Prime divisibility of binomial coefficients

Dr. Russ Woodroofe
Mississippi State University
rwoodroofe@math.msstate.edu

Binomial coefficients

We write

$$\binom{n}{k} \text{ to mean } \frac{n!}{k! \cdot (n-k)!}. \qquad \text{Say as ``}n \text{ choose } k\text{''}.$$

That is,

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1}{k \cdot (k-1) \cdot (k-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1 \ \cdot \ (n-k)!}.$$

Example:

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = \frac{120}{2 \cdot 6} = 10.$$

The numbers $\binom{n}{k}$ are called *binomial coefficients*.

We'll explain why soon.

Why care about $\binom{n}{k}$? At least 3 reasons.

Why care about $\binom{n}{k}$? At least 3 reasons.

Reason 1: $\binom{n}{k}$ is the number of ways to choose
a set of k elements out of
a set of n elements.

Example: Forming a team of 3 people from 6 people, you could:

Choose a 1st person (in 6 ways)
Choose a 2nd person (in 5 ways)
Choose a 3rd person (in 4 ways)

Then notice the order of choosing didn't matter.

Since there are $3! = 6$ ways of ordering 1st, 2nd, 3rd,

$$\text{total \# of possible teams } = \binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3!} = 20.$$

Why care about $\binom{n}{k}$? Choosing sets!

Example: Possible teams of 3 from 6 mathematicians
Alice, **B**ob, **C**harlie, **D**ee, **E**mma, and **F**rank.

ABC			
ABD			
ABE			
ABF			
ACD	BCD		
ACE	BCE		
ACF	BCF		
ADE	BDE	CDE	
ADF	BDF	CDF	
AEF	BEF	CEF	DEF

(You might notice that $\binom{6}{3} = \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$.)

Why care about $\binom{n}{k}$? Powers of $(1+x)$ coefficients!

Why care about $\binom{n}{k}$? Powers of $(1+x)$ coefficients!

Reason 2: $\displaystyle\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$.

Example: To find $(1+x)^6$, expand it out:

$$(1+x)\cdot(1+x)\cdot(1+x)\cdot(1+x)\cdot(1+x)\cdot(1+x)$$

To get an x^k choose a set of k of the factors A, B, C, D, E, F to provide an x in the FOIL algorithm.

$$\text{So } (1+x)^6 = 1 + \binom{6}{1}\cdot x + \binom{6}{2}\cdot x^2 + \binom{6}{3}\cdot x^3 + \binom{6}{4}\cdot x^4 + \dots$$
$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$$

Reason 2: $\displaystyle\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$.

Example: To find $(1+x)^6$, expand it out:

$$\underbrace{(1+x)}_{A} \cdot \underbrace{(1+x)}_{B} \cdot \underbrace{(1+x)}_{C} \cdot \underbrace{(1+x)}_{D} \cdot \underbrace{(1+x)}_{E} \cdot \underbrace{(1+x)}_{F}$$

To get an x^k choose a set of k of the factors A, B, C, D, E, F to provide an x in the FOIL algorithm.

$$\begin{aligned} \text{So } (1+x)^6 &= 1 + \binom{6}{1} \cdot x + \binom{6}{2} \cdot x^2 + \binom{6}{3} \cdot x^3 + \binom{6}{4} \cdot x^4 + \dots \\ &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6. \end{aligned}$$

Reason 2: $\displaystyle\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$. **(Binomial!)**

Example: To find $(1+x)^6$, expand it out:

$$\underbrace{(1+x)}_{A}\cdot\underbrace{(1+x)}_{B}\cdot\underbrace{(1+x)}_{C}\cdot\underbrace{(1+x)}_{D}\cdot\underbrace{(1+x)}_{E}\cdot\underbrace{(1+x)}_{F}$$

To get an x^k choose a set of k of the factors A, B, C, D, E, F to provide an x in the FOIL algorithm.

$$\text{So } (1+x)^6 = 1 + \binom{6}{1}\cdot x + \binom{6}{2}\cdot x^2 + \binom{6}{3}\cdot x^3 + \binom{6}{4}\cdot x^4 + \dots$$
$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$$

You might have seen the coefficients of $(1+x)^n$ before through *Pascal's triangle* (another approach to binomial coefficients).

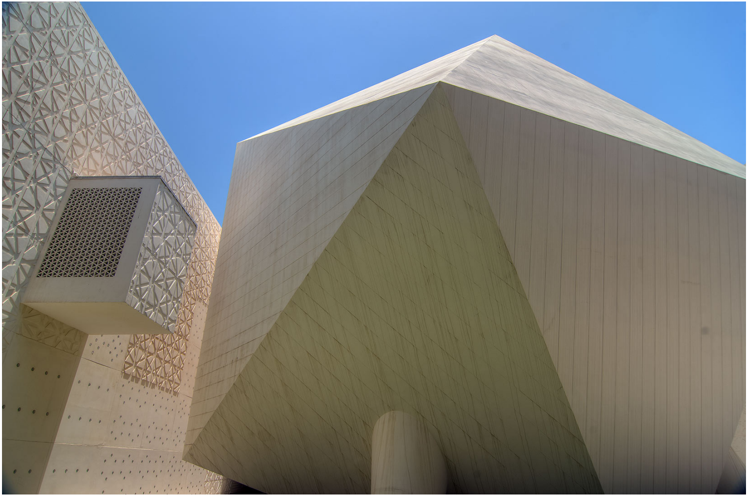
Here's a picture of Pascal:

(scan of a c.1690 painting, from Wikipedia)

Why care about $\binom{n}{k}$? Symmetry!

Reason 3: Symmetry!

Weill Cornell Medical College, Doha Qatar.
The icosohedral lecture hall and lattice work are both highly symmetric.



Why care about $\binom{n}{k}$? Symmetry!

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

$\#$ ways to rearrange a set with n elements?

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

\# ways to rearrange a set with n elements? $n!$

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

\# ways to rearrange a set with n elements? $n!$

\# ways to rearrange a set with n elements, so that the first k elements come before the last $n - k$?

Why care about $\binom{n}{k}$? Symmetry!

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

\# ways to rearrange a set with n elements? $n!$

\# ways to rearrange a set with n elements, so that the first k elements come before the last $n-k$? $k! \cdot (n-k)!$

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

\# ways to rearrange a set with n elements? $n!$

\# ways to rearrange a set with n elements, so that the first k elements come before the last $n-k$? $k! \cdot (n-k)!$

So $\binom{n}{k}$ is the ratio of the above two!

Why care about $\binom{n}{k}$? Symmetry!

Reason 3: Symmetry!

One kind of symmetry: rearranging (reordering) elements in a set.

Example: Rearrange ABCDEF to ACBDEF or to FEDCBA.

Question: How many rearrangements are there?

ways to rearrange a set with n elements? $n!$

ways to rearrange a set with n elements, so that the first k elements come before the last $n-k$? $k! \cdot (n-k)!$

So $\binom{n}{k}$ is the ratio of the above two! **(Group Theory.)**

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Motivation: Problems on symmetry.

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$$n = 6 : \quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$$n = 6: \quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

$$n = 15: \quad \binom{15}{1} = 15, \binom{15}{2} = 105, \binom{15}{3} = 455, \binom{15}{4} = 1365,$$

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$$n = 6: \quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ “work”.

$$n = 15: \quad 15, 105, 455, 1365, 3003, 5005, 6435, 5005, \ldots$$

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$n = 6:$ $\quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

$n = 15:$ $\quad 15, 105, 455, 1365, 3003, 5005, 6435, 5005, \ldots$

primes pairs $3, 5$ or $3, 13$ or $5, 13$ "work".

A problem of John Shareshian and myself

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$$n = 6: \quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

$$n = 15: \quad 15, 105, 455, 1365, 3003, 5005, 6435, 5005, \ldots$$

primes pairs $3, 5$ or $3, 13$ or $5, 13$ "work".

$n = 30$?

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$n = 6:\quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

$n = 15:\quad 15, 105, 455, 1365, 3003, 5005, 6435, 5005, \ldots$

primes pairs $3, 5$ or $3, 13$ or $5, 13$ "work".

$n = 30? \quad n = 31416?$

Our problem:
For a number n, must there be primes p, q so that

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}$$

are all divisible by at least one of the two primes?

Example:

$n = 6 : \quad \binom{6}{1} = 6, \binom{6}{2} = 15, \binom{6}{3} = 20, \ldots$

primes pairs $2, 3$ or $2, 5$ or $3, 5$ "work".

$n = 15 : \quad 15, 105, 455, 1365, 3003, 5005, 6435, 5005, \ldots$

primes pairs $3, 5$ or $3, 13$ or $5, 13$ "work".

$n = 30? \quad n = 31416? \quad n = 1000000?$

Primes dividing $\binom{30}{k}$

Example: Find two primes so at least one divides each of

$$\binom{30}{1}, \binom{30}{2}, \binom{30}{3}, \binom{30}{4}, \binom{30}{5}, \ldots.$$

Primes dividing $\binom{30}{k}$

Example: Find two primes so at least one divides each of

$$\binom{30}{1}, \binom{30}{2}, \binom{30}{3}, \binom{30}{4}, \binom{30}{5}, \ldots.$$

A nice feature of 30 is that $30 = 29 + 1$. (And 29 is prime.)

In particular:

29 does <u>not</u> divide $1!, 2!, 3!, \ldots,$ or $28!$,

but 29 divides $30!$ and $29!$.

So 29 divides all of the following

$$\binom{30}{2} = \frac{30 \cdot 29}{2!}, \quad \binom{30}{3} = \frac{30 \cdot 29 \cdot 28}{3!}, \quad \binom{30}{4} \ldots$$

Primes dividing $\binom{30}{k}$

Example: Find two primes so at least one divides each of

$$\binom{30}{1}, \binom{30}{2}, \binom{30}{3}, \binom{30}{4}, \binom{30}{5}, \ldots.$$

A nice feature of 30 is that $30 = 29 + 1$. (And 29 is prime.)

In particular:

29 does <u>not</u> divide $1!, 2!, 3!, \ldots,$ or $28!$,

but 29 divides $30!$ and $29!$.

So 29 divides all of the following

$$\binom{30}{2} = \frac{30 \cdot 29}{2!}, \quad \binom{30}{3} = \frac{30 \cdot 29 \cdot 28}{3!}, \quad \binom{30}{4} \ldots$$

And the primes pairs $2, 29$ or $3, 29$ or $5, 29$ all "work" for $n = 30$.

Main lemma

We can see from the example $n = 30$ that if $n - 1$ is a prime, then $n - 1$ and a divisor of n "work" for our problem.

With just a bit more work, Shareshian and I proved:

Lemma: If n is a positive integer and p, q are primes so that

p^a divides n, and also

$q < n < q + p^a$,

then at least one of p, q divide each binomial coefficient of n.

Example: $n = 30, q = 29, \quad \text{or} \quad n = 300, q = 293.$

Main lemma

We can see from the example $n = 30$ that if $n - 1$ is a prime, then $n - 1$ and a divisor of n "work" for our problem.

With just a bit more work, Shareshian and I proved:

Lemma: If n is a positive integer and p, q are primes so that

p^a divides n, and also

$q < n < q + p^a$,

then at least one of p, q divide each binomial coefficient of n.

Example: $n = 30, q = 29, \quad$ or $\quad n = 300, q = 293.$

Combining the Lemma with results about typical prime gaps, we show that there are prime pairs dividing every binomial coefficient of n for $99.999999999999999999999999999\,\%$ of numbers n.
(Some recent work on small prime gaps: Zhang's Theorem.)

Combining the Lemma with a computer, we find prime pairs dividing every binomial coefficient of n for n up to 1 billion.


$\sum_{n=1}^{\infty} n = -\frac{1}{12}$
Thank you!

Related book:

Concrete Mathematics, by R. Graham, D. Knuth, and O. Patashnik.

A readable introduction to mathematics around binomial coefficients and other nice topics. (Calculus level-ish.)

MISSISSIPPI STATE UNIVERSITY™
MATHEMATICS & STATISTICS

Dr. Russ Woodroofe / rwoodroofe@math.msstate.edu