The independence polynomial of claw free graphs and trees

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1 Introduction

1.1 A brief history

Independence polynomials have been studied in various papers before. Heilmann and Lieb proved the real-rootedness of the independence polynomials of line graphs in [7], while Schwenk showed in [10] that the edge independent sequence of a graph is unimodal. Hamidoune later showed in [6] that the independence polynomial of a claw free graph is unimodal. A particularly notable result is in [4], where Chudnovsky and Seymour showed that all the roots of the independence polynomials of claw free graphs are real. We will define the edge independent sequence, independence polymomial, and the claw in Section 1.2. In this paper, we will study independence polynomials in connection with trees and claw free graphs.

In Subsection 1.2, we will provide most of the definitions and tools needed throughout the paper. In Section 2, we will do some calculations to introduce the reader to independence polynomials of graphs. In Section 3 we will examine the following question given by Alavi, Malde, Schwenk, and Erdős as Problem 3 in [1]:

Question 1. *For trees (or perhaps forests) is the independence polynomial unimodal?*

We will discuss an unsuccessful approach to Question 1. In Section 4, we will look at the lexicographic product of two graphs to determine when it is claw free. In Section 5, we will work with independence polynomials that have a root of -1.

1.2 Definitions and some tools

For basic graph theory notation, we use the same practices as Matoušek and Nešetřil in [9].

The following are some additional definitions needed for this paper.

Definition 2. The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \cup G_2$ having as vertex set $V(G_1) \cup V(G_2)$, and as edge set $E(G_1) \cup E(G_2)$. Similarly, the *join* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having as vertex set $V(G_1) \cup V(G_2)$, and as edge set $E(G_1) \cup E(G_2) \cup \{e_1e_2 | e_1 \in E(G_1), e_2 \in E(G_2)\}$.

Example 3. $K_{3,4} \cong \overline{K}_3 + \overline{K}_4$

Definition 4. The *claw* is the graph $K_{1,3}$, and a graph is said to be *claw free* if it has no induced subgraph isomorphic to a claw.

Example 5. K_n is claw free because it has all possible edges.

Example 6. $P_2 + \overline{K}_3$ is not claw free because a vertex in P_2 together with the three vertices in $\overline{K}_3$ form a claw.

Definition 7. The *diameter* of a graph is the greatest distance between two vertices.

Definition 8. A polynomial $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is *unimodal* if $a_0 \leq a_1 \leq ... \leq a_{i-1} \leq a_i \geq a_{i+1} \geq ... \geq a_{n-1} \geq a_n$ for some $i \in \{0, ...n\}$.

Example 9. The polynomial $1 + 5x + 17x^2 + 12x^3$ is unimodal because $1 \leq 5 \leq 17 \geq 12$.

Example 10. The polynomial $1 + 7x + 8x^2 + 6x^3 + 13x^4$ is non-unimodal because $8 \geq 6 \leq 13$.

Definition 11. An *edge independent set* in G is a set of edges $E_1 \subset E(G)$ such that, for all edges $e, f \in E_1, e \cap f = \emptyset$.

Definition 12. An *independent set* in G is a set of pairwise non-adjacent vertices. An independent set of maximum size will be referred to as a *maximum independent set* of G, and the *independence number* of G, denoted by

$\alpha(G)$, is the cardinality of a maximum independent set in G. Let s_k be the number of independent sets of cardinality k in a graph G. The polynomial

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha(G)} x^{\alpha(G)}$$

is called the *independence polynomial* of G.

Levit and Mandrescu give the following two equalities relating to independence polynomials in [8]:

Lemma 13. *Let $G = (V, E)$ be a graph, $w \in V$. Then*

$$I(G;x) = I(G - w; x) + xI(G - N[w]; x)$$

Proof. $I(G-w;x)$ counts the independent sets that do not contain w. $xI(G-N[w];x)$ counts the independence sets that do contain w. □

Lemma 14. *Let $G = G_1 \cup G_2$. Then*

$$I(G;x) = I(G_1;x)I(G_2;x)$$

Sketch. This is not difficult to see with some knowledge of generating functions. See, for example, [2]. □

2 Independence polynomials of some families of graphs

We start by computing the independence polynomials of some well-known families of graphs to introduce the reader to calculating independence polynomials.

Definition 15. The *Fibonacci polynomials* are defined by the following recursion:

$$F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x)$$

In [8], Levit and Mandrescu give the following two relationships for the independence polynomials of path and cycle graphs:

Lemma 16. $I(P_n; x) = F_{n+1}(x)$

Lemma 17. $I(C_n; x) = F_{n-1}(x) + 2xF_{n-2}(x)$

These two facts got me interested in the independence polynomials of other infinite graph families.

Lemma 18. *Let G be a graph on n vertices. The constant term and linear terms of $I(G; x)$ are 1 and nx, respectively.*

Proof. There is one way to choose 0 independent vertices, so the constant term counts the empty set. When choosing an independent set of size one, any vertex is an independent set, so the linear term counts the number of vertices. □

Lemma 19. $I(K_n; x) = 1 + nx$.

Proof. K_n has no independent sets of size greater than 1, so $\forall k \in \mathbb{N}, s_k = 0, k \geq 2$. By Lemma 18, we have $1 + nx$. □

Lemma 20. *Thus,* $I(\overline{K}_n) = \sum_{i=0}^{n} \binom{n}{i} x^i$.

Proof. $\overline{K}_n$ has no edges, so any subset of $V(\overline{K}_n)$ is an independent set. Thus, $\forall k \in \mathbb{N}, s_k = \binom{n}{k}$. □

Proposition 21. $I(\overline{P}_n) = 1 + nx + (n-1)x^2$

Proof. The constant and linear terms can be determined by Lemma 18. There are $n-1$ edges in P_n, so there are $n-1$ independent sets of size 2 in $\overline{P}_n$. P_n has no subgraphs isomorphic to a complete graph of size greater than 2, so $\overline{P}_n$ has no independent sets of size greater than 2. □

Proposition 22. $I(\overline{C}_n) = 1 + nx + nx^2$

Proof. The constant and linear terms can be determined by Lemma 18.There are n edges in C_n, so there are n independent sets of size 2 in $\overline{C}_n$. C_n has no subgraphs isomorphic to a complete graph of size greater than 2, so $\overline{C}_n$ has no independent sets of size greater than 2. □

3 A class of polynomials

In this section we work toward finding an answer to Question 1. The approach we took to this question was to try to create a class of polynomials that contained all independence polynomials and also only contained unimodal polynomials. Unfortunately, we were unsuccessful.

Definition 23. Let $T = (V, E)$ be a tree and $v \in V$ be such that all neighbors of v except possibly one are leaves. Then we call v a *near-leaf.*

The following proposition encapsulates a useful property of trees that we will use to show our main property about the class of polynomials we define in this section.

Proposition 24. *Let $T = (V, E)$ be a tree with $\operatorname{diam} T \geq 2$. Then there exist at least two near-leaves in T.*

Proof. Let T be a tree. Choose $v, w \in V$ such that $\operatorname{dist}(v, w) = \operatorname{diam} T$. Then we claim that v and w are both leaves. Suppose not. Without loss of generality, let $\deg v > 1$. Then let $P = v, ..., w$ be the unique path from v to w. One of the neighbors of v is $v_1 \in P$; the others, say $\{n_1, n_2, ..., n_i\} \notin P$. Then choose one of those, say n_j, and we have $\operatorname{dist}(n_j, w) = \operatorname{diam} T + 1$. This is a contradiction. Thus, v and w are leaves.

Then v has a unique neighbor y, and w has a unique neighbor z. Without loss of generality, we look at y. If $\operatorname{diam} T = 2$, then $y = z$ and we have a star graph. Then all neighbors of y are leaves, and y is a near-leaf. Consider $\operatorname{diam} T \geq 3$. y has one neighbor y_1 such that $y_1 \in P, \deg y_1 > 1$. Then suppose y has another non-leaf neighbor $t \notin P$. Then t has at least one more neighbor t_1, and $\operatorname{dist}(v, t_1) = \operatorname{diam} T + 1$. This is a contradiction. A similar argument can be made for z and w. Thus, y and z are both near-leaves. $\square$

The following is the main purpose of this section:

Definition 25. Let $\mathcal{C}$ be the class of polynomials defined by the following recursion: $1 \in \mathcal{C}$, and $f + gx \in \mathcal{C}$ whenever f and g follow these rules:

1. $f, g \in \mathcal{C}$,
2. $\deg f \geq \deg g \geq \deg f - 1$, and
3. $f \geq g$ component-wise.

We were led to define $\mathcal{C}$ in such a way from thinking about Lemma 13. Our goal was to model the behavior of trees when choosing w to be a leaf. This provided the basis for the proof of Proposition 28.

The following lemmas are facts about $\mathcal{C}$ that are helpful in the proofs of Proposition 28 and Proposition 29 below.

Lemma 26. *Let f be a polynomial. If $f \in \mathcal{C}$, then $(1+x)f \in \mathcal{C}$.*

Proof. We know $f \in \mathcal{C}$, $\deg f = \deg f$, and $f = f$ component-wise. Thus, $f + xf = (1+x)f \in \mathcal{C}$. □

Lemma 27. *For all $n \in \mathbb{N}$, $1 + nx \in \mathcal{C}$.*

Proof. We proceed by induction.

Base Case: $1 + x(1) = 1 + x \in \mathcal{C}$.

Inductive Step: $1 + nx + x(1) = 1 + (n+1)x \in \mathcal{C}$. □

The following proposition is the reason we proved Proposition 24 above.

Proposition 28. *If T is a tree, then $I(T;x) \in \mathcal{C}$.*

Proof. Let T be a tree. We proceed by induction.

Base Case: By Lemma 27, $1 + x \in \mathcal{C}$. $1 + x = I(K_1;x)$, and K_1 is a tree.

Inductive Step: Choose a vertex v to be a leaf that is a neighbor of a near-leaf w of degree d. By Lemma 13, we have

$$I(T;x) = I(T-v;x) + x \cdot I(T-N[v];x).$$

We know $f = I(T-v;x)$ is a tree because we have only deleted a leaf. We know that $g = I(T-N[v];x) = I(T-v-w;x)$ is the disjoint union of a tree T_1 and $d-2$ independent vertices. Thus, by Lemma 14, $g = (1+x)^{d-2} I(T_1;x)$. Now we check that f and g follow the rules of $\mathcal{C}$.

1. By our inductive hypothesis and Lemma 26, $f, g \in \mathcal{C}$.

2. Suppose $\deg I(T;x) = n$.

   (a) Then suppose that all independent sets of size n contain v. Then these sets must not contain w. Thus, $\deg f = \deg g = n-1$.

   (b) Then suppose at least one independent set of size n does not contain v. Then it must contain w. Thus, $\deg f = n$, and $\deg g = n-1$. Therefore, $\deg f \geq \deg g \geq \deg f - 1$.

3. Every independent set in $I(T - N[v]; x)$ is also in $I(T - v; x)$. Thus, $f \geq g$ component-wise.

Therefore, $I(T; x) \in \mathcal{C}$. □

After determining that $\mathcal{C}$ contains all the independence polynomials of trees, we wanted to discover whether or not it contained any non-unimodal polynomials. Unfortunately, it does. We selected the following independence polynomial of a graph from [1]:

Proposition 29. *$\mathcal{C}$ contains a non-unimodal polynomial that is the independence polynomial of a graph:* $I(K_{17} + 4K_2; x) = 1 + 25x + 24x^2 + 32x^3 + 16x^4$

Proof. These steps follow the rules of $\mathcal{C}$ and yield the desired polynomial:

1. By 27, $1 + 2x \in \mathcal{C}$
   (a) By Lemma 26, $(1 + 2x)(1 + x) = 1 + 3x + 2x^2 \in \mathcal{C}$
   (b) $1 + 3x + 2x^2 + x(1 + 2x) = 1 + 4x + 4x^2 \in \mathcal{C}$
   (c) By Lemma 26, $(1 + 4x + 4x^2)(1 + x) = 1 + 5x + 8x^2 + 4x^3 \in \mathcal{C}$
   (d) $1 + 5x + 8x^2 + 4x^3 + x(1 + 4x + 4x^2) = 1 + 6x + 12x^2 + 8x^3 \in \mathcal{C}$. We'll call this g'.
2. By Lemma 27, $1 + 19x \in \mathcal{C}$
   (a) $1 + 19x + x(1 + 2x) = 1 + 20x + 2x^2 \in \mathcal{C}$
   (b) $1 + 20x + 2x^2 + x(1 + 2x) = 1 + 21x + 4x^2 \in \mathcal{C}$
   (c) $1 + 21x + 4x^2 + x(1 + 4x + 4x^2) = 1 + 22x + 8x^2 + 4x^3 \in \mathcal{C}$
   (d) $1 + 22x + 8x^2 + 4x^3 + x(1 + 4x + 4x^2) = 1 + 23x + 12x^2 + 8x^3 \in \mathcal{C}$
   (e) $1 + 23x + 12x^2 + 8x^3 + x(1 + 6x + 12x^2 + 8x^3) = 1 + 24x + 18x^2 + 20x^3 + 8x^4 \in \mathcal{C}$. We'll call this f'.
3. $f' + xg' = 1 + 25x + 24x^2 + 32x^3 + 16x^4 \in \mathcal{C}$

□

After learning that $\mathcal{C}$ does contain a non-unimodal polynomial, and in fact, such a polynomial that is the independence polynomial of a graph, we were led to ask the following question:

Question 30. *Is there any graph whose independence polynomial is not contained in $\mathcal{C}$?*

This question is more difficult than the analogous question with trees, because we can't form the proof around the existence of a near-leaf in an arbitrary graph.

4 Lexicographic product

Definition 31. Let $G = (\{A, B, C, ...\}, E(G)), H = (\{a, b, c, ...\}, E(H)\}$ be graphs. Define the *lexicographic product of* G *and* H to be

$$G[H] = (\{Aa, Ab, Ac, ..., Ba, Bb, Bc, ..., Ca, Cb, Cc, ...\}, E(G[H])),$$

where for some $W, Y \in V(G)$ and $x, z \in V(H)$ an edge $e = WxYz \in E(G[H])$ if either of the following is true:

1. $WY \in E(G)$
2. $W = Y$ and $xz \in E(H)$

Brown, Hickman, and Nowakowski showed in [3] the following equality relating to the independence polynomial of the lexicographic product of graphs:

Lemma 32. $I(G[H]; x) = I(G; I(H; x) - 1)$

With the work of Chudnovsky and Seymour in [4], it is natural to ask when the lexicographic product of two graphs would be claw free. In this section we show when it is claw free; we found it to be a restrictive relationship between the two graphs.

Definition 33. Let G and H be defined as in Definition 31, and let $i \in V(H)$ and $I \in V(G)$. We define a *copy of* G *in* $G[H]$ to be the induced subgraph on the vertices $Ai, Bi, Ci, ...$ and a *copy of* H *in* $G[H]$ to be the induced subgraph on the vertices $Ia, Ib, Ic, ...$ We write G^c to denote a copy of G in $G[H]$ and H^c to denote a copy of H in $G[H]$.

The following lemma is helpful in proving Proposition 35.

Lemma 34. *Let* $G = G_1 \cup G_2 \cup ... \cup G_a$ *be a graph with* a *connected components, and let* H *be a graph. Then*

$$G[H] \cong G_1[H] \cup G_2[H] \cup ... \cup G_a[H].$$

Proof. If $G_i \cap G_j = \emptyset$, then $G_i[H] \cap G_j[H] = \emptyset$. Therefore, $G[H] \cong G_1[H] \cup G_2[H] \cup ... \cup G_a[H]$. □

Proposition 35. *Suppose G and H are graphs. Then $G[H]$ is claw free if and only if one of the following is true:*

1. *G is an empty graph, and H is claw free.*
2. *G is claw free, and H is a complete graph.*
3. *$G \cong K_a \cup K_b \cup ...$ is the disjoint union of complete graphs, and H is co-triangle-free.*

Proof. $(\implies)$

We consider cases:

1. G is the empty graph, so $G[H]$ is the disjoint union of $|V(G)|$ copies of H. Thus, $G[H]$ is claw free.
2. $G[H]$ is the disjoint union of $|V(G)|$ copies of H with all edges added between two copies corresponding to edges in G. Suppose $G[H]$ has a claw. H is a complete graph, so $\alpha(H) = 1$. Therefore, each leaf vertex of the claw is in a different copy of H. Then G has a claw. This is a contradiction. Thus, $G[H]$ is claw free.
3. By 34, $G[H] \cong K_a[H] \cup K_b[H] \cup ...$, so if we show $K_i[H]$ is claw free for some $i \in \mathbb{N}$, then $G[H]$ is claw free. We consider cases:
   (a) $H = K_n, n \in \mathbb{N}$. Thus, each component of $G[H] \cong K_a[H] \cup K_b[H] \cup ... \cong K_{an} \cup K_{bn} \cup ...$ is claw free. Therefore, $G[H]$ is claw free.
   (b) $\alpha(H) = 2$. Suppose $K_i[H]$ has a claw. K_i is claw free, so the leaves of the claw must all be in the same copy of H. Thus, $\alpha(H) \geq 3$. This is a contradiction. Thus, $K_i[H]$ is claw free, and so $G[H]$ is claw free.

$(\impliedby)$

Suppose the negation of the hypothesis.

We consider cases:

1. Suppose G or H contains a claw. Then either the copy of G or the copy of H contains a claw. Thus, $G[H]$ contains a claw.

2. Suppose G has at least one edge AB and is not the disjoint union of complete graphs, and H has at least one non-edge xy. Then G also contains another edge BC that is connected to AB. Then the edges $\{BxAx, BxAy, BxCx\}$ are in $G[H]$, but the edges $\{AxAy, AyCx, CxAx\}$ are not in $G[H]$. Thus, $G[H]$ has a claw.

3. Suppose G has at least one edge AB, and the complement of H contains at least one triangle. Then H has an independent set of size 3 $\{x, y, z\}$. Then $G[H]$ contains the edges $\{AxBx, AxBy, AxBz\}$ but does not contain the edges $\{BxBy, ByBz, BxBz\}$. Thus, $G[H]$ has a claw. □

5 Independence polynomials with a root of -1

The independence polynomial at -1 has been studied in [3]. It is interesting because taking the independence complex gives a connection with topology. In particular, the graphs whose independence complexes collapse to a point have independence polynomials with a root of -1. In other words, their reduced Euler characteristic is -1.

Lemma 36. *Let G, H be graphs. If $I(G; x) = I(H; x)$, then $|V(G)| = |V(H)|$, and $|E(G)| = |E(H)|$.*

Proof. Suppose $I(G; x) = I(H; x) = 1 + ax + bx^2 + \dots$. Then $|V(G)| = a = |V(H)|$, and $|E(G)| = \binom{a}{2} - b = |E(H)|$. □

While considering how to answer Question 39, we were led to the following question:

Question 37. *Let G be a graph, and let $f = I(G; x)$ have a factor of $(1+x)$. Does $\frac{f}{1+x} = I(H; x)$ for some graph H?*

Counterexample. Let $f = I(P_7; x) = 1 + 7x + 15x^2 + 10x^3 + x^4$. $I(P_7; -1) = 1-7+15-10+1 = 0$, so $(1+x)$ is a factor of f. $\frac{f}{1+x} = 1+6x+9x^2+x^3$. This is not the independence polynomial for any graph, by inspection of graphs with six vertices and six edges. □

Lemma 38. *Let $G = (V, E)$ be a connected graph on n vertices and m edges with $(1 + x)$ a factor of $I(G; x)$. Then $n - 1 \leq m$.*

Proof. Suppose $m < (n-1)$. Then G has at least two connected components. This is a contradiction. Thus, $m \geq (n - 1)$. □

The following question could be proved using Question 37 and Lemma 36, if Question 37 had a positive answer. Nevertheless, it is still an open question.

Question 39. *Let G be defined as in Lemma 38. Is $m \leq \binom{n-1}{2}$ always true?*

The bound in Question 39 held for ten thousand randomly created graphs with a number of vertices in the range $[10, 100]$. It also holds for all graphs on up to nine vertices.

Fact 40. *The following table shows the number of connected graphs on n vertices whose independence polynomials have a root of -1:*

n	*Graphs*
≤ 3	0
4	1
5	6
6	38
7	277
8	3056
9	59768

The Sage [5] code that produced these results is available at my advisor, Russ Woodroofe's webpage: http://rwoodroofe.math.msstate.edu/advising.html
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