12 Homework 12 (Orthogonal Projection)

1. Let $\mathcal{P}_2$ denote a vector space of all polynomial of degree ≤ 2,

$$\mathcal{P}_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}.$$

(a) Is the following product $\langle p, q\rangle = p(1)q(1) + 2p(0)q(0) + p(-1)q(-1)$ inner product for $\mathcal{P}_2$?

(b) For subspace $\mathcal{L} \subseteq \mathcal{P}_2$ generated by $p_1(x) = 1$ and $p_2(x) = x$ find an orthogonal complement.

(c) Find the orthogonal projection of $p(x) = -2x^2 + x + 2$ on $\mathcal{L}$.

2. In inner product space $\mathbb{R}^4$, with standard inner product, let $\mathcal{M}$ denote subspace spanned by vectors $(2, 1, 0, 0)^\top$ and $(1, 1, 1, 1)^\top$. Find a basis for orthogonal complement of $\mathcal{M}$ and find the orthogonal projection of $\boldsymbol{a} = (3, -4, 5, -5)^\top$ onto $\mathcal{M}$.

3. Let $\mathcal{M} = \operatorname{span}\{\boldsymbol{a}, \boldsymbol{b}\}$ denote subspace of inner product space $\mathbb{R}^n$ (with standard inner product) spaned by vectors $\boldsymbol{a} = (0, 1, 2, ..., n-1)^\top$ and $\boldsymbol{b} = (1, 1, 1, ..., 1)^\top$. Find orthogonal complement $\mathcal{M}^\perp$ and find the orthogonal projection of $\boldsymbol{z}$ onto $\mathcal{M}$ where

$$\boldsymbol{z} = \left(\frac{1}{2}n(3-n), \frac{1}{2}n(n-1), 0, 0, ..., 0\right)^\top \in \mathbb{R}^n.$$

4. Find the orthogonal projection of $\boldsymbol{x} = (-12, -13, 5, 2)^\top$ onto $\mathcal{M}$ if we have that

$$\mathcal{M} = \operatorname{span}\left\{\begin{pmatrix}1\\-2\\2\\-3\end{pmatrix}, \begin{pmatrix}2\\-3\\2\\4\end{pmatrix}\right\} \subseteq \mathbb{R}^4$$

(with respect to standard inner product).

5. In inner product space $\mathcal{P}_3 = \{at^3 + bt^2 + ct + d \mid a, b, c, d \in \mathbb{R}\}$ of all polynomials of degree ≤ 3 with an inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t$$

let $\mathcal{M} = \operatorname{span}\{t, 1+t\}$ be a given subspace. Find the orthogonal projection of $r(t) = -5t^3 - 12t^2 + 6t + 6$ onto $\mathcal{M}$.

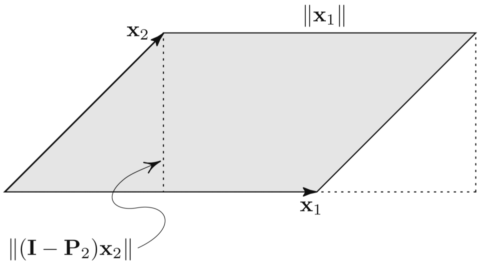
6. Space $\mathcal{L}$ is defined as set of solutions for the following system

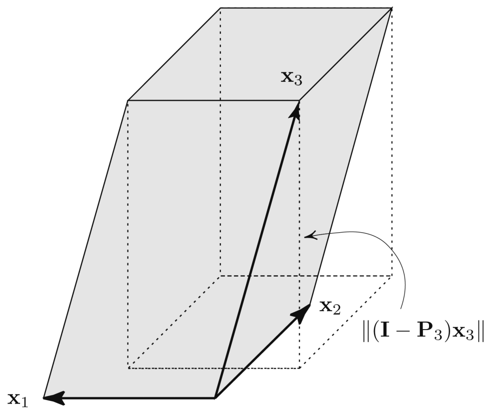
$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 - 9x_4 &= 0 \;. \end{aligned}$$

Find the orthogonal projection of $x = (7, -4, -1, 2)^\top$ onto $\mathcal{L}$ in $\mathbb{R}^4$.

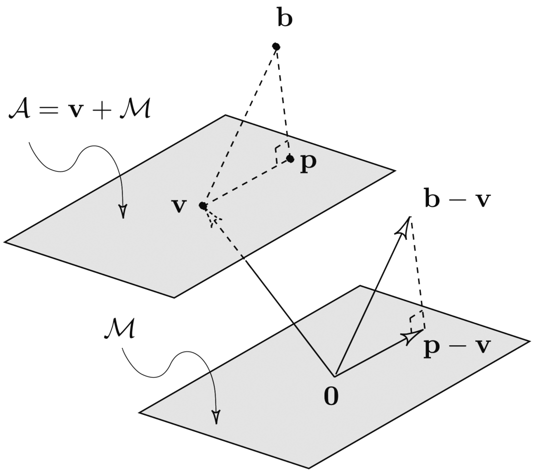
7. (challenge) (Volume, Gram-Schmidt, and QR). A solid in $\mathbb{R}^m$ with parallel opposing faces whose adjacent sides are defined by vectors from a linearly independent set $\{x_1, x_2, ..., x_n\}$ is called an n-dimensional parallelepiped. As shown in the shaded portions of figure below, a two-dimensional parallelepiped is a parallelogram, and a three-dimensional parallelepiped is a skewed rectangular box.

Problem: Determine the volumes of a two-dimensional and a three-dimensional parallelepiped, and then make the natural extension to define the volume of an n-dimensional parallelepiped.





8. Affine Projections. If $\boldsymbol{v} \neq \mathbf{0}$ is a vector in a space $\mathcal{V}$, and if $\mathcal{M}$ is a subspace of $\mathcal{V}$, then the set of points $\boldsymbol{A} = \boldsymbol{v} + \mathcal{M}$ is called an **affine space** in $\mathcal{V}$. Strictly speaking, $\boldsymbol{A}$ is not a subspace (e.g., it doesn't contain 0), but, as depicted in figure below, $\boldsymbol{A}$ is the translate of a subspace - i.e., $\boldsymbol{A}$ is just a copy of $\mathcal{M}$ that has been translated away from the origin through $\boldsymbol{v}$. Consequently, notions such as projection onto $\boldsymbol{A}$ and points closest to $\boldsymbol{A}$ are analogous to the corresponding concepts for subspaces.



Problem: For $\boldsymbol{b} \in \mathcal{V}$, determine the point $\boldsymbol{p}$ in $\boldsymbol{A} = \boldsymbol{v} + \mathcal{M}$ that is closest to $\boldsymbol{b}$. In other words, explain how to project $\boldsymbol{b}$ orthogonally onto $\boldsymbol{A}$.