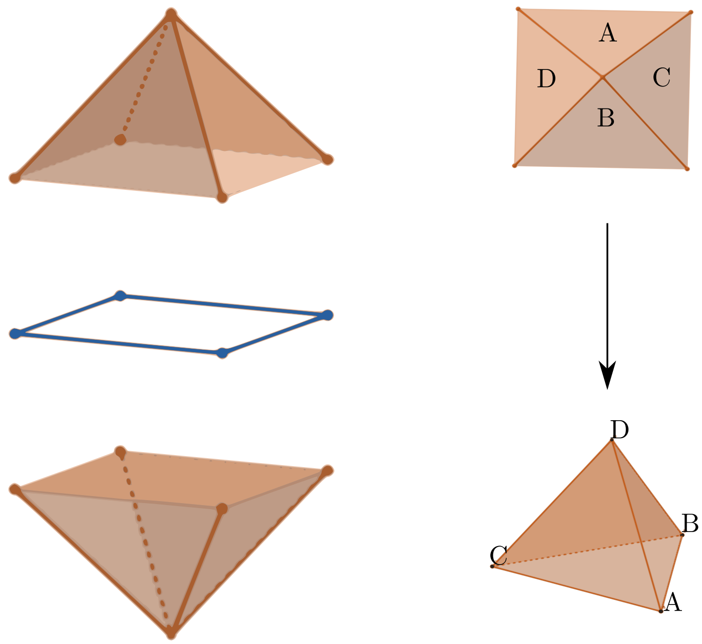

Figure 3: The process of obtaining the cross-dual of S

It is not hard to see that Example 4.1 can be generalized to the n-dimensional cross-polytope ($n \geq 1$) embedded into the $(n+1)$-dimensional cross-polytope. In this situation, taking the cross-dual yields two disjoint $(n+1)$-dimensional simplices.
An alternative and closely related definition, can be obtained by looking at the face lattice of the cross-polytope:

Definition 4.3. *Let X and Δ be as in Definition 4.2. Then $L(\Delta)$ is a subposet of $L(X)$ and we call the $(L(\Delta)^c)^{\updownarrow}$ the poset cross-dual of Δ and denote it Δ^{pcd}.*

Example 4.2. *Let S and X be as in Example 4.1. Then the poset cross-dual of S will be the face lattice of a cubical complex consisting of two disjoint squares. See Figure 4.*

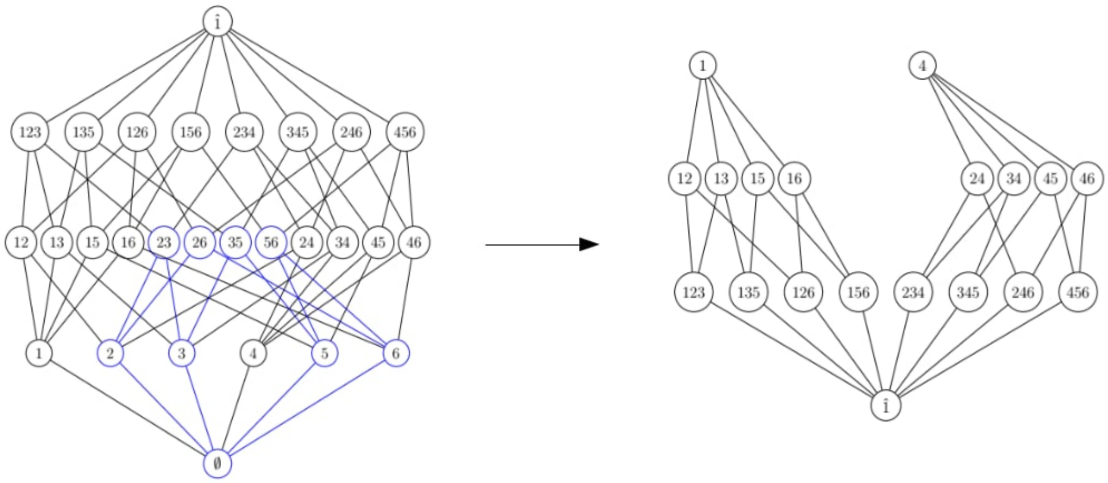

Figure 4: The process of obtaining the poset cross-dual of S

The following proposition tells characterizes situations when this phenomenon occurs:

Proposition 4.1. *Let Δ be a complex embeddable into the n-dimensional cross-polytope X. Then cross-dual of Δ is disconnected if and only if in Δ there exists facets $\sigma_1, \sigma_2, \ldots, \sigma_k$ of X such that $\partial(\bigcup_{i=1}^k \sigma_i) \subset \Delta$ and $\bigcup_{i=1}^k \sigma_i \setminus \Delta \neq \emptyset$.*

Proof. One direction is easy. For the other direction assume that Δ is such that Δ^{cd} is disconnected. This means that the order complex of Δ^{pcd} is disconnected as well. Thus, in the top level of Δ^{pcd} we will have at least two elements whose join is the zero element of the poset. Thus in the first level of Δ there exist two disjoint sets A and B such that for any $x \in A$ and any $y \in B$ there is no meet. These correspond to the connected components of $|X| \setminus |\Delta|$. These components are union of facets of X (or facets without part of the boundary to be precise). Thus in order to achieve separation, Δ must contain their original intersection, which is exactly the boundary of one of the components. $\square$

Of course, we need to show that our definitions yield the "same" object. Consider X and Δ as before. Now each vertex in Δ^{cd} will correspond to an element of the facet cover of Δ which is exactly a facet missing from Δ. On the other hand, vertex in Δ^{pcd} will be a maximal element in $L(X) \setminus \{\hat{1}\}$ i.e., a facet of X which is not in Δ. Thus we are working on the same vertex set. We now look at the ways faces will be formed in both of these objects. In order to get a k-dimensional face in Δ^{pcd} we need to have $k+1$ chains of length k in $(L(X) \setminus L(\Delta))$. These chains have to share the same origin and finish with maximal elements of $L(X) \setminus \{\hat{1}\}$. Similarly, a k-dimensional face is formed in Δ^{cd} whenever we have k elements in the face cover intersecting. So there is a way of "crossing" between the two objects. We will now state and prove a theorem formalizing this:

Theorem 4.1. *Let Δ be a simplicial complex embedded into a n-dimensional cross-polytope X. Then, the geometric realizations of the cross-dual and the poset cross-dual of Δ are homotopy equivalent.*

Proof. We first note that by the Nerve lemma $|X| \setminus |\Delta|$ is homotopy equivalent to Δ^{cd}. Next, by Lemma 3.3 we know that $|X| \setminus |\Delta|$ is homotopy equivalent to $|\Delta^*|$. Thus, as homotopy equivalence is an equivalence relation we get the desired result. □

One may ask: "Why do we need two definitions if they end up doing the same job?" An answer to that question is that each of them comes with its own set of advantages and disadvantages. The cross-dual inflates the dimension. Indeed, if a complex is embedded into the n-dimensional cross-polytope, its cross-dual can have dimension as high as 2^{n-1}. An advantage is that the cross-dual is always simplicial on at most 2^n vertices. On the other hand, the poset cross-dual keeps the dimension as low as possible. Its disadvantage is that it yields a face poset of a cubical complex. Cubical complexes aren't always as easy to deal with as simplicial complexes. One way to fix this is to look at its order complex. However, then the number of vertices can become very high, as the order complex of a cubical face lattice is its barycentric subdivision.

4.1 Cross-polytopal Bier spheres and many more vertex decomposable spheres

Now we will consider the poset construction of Bier spheres from [3]. The classical Bier sphere construction uses the combinatorial Alexander dual and deleted join in order to make a sphere. The construction of Bier posets gave an analogous description that uses the poset description of the combinatorial Alexander dual and ideals of the boolean lattice. We will work on the face lattice of the n-dimensional cross-polytope, denoted O_n. In this setting, looking at $\mathrm{Bier}(P, O_n)$ corresponds to looking at the ideal of O_n combined with its poset cross-dual. As before, this construction will yield a sphere. We will show that this sphere is always vertex-decomposable and thus shellable. Indeed, our proof will also give us the k-decomposability of any Bier sphere constructed from a k-decomposable simplicial sphere.
First, we will show that the n-dimensional cross-polytope is vertex-decomposable. For that we need the following lemma, whose proof is completely straightforward:

Lemma 4.1. *Δ is vertex-decomposable if and only if cone(Δ) is vertex-decomposable.*

Proposition 4.2. *The n-dimensional cross-polytope is vertex-decomposable*

Proof. We proceed by induction on n: For $n = 1$, everything is clear. We now assume that the statement is true for $(n-1)$. In general, any vertex is easily seen to be a shedding vertex. The link of the vertex is an $(n-1)$-dimensional cross-polytope. The deletion is the cone over an $(n-1)$-dimensional cross-polytope. Both are vertex-decomposable by the lemma/inductive step. □

We can now proceed to the main theorem:

Theorem 4.2. *Let Δ be a k-decomposable simplicial sphere, and denote by X the face lattice of Δ. For a proper ideal I of X, the Bier sphere* $\mathrm{Bier}(X, I)$ *is k-decomposable.*

Proof. To begin, we note that in their paper Björner et al. proved that the order complex of $\mathrm{Bier}(P, I)$ is a stellar subdivision of the order complex of P [3, Theorem 2.2]. Also, by [11, Theorem 2.7] k-decomposability is preserved under stellar subdivision. This completes the proof. □

We get the following as a corollary easily:

Corollary 4.1. *For any simplicial complex* Δ *embeddable in the* n*-dimensional cross-polytope,* $\mathrm{Bier}(O_n, L(\Delta))$ *is vertex-decomposable.*

Corollary 4.2. *For any ideal* I *of a Boolean lattice* B_n*,* $\mathrm{Bier}(B_n, I)$ *is vertex-decomposable and hence shellable.*

The proof of this corollary is immediate from Theorem 4.2 and offers an improvement on the original proof of shellability from [3]. Even further, this gives us a method of constructing many vertex-decomposable non-polytopal spheres. The number of these spheres is much greater than the number of shellable spheres from the original paper. To see this, we only need to notice the number of complexes embeddable into the cross-polytope is bigger than the number of complexes embeddable into the simplex. This gives us the following corollary:

Corollary 4.3. *Let* A_n *be the set of all* n*-dimensional vertex-decomposable simplicial complexes. Let* S_n *be the same set with the additional requirement that complexes are polytopal. Then*

$$\lim_{n\to\infty} \frac{|S_n|}{|A_n|} = 0.$$

In the original paper, Björner et al. left an open question. They asked if there was a way to adapt Bier's construction to make many shellable $(n-2)$ spheres on more than $2n$ vertices. This was first answered by Čukić and Delucchi in [18]. Our particular construction gives a nice way of constructing many such vertex-decomposable spheres. Simply let P be an ideal of O_n such that there exists more than 4 intervals $[x, y]$ where $x \in P$, $x \notin P$ will yield such a sphere.

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