HOMOLOGY USING LINEAR ALGEBRA

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ABSTRACT. We will give an introduction to Homology Theory understandable to a student who has taken Linear Algebra. We will give basic examples using simple row reduction to compute homologies. Our purpose here is not to show anything new but to show undergraduate students that homology computations can be done using simple techniques they learned in their Linear Algebra course and to generate further interest in their study of the subject.

1. Introduction

Homology is a mathematical method for defining holes in a shape. To compute a homology, we begin with a graph containing v vertices and x edges. We give the edges a forward orientation for convenience, as it makes it easier to label the signed incidence matrix. A *signed incidence matrix*, D^1, is a matrix such that $D^1_{ij} = 1$ if edge x_j leaves vertex v_i, -1 if edge x_j enters vertex v_i, and 0 otherwise, with each column representing an edge and each row representing a vertex. The reader can skip ahead to Section 3 to see examples of this matrix.

The important idea here is that a cycle in a graph gives a linear dependency relation in the incidence matrix D^1. This means that the number of cycles in the graph is given by the number of linear dependencies in the incidence matrix.

A much more advanced explanation of this information is available in a standard graduate textbook on Algebraic Topology, such as in Allen Hatcher's book [2].

2. Background linear algebra

We give in this section a reminder of some of the Linear Algebra necessary to compute homologies. Readers who do not want a review may feel free to skip over this section and continue on to Section 3. Those who would like to study more Linear Algebra should reference their favorite text. There are also plenty of free texts and courses offered online.

We begin with one of the most important theorems in Linear Algebra, the Rank-Nullity Theorem [3].

Let A be a matrix.

(1) The dimension of the column space of A is called the *rank* of A and is denoted $\mathrm{rank}(A)$.
(2) The dimension of the null space of A is called the *nullity* of A and is denoted $\mathrm{null}(A)$.

Theorem 1 (Rank-Nullity Theorem)**.** *If A is any $m \times n$ matrix, then*

$$\mathrm{rank}(A) + \mathrm{null}(A) = n.$$

This theorem can be applied to linear maps as well. A *linear map*, or *linear transformation*, is a mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ that preserves the operations of addition and scalar multiplication.

For example, say we have vectors $\vec{a}$, $\vec{b} \in \mathbb{R}^n$, a scalar c, a transformation T. Then,

$$\begin{aligned} T(\vec{a} + \vec{b}) &= T(\vec{a}) + T(\vec{b}) \text{ and} \\ T(c\,\vec{a}) &= cT(\vec{a}). \end{aligned}$$

To apply this to matrices, we take the identity matrix I_n,

$$\begin{array}{cccc} \vec{e_1} & \vec{e_2} & \dots & \vec{e_n} \end{array}$$
$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The vectors $\left\{\vec{e_1}, \vec{e_2}, \dots \vec{e_n}\right\}$ are something called the standard basis for $\mathbb{R}^n$. Being a *basis* means that the vectors must span $\mathbb{R}^n$ and they must be linearly independent.

Next, we multiply a vector $\vec{x}$ by our standard basis, so

$$\vec{x} = x_1\,\vec{e_1} + x_2\,\vec{e_2} + \dots + x_n\,\vec{e_n} .$$

A linear transformation, T, of $\vec{x}$,

$$\begin{aligned} T(\vec{x}) &= T(x_1\,\vec{e_1} + x_2\,\vec{e_2} + \dots + x_n\,\vec{e_n}) \\ &= T(x_1\,\vec{e_1}) + T(x_2\,\vec{e_2}) + \dots T(x_n\,\vec{e_n}) \\ &= x_1T(\vec{e_1}) + x_2T(\vec{e_2}) + \dots x_nT(\vec{e_n}). \end{aligned}$$

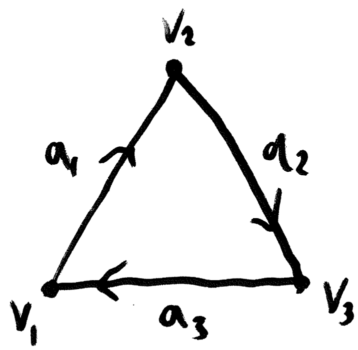

FIGURE 3.1. A simple triangle graph

If we put this in matrix format, we have

$$T(\vec{x}) \;=\; \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \dots & T(\vec{e_n}) \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

All linear transformations can be represented in this way.

Corollary 2 (Rank-Nullity for Linear Maps)**.** *Let V and W be vector spaces over some field, and let $T : V \longrightarrow W$ be a linear map. Then the rank of T is the dimension of the image of T and the nullity of T is the dimension of the kernel of T, so we have*

$$\dim(\operatorname{Im}(T)) + \dim(\ker(T)) = \dim(V).$$

3. Graphs and homology

We will begin by giving some example graphs and looking at how to set up the signed incidence matrices. We will then row reduce the matrices and find the ranks and nullities before we actually move on to computing the homologies.

Example 3 (Triangle Example)**.** We'll start by giving an example of the signed incidence matrix and how to set it up using the graph pictured in Figure 3.1. We set up D^1 as we described. So we have,

$$D^1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Notice that the -1 in the first row, final column represents the final edge re-entering the starting vertex. Next, we row reduce this matrix

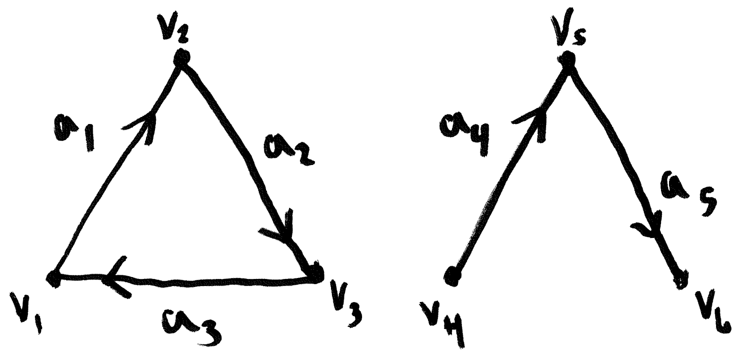

FIGURE 3.2. A disjoint graph

and count the linearly independent columns, or pivots. This gives,

$$\begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since we have 2 pivot columns, we can say that

$$\operatorname{rank}(D^1) = 2.$$

Using the Rank-Nullity Theorem (1), we know then that

$$\operatorname{null}(D^1) = 1.$$

Example 4 (Disjoint Example)**.** Next, we give an example of the signed incidence matrix of a disjoint graph, pictured in Figure 3.2.

$$D^1 = \begin{array}{c c} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Notice here that graph being disjoint gives us a block matrix with two submatrices. Row reducing, we have

$$\begin{array}{c c} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In this case, we have 4 pivot columns, so

$$\operatorname{rank}(D^1) = 4.$$

By the Rank-Nullity Theorem (1), this gives

$$\operatorname{null}(D^1) = 1.$$

The nullities, or linearly dependent columns, of our incidence matrices tell us about the number of cycles in each graph. We see that in each case, our matrices tell us that we have 1 cycle, and we can confirm this by looking at our figures. Notice also that $\#\text{vertices} - \operatorname{rank}(D^1)$ gives the number of components of each example graph.

These incidence matrices are linear transformations (defined in Section 2) between the edge space(or C_1) to the vertex space(or C_0). In other words, it maps each edge to the vertices that are incident, or attached, to it. The edge space and vertex space are vector spaces defined in terms of the edge and vector sets. We call this the map of boundary 1, ∂_1, or the one-dimensional boundary. In basic terms this boundary refers to the cycle itself. We write,

$$\partial_1 : C_1 \longrightarrow C_0.$$

Boundary maps are explained in more detail in Section 4 but for now, we carry on.

To compute our homologies, we utilize our corollary of the Rank-Nullity Theorem (2) which can be seen in Section 2. In our case, V would be our edge space, C_1, W would be our vertex space, C_0, and the map T would be our boundary map ∂_1. Using this corollary, we are able to compute the dimension of our homology, $\tilde{H}_i$. Once again, more complete reasoning for this is given in Section 4.

$$\dim(\tilde{H}_i) = \dim(\ker \partial_i) - \dim(\operatorname{Im} \partial_{i+1})$$

Returning to Example 3, we can now complete our computation for $\tilde{H}_1$.

When we left off, we had the rank and nullity of our incidence matrix. We now know that the rank is the dimension of the image of ∂_1 and the nullity is the dimension of the kernel of ∂_1. So we have,

$$\begin{aligned}\dim(\operatorname{Im}\partial_1) &= 2 \text{ and}\\ \dim(\ker\partial_1) &= 1\end{aligned}$$

Now we require $\dim(\operatorname{Im}\partial_2)$ to complete our computation. In our current examples, since neither are 2-dimensional, this will have to be 0. (We will have an example later on where we compute $\tilde{H}_1$ of the Mobius Strip and this will not be the case.) So now we have,

$$\dim(\operatorname{Im}\partial_2) = 0.$$

Using our definition for the dimension of $\tilde{H}_i$, we have

$$\begin{aligned}\dim(\tilde{H}_1) &= \dim(\ker\partial_1) - \dim(\operatorname{Im}\partial_2), \text{ so}\\ \dim(\tilde{H}_1) &= 1 - 0 = 1.\end{aligned}$$

Thus the dimension of our $\tilde{H}_1$ is 1.

We now return to Example 4 to see whether there is any difference. Once again, we already have our rank and nullity, so

$$\begin{aligned}\dim(\operatorname{Im}\partial_1) &= 4 \text{ and}\\ \dim(\ker\partial_1) &= 1.\end{aligned}$$

Our graph is still not 2-dimensional, and so

$$\begin{aligned}\dim(\operatorname{Im}\partial_2) &= 0 \text{ and so}\\ \dim(\tilde{H}_1) &= 1 - 0 = 1.\end{aligned}$$

As we stated in the introduction, homology defines holes in a shape. In the case of $\tilde{H}_1$, this refers to the number of cycles in our graphs. Had the second component in our disjoint graph been a cycle, the dimension of $\tilde{H}_1$ would have been 2 rather than 1.

Another important type of graph is a *tree*, which is a connected graph containing no cycles. That is, there is no place in the graph where any edge returns to a previous vertex. Also, every tree containing n vertices will have $n-1$ edges. An example of this type of graph is given in Figure 3.3.

Lemma 5. *Every tree has a vertex of degree* 1.

The *degree of a vertex* is the number of edges which are incident to that vertex. What this theorem is telling us is that no matter how large or how small a tree is, it will always contain at least one vertex

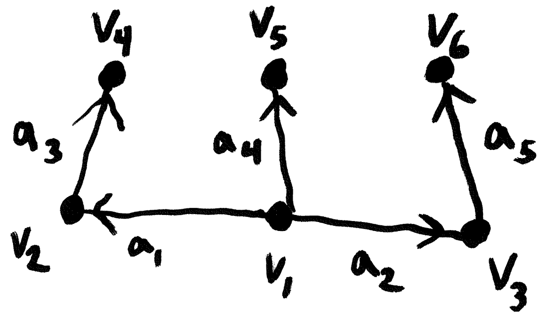

FIGURE 3.3. A tree graph

which is only attached to a single edge. This comes as a result of trees containing no cycles.

Example 6 (Tree Example)**.** We give an example of this type of graph and will once again compute $\tilde{H}_1$.

We begin by setting up the signed incidence matrix. We have,

$$D^1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Row reducing this, we are left with

$$D^1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

This gives

$$\begin{aligned} \text{rank}(D^1) &= \dim(\operatorname{Im} \partial_1) = 5 \text{ and} \\ \text{null}(D^1) &= \dim(\ker \partial_1) = 0 \end{aligned}$$

Recall that we said that the nullity tells us about the number of cycles in the graph. Since a tree has no cycles, the nullity ends up being 0.

Next we need $\dim(\operatorname{Im}\partial_2)$, which is again 0 in this 1-dimensional graph, so

$$\dim(\operatorname{Im}\partial_2) = 0.$$

Now we complete the computation.

$$\dim(\tilde{H}_1) = \dim(\ker\partial_1) - \dim(\operatorname{Im}\partial_2) = 0 - 0 = 0.$$

This tells us that the tree has $\tilde{H}_1 = 0$, which makes sense, as tree graphs do not have any cycles. Of course, this also tells us something about trees in general.

Theorem 7. *If a graph T is a tree, then $\tilde{H}_1 = 0$.*

Proof. Let D^1 be the incidence matrix representing any tree T. Since T is a tree, if D^1 contains n columns, then D^1 must contain $n+1$ rows and each column will have only one 1 and one -1 in some row beneath it. When D^1 is row reduced, there will be exactly n linearly independent columns. That is, $\operatorname{rank}(D^1) = n$.

Adding a new branch to this tree equates to adding one new edge connecting to a new vertex. In D^1, this means adding one new column and row. In the new column, since the new vertex has only one edge attached, this gives a -1 in the row representing that edge and a 1 in the row representing the vertex that we branched from. Row reducing D^1, this new column is still linearly independent.

Because each column in D^1 is linearly independent, the final row will always contain only 0s after row reduction. This means that the number of linearly independent columns will always be equal to the total number of columns in D^1. That is,

$$\dim(\operatorname{Im}\partial_1) = \dim(D^1).$$

By the Rank-Nullity Theorem (2),

$$\begin{aligned} \dim(\ker\partial_1) &= 0 \text{ and so} \\ \dim(\tilde{H}_1) &= 0. \end{aligned}$$

□

In order to compute $\tilde{H}_0$ of these graphs, we'll also need some information about the zero-dimensional boundary (∂_0). Boundary 0 maps the vertex space into our field $\mathbb{R}$. We think of this 1-dimensional vector space as representing the empty set, and call it C_{-1}. So we have,

$$\partial_0 : C_0 \longrightarrow C_{-1}.$$

Again, this is explained in further detail in Section 4, but for now we compute $\tilde{H}_0$ for our previous examples.

Returning first to Example 3, we recall that

$$\dim(\operatorname{Im}\partial_1) = 2.$$

To get the $\dim(\ker\partial_0)$, we can simply count the number of vertices in our graph, and subtract by one. Then

$$\begin{aligned}\dim(\ker\partial_0) &= 3-1=2, \text{ so}\\ \dim(\tilde{H}_0) &= 2-2=0.\end{aligned}$$

So we see that $\tilde{H}_0 = 0$. Before we find out what this means, we take a look now at Example 4 to see whether there is a difference.

Recall that

$$\dim(\operatorname{Im}\partial_1) = 4.$$

Again we count our total vertices and subtract by one, so

$$\begin{aligned}\dim(\ker\partial_0) &= 6-1=5 \text{ so,}\\ \dim(\tilde{H}_0) &= 5-4=1.\end{aligned}$$

We might now be asking ourselves: "Why is there a difference between $\tilde{H}_0$ of the disjoint graph and the non-disjoint graph?"

This comes as a result of what $\tilde{H}_0$ actually measures. We stated before that homology is a method of defining "holes" in a shape. In the case of $\tilde{H}_0$ these holes refer to connectedness. In other words, a "hole" in 0 dimensions is just a gap between components, such as in the graph in Example 4. If we had added another disjoint piece to this graph, we would have had $\tilde{H}_0 = 2$ and so on as more are added. In Example 3, $\tilde{H}_0$ would always be 0 because there is only a single connected piece, so there would be no holes to find. The same is true of Example 6.

In fact, these ideas yield important theorems in homology.

Theorem 8 (Cycle Graph Theorem). *Let G be a cycle graph. Then*

$$\tilde{H}_0 = 0.$$

Lemma 9 (Zero-Homology of a Tree). *Let a graph T be a tree. Then*

$$\tilde{H}_0 = 0.$$

Proof. We give a brief proof. Of course, the total number of vertices is equal to the number of rows in our matrix. That is, $n+1$. This means that $\dim(\ker\partial_0) = n$, which is the total number of columns, or $\dim(V)$. Thus,

$$\begin{aligned}\dim(\ker\partial_0) &= \dim(V) = &&\dim(\operatorname{Im}\partial_1) \text{ and so}\\ \dim(\tilde{H}_0) &= \dim(\ker\partial_0) &&- \dim(\operatorname{Im}\partial_1) = 0, \text{ so}\\ &\tilde{H}_0 = 0.\end{aligned}$$

□

Theorem 10 (Connectivity Theorem). *Let G be a connected graph. Then*

$$\tilde{H}_0 = 0.$$

Proof. Note that every connected graph contains a spanning tree.[1] A *spanning tree* in a graph G is a subgraph of G that includes all the vertices of G and is also a tree. We know that for any tree, we have $\tilde{H}_0 = 0$ and if we have v vertices, we have $v - 1$ edges in the tree which gives a matrix with v rows and $v - 1$ columns. Adding any of the original edges back into the spanning tree means adding a column but no new row since our spanning tree included all vertices. This ultimately means that the number of linearly dependent columns, that is, $\dim(\ker \partial_1)$ will increase but the number of linearly independent columns, or $\dim(\operatorname{Im} \partial_1)$ will stay the same. Since our number of vertices remains the same as well, $\dim(\ker \partial_0)$ does not change either. So, just as in our spanning tree,

$$\dim(\ker \partial_0) = \dim(\operatorname{Im} \partial_1), \text{ so}$$
$$\tilde{H}_0 = 0.$$

□

Theorem 10 tells us about any connected graph. That is, regardless of whether the graph is a tree, a cycle graph, or anything in between. As long as there only one component in the graph, $\tilde{H}_0 = 0$.

Theorem 11 (Disjoint Graph Theorem). *Let G be a graph. Then $\tilde{H}_0$ is equivalent to one less than the number of connected components. That is,*

$$\tilde{H}_0 = n - 1$$

where n is the total number of components in the graph.

4. Boundary maps and chain complexes

We now provide a deeper explanation on boundaries and their association with chain complexes. As we stated in Section 3, we begin with our boundary map,

$$\partial_1 : C_1 \longrightarrow C_0$$

which maps the edge space into the vertex space. But what does this mean?

Example (3 continued). Looking at our first example with edges $a_1, a_2,$ and a_3 and vertices $v_1, v_2,$ and v_3, our map of ∂_1 would be

$$\begin{aligned} \partial_1: \quad a_1 &\longrightarrow v_2 - v_1 \\ a_2 &\longrightarrow v_3 - v_2 \\ a_3 &\longrightarrow v_1 - v_3 \end{aligned}$$

We see that if we were to combine our 3 edges, we end up with 0,

$$(v_2 - v_1) + (v_3 - v_2) + (v_1 - v_3) = 0.$$

Next, we take a look at our map of ∂_0,

$$\partial_0 : C_0 \longrightarrow C_{-1}$$

which maps the vertex space into the field $\mathbb{R}$, or the 1-dimensional vector space representing the empty set.

We look again to our first example with vertices $v_1, v_2,$ and v_3. After row-reducing, we were left with

$$\begin{array}{cc} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} .$$

This tells us that,

$$\begin{aligned} v_1 &= v_3 \text{ and} \\ v_2 &= v_3 \text{ so,} \\ v_1 &= v_2 = v_3. \end{aligned}$$

Then, our map of ∂_0 can be written,

$$\begin{aligned} \partial_0: \quad v_1 &\longrightarrow 1 \\ v_2 &\longrightarrow 1 \\ v_3 &\longrightarrow 1. \end{aligned}$$

We can combine our boundary map for ∂_0 with our map for ∂_1 to get,

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} 0$$

where 0 is the trivial vector space.

This is something called a *chain complex*, or a sequence of vector spaces C_i and maps, or functions, ∂_i such that

$$\partial_i(\partial_{i+1}(x)) = 0 \text{ for all } x \in C_{i+1}.$$

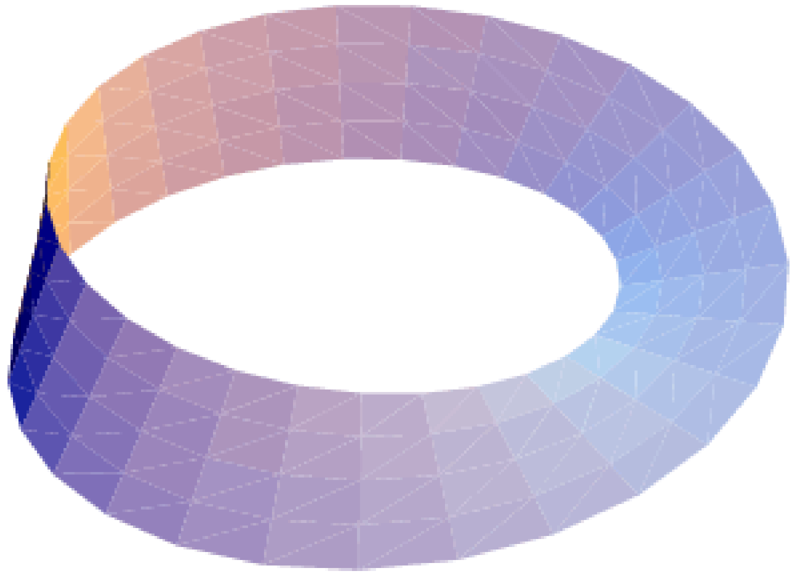

FIGURE 5.1. The Mobius Strip (figure due to Weisstein [4])

So the image of ∂_{i+1} is contained in the kernel of ∂_i. That is,

$$\operatorname{Im}(\partial_{i+1}) \subseteq \ker(\partial_i).$$

The homology of the chain complex is another sequence of vector spaces,

$$\begin{aligned} \tilde{H}_i(C) &= \ker(\partial_i)/\operatorname{Im}(\partial_{i+1}) \text{ with} \\ \dim(\tilde{H}_i) &= \dim(\ker\partial_i) - \dim(\operatorname{Im}\partial_{i+1}). \end{aligned}$$

We can, of course, add on more boundary maps to our chain complex. In the next section, we will be working with ∂_2, with the map

$$\partial_2 : C_2 \longrightarrow C_1.$$

This maps the triangle space (C_2) into our edge space (C_1). This extends our chain complex,

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} 0.$$

5. A 2-DIMENSIONAL EXAMPLE

In the Section 3, we mentioned that $\dim(\operatorname{Im}\partial_2)$ would have to be 0 in the graphs we were using. This is not necessarily always the case. We come now to the famous Mobius Strip and will compute its $\tilde{H}_1$.

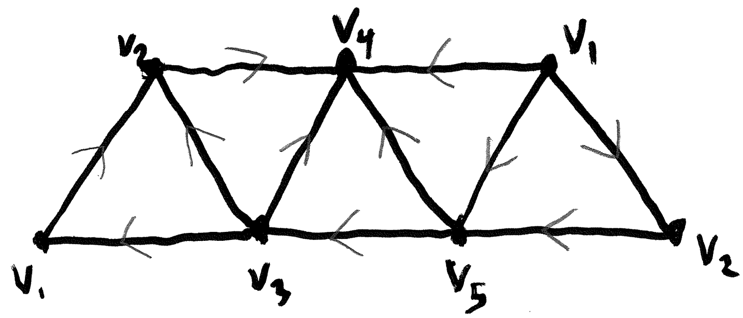

FIGURE 5.2. Triangulation of the Mobius Strip

The Mobius Strip is a surface with one continuous side formed by joining the ends of a rectangular strip after twisting one side 180 degrees.

To compute the homology, we first provide a triangulation (Figure 5.2). Notice that the first edge on the triangulation is the same as the final edge. It is simply reversed so that when the two are connected, they form the Mobius Strip.

We begin now in the same way as the first examples, with the signed incidence matrix between the vertices and edges for this graph.

$$D^1 = \begin{array}{cc} & \begin{array}{cccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

Once more, we row reduce and find our rank and nullity.

$$D^1 = \begin{array}{cc} & \begin{array}{cccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

This gives

$$\begin{aligned} \mathrm{rank}(D^1) &= 4 \\ \mathrm{null}(D^1) &= 6. \end{aligned}$$

Next, we need $\dim(\mathrm{Im}\,\partial_2)$. Luckily for us, this time we have a method to find it. To compute this, we create a new signed incidence matrix, this time between the edges and triangles in the graph rather than vertices and edges. This gives,

$$D^2 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Also luckily, this still only requires us to use a simple row reduction procedure. So we have,

$$D^2 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Now we see that,

$$\begin{aligned} \dim(\operatorname{Im} \partial_2) &= 5 \text{ so,} \\ \dim(\tilde{H}_1) &= 6 - 5 = 1. \end{aligned}$$

This tells us that the Mobius Strip has one 1-dimensional hole, or cycle. We should remember not to let the cycles in the triangulation confuse us, as these triangle are filled in, whereas in our earlier examples, the triangles were empty.

We can also see immediately here that $\dim(\tilde{H}_2)$ is going to be 0. Since $\dim(\operatorname{Im} \partial_2) = 5 = \dim(D^2)$, it follows that $\dim(\ker \partial_2)$ must be 0.

It is possible to compute even higher dimensional homologies for objects such as the sphere or the torus. This is left as an exercise for the reader.

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