11 Homework 11 (Orthogonal Decomposition)

1. Compute a URV factorization for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 5 \end{bmatrix}$.

2. Basis for a vector space

$$\mathcal{L} = \left\{ (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \,|\, x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0 \right\}$$

is $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\}$. Find orthogonal complement of $\mathcal{L}$ (with respect to standard inner product $\langle x, y \rangle = x^\top y$).

3. In inner product space $\mathcal{P}_2 = \{p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ of all polynomials of degree less or equal 2 with inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, \mathrm{d}x$ let $\mathcal{M}$ be a given subspace defined with

$$\mathcal{M} = \operatorname{span}\{x^2 - 1, x + 1\}.$$

Find a basis for $\mathcal{M}^\perp$, and write a polynomial $p(x) = 2x^2 + x + 5$ in form $p = p_1 + p_2$, where $p_1 \in \mathcal{M}$, $p_2 \in \mathcal{M}^\perp$.

4. In inner product space $\mathbb{R}^4$, with inner product defined with

$$\langle x, y \rangle = x_1y_1 + 2x_2y_2 + x_3y_3 + 2x_4y_4$$

let $\mathcal{V}$ be a given subspace spanned with vectors $v_1 = (1, 0, 1, 0)^\top$ and $v_2 = (1, 0, 1, 1)^\top$. Write vector $x = (4, 2, 2, 4)^\top$ in form $x = v + w$, where $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

5. Let $\mathcal{M}$ denote subspace of inner product space $\mathrm{Mat}_{2\times2}(\mathbb{R})$ spanned with matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. Find a basis for orthogonal complement of $\mathcal{M}$, and write the matrix $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in the form $X = Y_1 + Y_2$, where $Y_1 \in \mathcal{M}$, and $Y_2 \in \mathcal{M}^\perp$. (Standard inner product in $\mathrm{Mat}_{2\times2}(\mathbb{R})$ is defined with $\langle A, B \rangle = \mathrm{trag}(AB^\top)$).

6. In inner product space $\mathcal{P}_3 = \{at^3 + bt^2 + ct + d \,|\, a, b, c, d \in \mathbb{R}\}$ of all polynomials of degree ≤ 3 with inner product defined with

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t$$

Let $\mathcal{M} = \operatorname{span}\{1 + t, 1\}$ be a given subspace. Find a basis for $\mathcal{M}^\perp$.

7. (challenge) (Singular Value Decomposition) For each $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ of rank r, show that there are orthogonal matrices $U \in \mathrm{Mat}_{m\times m}(\mathbb{R})$, $V \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and a diagonal matrix $D_{r\times r} = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$ such that

$$A = U \begin{pmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{m\times n} V^\top \qquad \text{with} \quad \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0. \tag{2}$$

Remark. The σ_i's are called the nonzero *singular values* of A. When $r < p = \min\{m, n\}$, A is said to have $p - r$ additional zero singular values. The factorization (2) is called a *singular value decomposition* of A, and the columns in U and V are called left-hand and right-hand *singular vectors* for A, respectively.