7 Homework 7 (Invariant Subspaces)

1. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be a given operator defined with

$$T(a + bt + ct^2) = a + b + c + (a + 3b)t + (a - b + 2c)t^2.$$

Find all one-dimensional subspaces of $\mathcal{P}_2$ that are invariant under T.

2. Let $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be a given linear operator defined with

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{pmatrix} a - b & 4a - 4b \\ -a + 2b + c & b + c \end{pmatrix}$$

Find all one-dimensional subspaces that are invariant under T.

3. Let T be a linear operator on a finite-dimensional vector space $\mathcal{V}$, and let $\mathcal{B}$ and $\mathcal{B}'$ be any two bases for $\mathcal{V}$. Show that $\det([T]_{\mathcal{B}}) = \det([T]_{\mathcal{B}'})$.

4. Let $\mathcal{V}$ and $\mathcal{W}$ denote given vector spaces and let $T : \mathcal{V} \to \mathcal{W}$ be a given linear transformation. Show that T is one-on-one (injective) if and only if $\ker(T) = \{\mathbf{0}\}$.

Recall: eigenvectors and eigenvalues. Let A be a given matrix. Scalars λ for which $(A - \lambda I)$ is singular are called the *eigenvalues* of A, and the nonzero vectors in $\ker(A - \lambda I)$ are known as the associated *eigenvectors* for A.

Let T be a linear operator on a vector space $\mathcal{V}$. A nonzero element $x \in \mathcal{V}$ is called an eigenvector of T if there exists a scalar λ such that $T(x) = \lambda x$. The scalar λ is called the eigenvalue corresponding to the eigenvector x.

5. Let T be a linear operator on a finite-dimensional vectorÂ·space $\mathcal{V}$ over a field $\mathbb{F}$ and let $\mathcal{B}$ be a basis of $\mathcal{V}$. Show that a scalar λ is an eigenvalue of T if and only if $\det([T]_{\mathcal{B}} - \lambda I) = 0$. With another words show that there exists nonzero vector $x \in \mathcal{V}$ such that $T(x) = \lambda x$ if and only if $\det([T]_{\mathcal{B}} - \lambda I) = 0$.

6. Let T be a linear operator on a finite-dimensional vector space $\mathcal{V}$, and let x denote nonzero element of $\mathcal{V}$. The subspace

$$\mathcal{W} = \mathrm{span}(\{x, T(x), T^2(x), ...\})$$

is called the *T-cyclic subspace of $\mathcal{V}$ generated by x*.

(a) Show that $\mathcal{W}$ is T-invariant.

(b) If $\dim(\mathcal{W}) = k \geq 1$ show that then

$$\{x, T(x), T^2(x), ..., T^{k-1}(x)\}$$

is a basis of $\mathcal{W}$.

7. Let T be a linear operator on a finite-dimensional vector space $\mathcal{V}$, and let $\mathcal{W}$ be a T-invariant subspace of $\mathcal{V}$. Assume that $\mathcal{B}$ and $\mathcal{B}'$ are basis for $\mathcal{V}$ and $\mathcal{W}$, respectively. Show that the polynomial $g(x) = \det([T_{/\mathcal{W}}]_{\mathcal{B}'} - xI)$ divides the polynomial $p(x) = \det([T]_{\mathcal{B}} - xI)$.

8. (The Cayley-Hamilton Theorem) Let T be a linear operator on finite dimensional vector space $\mathcal{V}$, let $\mathcal{B}$ denote a basis of $\mathcal{V}$ and let $f(x) = \det([T]_{\mathcal{B}} - xI)$ be the given polynomial. Show that then

$$f(T) = T_0 \qquad \text{(the zero transformation)}$$

(i.e. $T_0(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x} \in \mathcal{V}$).