UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

Master's Thesis
(Magistrsko delo)

The University Timetabling Problem – Complexity and an Integer Linear Programming Formulation: a Case Study of UP FAMNIT

(Problem urnika na univerzi – računska zahtevnost in formulacija s celoštevilskim linearnim programom: študija primera UP FAMNIT)

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Izvleček:
V magistrskem delu je obravnavan problem univerzitetnega urnika. Problem urnika določa prirejanje ustreznega časovnega intervala vsakemu elementu določene množice objektov; v primeru univerzitetnega urnika je to množica predavanj. Problem urnika je v splošnem *NP*-težek problem in ga je težko rešiti do optimalnosti. V delu predstavimo nekaj različnih pristopov za reševanje problema urnika, kot so barvanja grafov, hevristike, celoštevilsko linearno programiranje, nevronske mreže itn. Definiramo problem, ki posploši problem urnika na Fakulteti za matematiko, naravoslovje in informacijske tehnologije Univerze na Primorskem (UP FAMNIT), ter s pomočjo polinomskih prevedb dokažemo *NP*-polnost problema. Nato na podlagi opisa študijskega procesa na fakulteti izpeljemo matematičen model s celoštevilskim linearnim programom, ki reši problem urnika na UP FAMNIT. Model je implementiran z uporabo programskega paketa Zimpl in rešen na konkretnih podatkih, in sicer za spomladanski semester študijskega leta 2016/17, z uporabo programskega paketa Gurobi. Dobljena rešitev je interpretirana in primerjana z ročno sestavljenim urnikom.

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Abstract: In the Master's thesis we consider a university timetabling problem, the problem of assigning courses to time intervals with respect to certain conditions. The problem is known to be *NP*-hard so no efficient solution methods are known for it. In the thesis we describe few various approaches for solving timetabling problems, such as graph colouring, integer linear programming, neural networks, heuristics, etc. We define the Famnit Timetable Design problem as a natural generalization of the actual timetabling problem for the Faculty of Mathematics, Natural Sciences and Information Technologies at the University of Primorska (UP FAMNIT) and prove that the problem is *NP*-complete. Using the description of the teaching process at the analysed institution we develop a mathematical model based on integer linear programming for solving the Famnit Timetable Design problem. The model is implemented using programming language Zimpl and evaluated using Gurobi software. The implementation is tested on the real input data for the Spring semester of the 2016/17 academic year. A timetable representing the results of the implementation is commented and compared with the one made by hand.

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List of abbreviations

BTD	Binary Timetable Design
e.g.	for example
FTD	Famnit Timetable Design
i.e.	that is
ILP	integer linear programming
LP	linear programming
MILP	mixed integer linear programming
s.t.	such that
TD	Timetable Design
UTP	university timetabling problem

1 Introduction

A problem of assigning some objects to available resources in order to complete some tasks is called a *scheduling problem.* Scheduling problems arise in arranging sport matches, arranging workers to jobs, scheduling flights, assigning events to some time intervals, etc. In this work we consider the problem of assigning some events to time intervals, generally known as the *timetabling problem.*

Research considering the timetabling problem started during the 1950s (see, e.g, [56]) and until now there are many papers considering various timetabling problems (see, e.g., [49]). During the years various definitions of timetabling problem have been formulated (see, e.g., [9, 17, 45, 49]). Here we introduce the one given by Wren [56]:

Definition 1.1. *Timetabling* is the allocation, subject to constraints, of given resources to objects being placed in space or time, in such a way as to satisfy as nearly as possible a set of desirable objectives.

Timetabling is a widely known problem that cannot be efficiently solved. In many institutions it takes a lot of time to prepare by hand a timetable satisfying given requirements and resources' restrictions, so various mathematical models have been developed for solving these problems using computer.

In many areas of human activity timetables need to be determined, for example by educational institutions, transport companies, for sport competitions, for production and manufacturing, and so on.

In this work we consider the educational timetabling problem, or more precisely a university timetabling problem (UTP). In the literature this problem is separated into two major categories, with respect to objects that are supposed to be scheduled (see Burke et al. [9]):

- *Course timetabling problem.* Objects representing courses are allocated to resources representing time intervals and available classrooms. Allocation of these objects has to be done with respect to some constraints, defined by the teaching process at the concerned institution.

- *Exam timetabling problem.* Objects representing exams are allocated to resources representing time intervals and available classrooms, with respect to some requirements. For each course there is exactly one exam to be scheduled.

There are significant differences between these two types of educational timetabling problems. For example, two distinct exams can be scheduled in the same classroom and at the same time interval, while for courses this is not possible. Exams should be uniformly distributed during the examination period, while for the courses this is not always the case. It follows that a single model representing both course and exam timetabling problems cannot be developed. Here we consider the course timetabling problem.

In Chapter 2 we give an overview of some fundamental theoretical results used in the thesis. In Chapter 3 we introduce the TIMETABLE DESIGN problem, which represents a general definition of the timetabling problem. We derive results concerning computational complexity of timetabling problems and give a literature overview of the approaches used for solving timetabling problems. Chapter 4 is devoted to the description and formal definition of the timetabling problem for the Faculty of Mathematics, Natural Sciences and Information Technologies at the University of Primorska (abbreviated UP FAMNIT).

In the literature various timetabling problems are often assumed to be *NP*-hard, without any proof guaranteeing that. A result developed in this thesis proves that FAMNIT TIMETABLE DESIGN (FTD), a problem defined in Section 4.2, is *NP*-complete. Following this result, we develop in Chapter 6 an integer linear programming model for the FTD problem.

The model is implemented with real data input from the Spring semester of academic year 2016/17 using the programming software Zimpl (see [34]) and Gurobi Optimizer (see [42]). Chapter 7 contains results of implementation, as well as their interpretation and comparison with a timetable prepared manually.

2 Theoretical background

In this chapter we recall fundamental definitions and results used in the thesis. In the first section we recall various classes of problems, defined with respect to the computational complexity of a problem. The second section is devoted to linear optimization problems, while the last one contains basic concepts of integer linear optimization.

2.1 Complexity classes

When speaking about some specific problem and determining its solution, one of the first things one has in mind is what time does it take to solve the problem. Here under the word "problem" we consider some decision problem, that is, a problem that for a given input asks whether the answer to some question is yes or no. A simple example of such a problem is: given a natural number n, determine if n is a prime number. Clearly, the time necessary to solve a given problem depends on input size. The *running time* of an algorithm is standardly defined as the function mapping a given positive integer n to the maximum number of arithmetic operations and comparisons that the algorithm performs on an input instance of size n. In complexity theory, problems are divided into classes with respect to the running time of algorithms that solve them.

Definition 2.1. A problem Π is *solvable in polynomial time* if there exists an algorithm that solves Π in time that is bounded by a polynomial function of input size.

A fundamental complexity class is class P, which consists of problems solvable in polynomial time. Class P is considered to be the set of problems that can be solved efficiently. A decision problem Π for which it may not be known whether there exists a polynomial time algorithm that solves it, but for any input I such that $\Pi(I)$ gives answer yes, there exists a certificate C such that using C, the fact that $\Pi(I)$ gives answer yes can be verified in time polynomial in the size of input I, is said to be *solvable in non-deterministic polynomial time*. Such problems define the complexity class *NP*. Clearly, a yes instance of any polynomial-time solvable problem Π can be verified in polynomial time, so it is true that $P \subseteq NP$. Whether the converse inclusion holds is far from trivial and is a major open question, with a conjecture that $P \neq NP$ [24]. Thus, complexity theory is developed under the assumption that there exist problems that are in *NP* and not in P. In particular, the conjectured set theoretic difference between classes *NP* and *P* represents a set of problems that are of special interest for research. We now define the class of *NP*-hard problems.

Definition 2.2. A problem Π is said to be *NP*-hard if the existence of a polynomial time algorithm that solves Π implies the existence of a polynomial time algorithm for any problem in the class *NP*.

In other words, the above definition says that the existence of a polynomial-time algorithm for any *NP*-hard problem implies equality of classes *P* and *NP*. Among the *NP*-hard problems, problems belonging to the class *NP* are of special interest. We say that a problem Π is *NP-complete* if it is in *NP* and if every problem in *NP* polynomially reduces to Π (see Definition 2.3). Clearly, every *NP*-complete problem is *NP*-hard, but there are also *NP*-hard problems that are not in *NP*. In particular, an *NP*-hard problem does not have to be a decision problem.

It is not at all obvious that the set of *NP*-complete problems is non empty, but under the assumption that this is the case, we can say that *NP*-complete problems are the hardest problems in *NP* [24]. Existence of a polynomial time algorithm for any one of them implies existence of a polynomial time algorithm for all of them. The family of known *NP*-complete problems is growing rapidly, so nowadays there are thousands of problems proved to be *NP*-complete. A conjectured relationships between complexity classes mentioned here is displayed in Figure 1.

One of the fundamental results considering the class of *NP*-complete problems is known as Cook's Theorem [13]. That is the result proving *NP*-completeness of a problem called SATISFIABILITY and represents the first *NP*-completeness proof in the literature, guaranteeing that the set of *NP*-complete problems is non-empty. The SATISFIABILITY problem can be defined as follows:

SATISFIABILITY

Instance:	A set U of binary variables $x_1, x_2, \dots, x_n$, a collection C of clauses representing disjunctions of elements in U or their negations.
Question:	Is there a satisfying truth assignment for C?

In order to prove that some problem Π belongs to the class of *NP*-complete problems, it suffices to show that $\Pi \in NP$ and that Π is at least as hard as some other problem in *NP*, or, in other words, that using a polynomial time algorithm for problem Π we can construct a polynomial time algorithm for some problem in *NP*. Such a correspondence between two problems is called a polynomial reduction.

Definition 2.3. A decision problem Π_1 can be *polynomially reduced* to a decision problem Π_2 if there exists a function f that, given an input I_1 for Π_1, constructs an input $I_2 = f(I_1)$ for Π_2 and has the following properties:

1. $f(I_1)$ can be computed in time that is polynomial in the size of I_1,
2. problem Π_1 has answer yes for input I_1 if and only if problem Π_2 has answer yes for input $f(I_1)$.

Such a reduction is also called *Karp's reduction* [31].

If the problem Π_1 polynomially reduces to Π_2, we denote that by $\Pi_1 \propto \Pi_2$. Problems Π_1 and Π_2 are said to be *polynomially equivalent* whenever a reduction can be constructed in both directions, that is, if $\Pi_1 \propto \Pi_2$ and $\Pi_2 \propto \Pi_1$. If we have problems Π_1, Π_2, Π_3, where $\Pi_1 \propto \Pi_2$ and $\Pi_2 \propto \Pi_3$, this means that the problem Π_2 is at least as hard as Π_1 and problem Π_3 is at least as hard as Π_2. Obviously, then, Π_3 is at least as hard as Π_1 and a polynomial reduction $\Pi_1 \propto \Pi_3$ can be constructed. Thus the existence of a polynomial reduction between two decision problems is a transitive relation. The following theorem characterizes *NP*-complete problems using the notion of polynomial reduction.

Theorem 2.4 (Garey et al. [24])**.** *A problem* Π *is NP-complete if and only if it is in NP and there exists an NP-complete problem that polynomially reduces to* Π*.*

Proof. Necessity of the condition follows immediately from Cook's theorem. If a problem Π is *NP*-complete, it belongs to the class *NP* by definition. The set of polynomials is closed under composition, so if $\Pi_1 \propto \Pi$ and there is an algorithm that solves Π in polynomial time, then composing the algorithm with a polynomial reduction from Π_1 to Π yields a polynomial-time algorithm that solves Π_1. Correctness of the theorem follows directly from the transitivity of polynomial reductions and from Cook's theorem, guaranteeing the existence of at least one *NP*-complete problem. $\square$

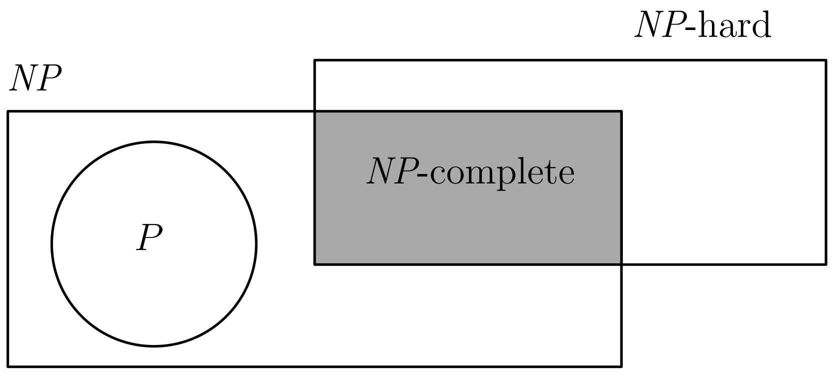


Figure 1: Conjectured relationships between some complexity classes [24].

Theorem 2.4 will be used in Chapter 5 for a proof of *NP*-completeness. Informally speaking, whenever a problem Π is proved to be *NP*-complete, the probability of existence of a polynomial-time algorithm for solving it becomes very small. The number of steps needed to solve some *NP*-complete problem using existing algorithms grows very fast as the size of input increases. Under the assumption that $P \neq NP$, finding solutions to large instances of such problems can be very difficult to do in real time.

Observe that besides problems that are characterized as decision problems and are supposed to give answer yes or no to some question, there are problems trying to find a solution that is good enough, with respect to some measure, usually represented by the value of some function. Such a problem Π is called an *optimization problem* and is defined by a (usually implicitly given) set of feasible solutions $\mathcal{D}$ and an *objective function* $f : \mathcal{D} \to \mathbb{R}$, which measures the quality of the each element in $\mathcal{D}$. An optimization problem can be referred to as *minimization* or *maximization* problem, with respect to whether function f is supposed to get either the smallest or the largest possible value. Whenever it is not specified if f should be minimized, or maximized (or this will be clear from the context), we will say that f is supposed to be "optimized". An element $\tilde{\mathbf{x}} \in \mathcal{D}$ is said to be an *optimal solution* of problem Π if $f(\tilde{\mathbf{x}}) = opt\{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{D}\}$, where $opt \in \{min, max\}$. In the following example we present two problems that are classified as a decision and an optimization problem, respectively.

Example 2.5. A Hamiltonian cycle in a graph $G = (V, E)$ is defined as a cycle C that is a subgraph of G and visits each vertex of G exactly once. Determining whether a graph G contains a Hamiltonian cycle is the following decision problem.

HAMILTONIAN CYCLE
Instance: A graph $G = (V, E)$.
Question: Does G contain a Hamiltonian cycle?

Clearly, the answer to the above question is either "yes" or "no", depending on the existence of a Hamiltonian cycle in G. An example of an optimization problem is a problem that asks about the longest cycle in the graph G, known as the LONGEST CYCLE problem.

LONGEST CYCLE
Instance: A graph $G = (V, E)$.
Question: Find a longest cycle in G.

This problem tries to find a cycle C in G of maximum possible length and not just to answer if a cycle of some length exists or not.

2.2 Linear Programming

A *linear optimization problem* is defined as minimization or maximization of some linear function subject to linear constraints. In Example 2.6 we have a simple problem of minimizing the sum of two numbers, with respect to linear inequalities, which have to be satisfied. Any assignment of values to variables x_1 and x_2 that satisfies the given constraints represents a feasible solution.

Example 2.6.

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{subject to} & -x_1 & + & 3x_2 & \leq & 9 \\ & 6x_1 & - & x_2 & \leq & 24 \\ & & & x_1, \quad x_2 & \geq & 0 \end{array}$$

An optimal solution of this problem is $(x_1, x_2) = (\frac{81}{17}, \frac{78}{17})$, with objective function value equal to $\frac{159}{17}$.

The set of all feasible solutions is said to be a feasible region. If no feasible solution exists, the problem is said to be infeasible. Otherwise, a feasible solution that optimizes the objective function is called an optimal solution. If the problem is not infeasible and no optimal solution exist, we say that the problem is unbounded.

A linear optimization problem is often referred to as a linear programming problem,or simply as a linear program. Depending on the nature of the problem, a linear programming problem can be defined as a minimization or a maximization problem, with constraints represented by equalities or inequalities. For simplicity, we define a linear program (abbreviated LP) in its standard form, using standard concepts of linear algebra.

Definition 2.7. *A linear program in standard form* is defined as

$$\begin{array}{rrcl} \text{minimize} & \mathbf{c^T x} & & \\ \text{subject to} & \mathbf{Ax} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}, \end{array}$$

where $\mathbf{A}$ is a matrix from $\mathbb{R}^{m \times n}$, $\mathbf{b}$ and $\mathbf{c}$ vectors from $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and $\mathbf{x}$ a vector of variables, $\mathbf{x} = (x_1, \ldots, x_n)^T$.

Another widely used formulation of linear programming problems has constraints in the form $\mathbf{Ax} \leq \mathbf{b}$ instead of equalities and no nonnegativity constraints, but that can be simply converted into standard form by adding nonnegative variables u_i to each row $\mathbf{a_i^T x} \leq b_i$ in order to reach equality. Variables u_i are called *slack variables.*

Lemma 2.8. *Given a matrix* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and a vector* $\mathbf{b} \in \mathbb{R}^m$*, consider the following two problems:*

Π_1) *Does the system* $\mathbf{Ax} \leq \mathbf{b}$ *have a solution?*

Π_2) *Does the system* $\mathbf{Ax} = \mathbf{b}$ *have a nonnegative solution?*

Problems Π_1 *and* Π_2 *are polynomially equivalent.*

Proof. In order to prove polynomial equivalence of problems Π_1 and Π_2 we have to construct polynomial reductions in both directions.
$\Pi_1 \propto \Pi_2$: Let $I_1 = (\mathbf{A}, \mathbf{b})$ be input for problem Π_1. We construct input $I_2 = (\mathbf{A}', \mathbf{b}')$ for Π_2 as $(\mathbf{A}', \mathbf{b}') = ([\mathbf{A}\ -\mathbf{A}\ \mathbf{I}], \mathbf{b})$, where $\mathbf{I} \in \mathbb{R}^{m\times m}$. Assume that for $\mathbf{x} \in \mathbb{R}^n$ it holds that $\mathbf{Ax} \leq \mathbf{b}$. Let us define vectors $\mathbf{x}^+, \mathbf{x}^- \in \mathbb{R}^n$ in the following way:

$$x_i^+ = \begin{cases} x_i, & \text{if } x_i \geq 0, \\ 0, & \text{if } x_i < 0 \end{cases} \qquad x_i^- = \begin{cases} 0, & \text{if } x_i \geq 0, \\ -x_i, & \text{if } x_i < 0 \end{cases} \tag{2.1}$$

where x_i, x_i^+, x_i^- denote the i-th component of vectors $\mathbf{x}, \mathbf{x}^+, \mathbf{x}^-$, respectively. Denote by $\mathbf{y}$ vector representing the difference $\mathbf{b} - \mathbf{Ax}$. Clearly, it holds that $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ and that vectors $\mathbf{x}^+, \mathbf{x}^-, \mathbf{y}$ are nonnegative. We claim that the vector $\mathbf{w} \in \mathbb{R}^{2n+m}$, defined as

$$\mathbf{w} = \begin{cases} x_i^+, & \text{if } 0 \leq i \leq n, \\ x_i^-, & \text{if } n+1 < i \leq 2n, \\ y_i, & \text{if } 2n+1 \leq i \leq 2n+m, \end{cases}$$

is a solution of the problem Π_2. This follows directly from the following equation:

$$\mathbf{A}'\mathbf{w} = [\mathbf{A}\ -\mathbf{A}\ \mathbf{I}] \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{y} \end{pmatrix} = \mathbf{Ax}^+ - \mathbf{Ax}^- + \mathbf{Iy} = \mathbf{A}\left(\mathbf{x}^+ - \mathbf{x}^-\right) + \mathbf{y} = \mathbf{Ax} + \mathbf{y} = \mathbf{b}.$$

For the other direction, assuming that for $\mathbf{w} \geq \mathbf{0}$ it is true that $\mathbf{A}'\mathbf{w} = \mathbf{b}$, we get

$$\mathbf{b} = \mathbf{A}'\mathbf{w} = \mathbf{Ax}^+ - \mathbf{Ax}^- + \mathbf{Iy} = \mathbf{Ax} + \mathbf{y},$$

and because of the nonnegativity of $\mathbf{y}$ it follows that $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{x}$ is unrestricted. Thus the system $\mathbf{A}'\mathbf{w} = \mathbf{b}'$ constructed this way has a nonnegative solution if and only if the system $\mathbf{Ax} \leq \mathbf{b}$ has a solution.

$\Pi_2 \propto \Pi_1$: Let now $I_2 = (\mathbf{A}, \mathbf{b})$ be input for problem Π_2. We construct input I_1 for Π_1 to be $I_1 = (\mathbf{A}', \mathbf{b}') = \left(\left[\mathbf{A^T}\ -\mathbf{A^T}\ -\mathbf{I}\right]^{\mathbf{T}}, \left[\mathbf{b^T}\ -\mathbf{b^T}\ \mathbf{0}\right]^{\mathbf{T}}\right)$. Clearly, constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ are equivalent to constraints $\mathbf{Ax} \leq \mathbf{b}$, $-\mathbf{Ax} \leq -\mathbf{b}$, $-\mathbf{x} \leq \mathbf{0}$, so equivalence of constraints $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{A}'\mathbf{x} \leq \mathbf{b}'$ follows immediately.
Existence of polynomial reductions in both directions implies the claimed polynomial equivalence of problems Π_1 and Π_2. □

Let S be a subset of the space $\mathbb{R}^n$. An element $\mathbf{x} \in S$ is a *convex combination* of elements $\mathbf{x_1}, \ldots, \mathbf{x_d} \in S$ if it can be written in the form $\mathbf{x} = \alpha_1\mathbf{x_1} + \cdots + \alpha_d\mathbf{x_d}$, where $\sum_{i=1}^{d} \alpha_i = 1$ and $\alpha_i \geq 0$ for every $i = 1, \ldots, d$. If condition $\sum_{i=1}^{d} \alpha_i = 1$ is omitted, $\mathbf{x}$ is said to be a *conical combination* of elements $\mathbf{x_1}, \ldots, \mathbf{x_d}$. We say that a set S is *convex* if it contains all convex combinations of its elements. Similarly, if a set S contains all conical combinations of elements in S, then S is called a *(convex) cone. A convex hull* of a set S is the smallest convex set containing S, while a *cone generated by* S is the smallest cone containing S.

A *polyhedron* $P \subseteq \mathbb{R}^n$ is a set that can be characterised as $\{\mathbf{x} \mid \mathbf{Ax} \leq \mathbf{b}\}$ for some matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$. A *hyperplane* and a *linear halfspace* are sets defined as $\{\mathbf{x} \mid \mathbf{a^T x} = \delta\}$ and $\{\mathbf{x} \mid \mathbf{a^T x} \leq \delta\}$, where $\mathbf{a}$ is a nonzero vector and δ a scalar. A point $\mathbf{x} \in P$ is said to be an *extreme point* of P if for any two points $\mathbf{x_1}, \mathbf{x_2} \in P$ and scalar $\lambda \in (0,1)$ equality $\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2} = \mathbf{x}$ implies that $\mathbf{x} = \mathbf{x_1} = \mathbf{x_2}$.

Observe that there is a connection between the definition of a polyhedron and the feasible region of a linear program. Based on that, a linear program can be represented graphically, where the optimization of linear function is equivalent to looking for a point representing the intersection of a hyperplane corresponding to the objective function, moved in the direction of its normal vector as much as possible, and of a polyhedron P [52]. During the development of approaches for solving linear programming problems some important results concerning optimal solutions were derived. One of these results says that the optimal value of a linear program in standard form, if it exists, is achieved by some extreme point of P where P is the polyhedron representing the set of feasible solutions of a linear programm. This statement is proved within the proof of correctness of simplex algorithm for solving linear programs (see [14]).

In the rest of the thesis, we will denote by $\mathbf{I}$ and $\mathbf{0}$ the identity matrix and the zero vector of dimensions that will be clear from the context, respectively. In order to derive results concerning feasibility of a linear program, we introduce the following theorem.

Theorem 2.9 (see, e.g., Schrijver [51]). *Let* $\mathbf{a_1}, \ldots \mathbf{a_m}, \mathbf{b}$ *be vectors in* $\mathbb{R}^n$. *Then either* $\mathbf{b}$ *belongs to the convex cone generated by vectors* $\mathbf{a_1}, \ldots, \mathbf{a_m}$*, or there exists a hyperplane* $\{\mathbf{x} \mid \mathbf{c^T x} = 0\}$*, such that* $\mathbf{c^T b} < 0$ *and* $\mathbf{c^T a_i} \geq 0$ *for* $i = 1, \ldots, m$.

Theorem 2.9 states that, if $\mathbf{b}$ is not a nonnegative linear combination of vectors $\mathbf{a_i}$, then there is a hyperplane that separates $\mathbf{b}$ from all vectors $\mathbf{a_i}$, $i = 1, \ldots, m$. A proof of Theorem 2.9, often called the “Fundamental theorem of linear inequalities”, can be found in Schrijver's comprehensive monograph [51]. Using Theorem 2.9 we can prove a basic result concerning feasibility of the linear program, known as Farkas' lemma.

Theorem 2.10 (Farkas' lemma [22]). *Given a matrix* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and a vector* $\mathbf{b} \in \mathbb{R}^m$*, there exists a vector* $\mathbf{x} \in \mathbb{R}^n$ *that satisfies* $\mathbf{Ax} \leq \mathbf{b}$ *if and only if for each nonnegative vector* $\mathbf{y} \in \mathbb{R}^m$ *with* $\mathbf{A^T y} = \mathbf{0}$ *we have* $\mathbf{b^T y} \geq 0$.

Proof. ($\Rightarrow$) Suppose that there exists a vector $\mathbf{x}$ such that $\mathbf{Ax} \leq \mathbf{b}$. Then, for every $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{A^T y} = \mathbf{0}$, we have $\mathbf{b^T} - (\mathbf{Ax})^{\mathbf{T}} \geq \mathbf{0}$. Multiplying the last inequality by nonnegative vector $\mathbf{y}$ gives us the desired result $\mathbf{b^T y} \geq \mathbf{x^T A^T y} = \mathbf{x^T 0} = 0$.
($\Leftarrow$) Let $\mathbf{b^T y} \geq \mathbf{0}$ whenever $\mathbf{y}$ is a nonnegative vector such that $\mathbf{A^T y} = \mathbf{0}$. Assume for a contradiction contrary that the system $\mathbf{Ax} \leq \mathbf{b}$ does not have a feasible solution. From the proof of Lemma 2.8 it follows that the system $\mathbf{W\bar{x}} = \mathbf{b}$, where $\mathbf{W} = [\mathbf{A} \; -\mathbf{A} \; \mathbf{I}]$ and $\mathbf{\bar{x}}$ is a vector of variables, has no nonnegative solution. If the columns of $\mathbf{W}$ are denoted by $\mathbf{w_1}, \ldots \mathbf{w_{2n+m}}$, then $\mathbf{b}$ cannot be a nonnegative linear combination of them, thus $\mathbf{b}$ is not an element of the cone generated by vectors $\mathbf{w_1}, \ldots, \mathbf{w_{2m+n}}$ (remember that the first n of them are exactly the columns of matrix $\mathbf{A}$). From Theorem 2.9 it follows that there is a hyperplane $\{\mathbf{x} \mid \mathbf{y^T x} = 0\}$ such that $\mathbf{b^T y} < 0$ and $\mathbf{y^T w_i} \geq 0$, $i = 1, \ldots, 2n + m$. A set of inequalities $\mathbf{y^T w_i} \geq 0$, $i = 1, \ldots, 2n + m$ is equivalent to the inequality $\mathbf{y^T W} \geq \mathbf{0}$. If we apply the definition of the matrix $\mathbf{W}$, we get the inequality $\mathbf{y^T}[\mathbf{A} \; - \mathbf{A} \; \mathbf{I}] \geq \mathbf{0}$, or equivalently:

$$\mathbf{y^T A} \geq \mathbf{0}, \; -\mathbf{y^T A} \geq \mathbf{0}, \; \mathbf{y^T} \geq \mathbf{0}.$$

Clearly, it is true that $\mathbf{y^T A} = \mathbf{0}$. Since $\mathbf{y}$ is a nonnegative vector and $\mathbf{b^T y} \leq 0$, we have a contradiction, so the statement is proved. □

Example 2.11. Suppose we have a following linear program:

$$\begin{array}{rrcrcr} \min & x_1 & + & x_2 & & \\ \text{subject to} & -x_1 & + & 2x_2 & \leq & 0 \\ & 2x_1 & - & x_2 & \leq & 2 \\ & & - & x_2 & \leq & -2. \end{array}$$

In this LP we have

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \\ 0 & -1 \end{pmatrix}.$$

If we take $\mathbf{y} = (2, 1, 3)^T$, what we get is $\mathbf{A^T y} = \mathbf{0}$ and $\mathbf{b^T y} = -4 < 0$. From Theorem 2.10 it follows that the system $\mathbf{Ax} \leq \mathbf{b}$ is infeasible. This means that there is a hyperplane H defined by positive normal vector $\mathbf{y}$ that contains vectors defined as columns of $\mathbf{A}$. Moreover, it holds that vectors $\mathbf{y}$ and $\mathbf{b}$ belong to distinct half-spaces with respect to hyperplane H. Since our example is in space of dimension three, the hyperplane is of dimension two, that is, it is a plane. In Figure 2 we can see the plane H generated by normal vector $\mathbf{y} = (2, 1, 3)^T$ and containing vectors $\mathbf{a_1}, \mathbf{a_2}$. As expected, vectors $\mathbf{y}$ and $\mathbf{b}$ belong to the distinct halfspaces with respect to the hyperplane H.

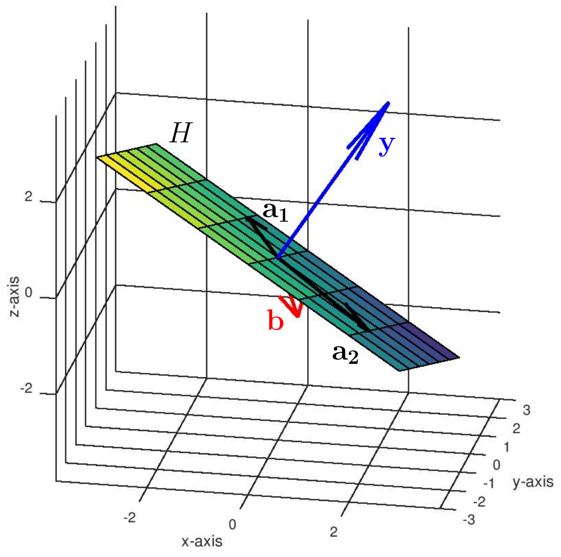


Figure 2: A plane through the origin with normal vector $\mathbf{y}$ (blue) containing vectors $\mathbf{a_1}, \mathbf{a_2}$ (black) and separating vector $\mathbf{b}$ (red) from vector $\mathbf{y}$.

Some of the most important results concerning linear programming consider pairs of dual programs, and are known in literature as the Duality Theory of Linear Programming.

Definition 2.12. Given a linear program

$$\begin{array}{lrcl} \text{minimize} & \mathbf{c^T x} & & \\ \text{subject to} & \mathbf{Ax} & \leq & \mathbf{b}, \end{array} \tag{2.2}$$

its *dual linear program* is defined as

$$\begin{array}{lrcl} \text{maximize} & \mathbf{b^T y} & & \\ \text{subject to} & \mathbf{A^T y} & = & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array} \tag{2.3}$$

The initial linear program is said to be *primal*.

In particular, for every row $\mathbf{a_i^T}$ of matrix $\mathbf{A}$ there is one dual variable y_i. All restrictions on variable y_i are derived from the corresponding constraint (see, e.g., [43]) and such dependencies between dual variables and corresponding row constraints are, for the general case, presented in Table 1.

Constraint in the primal	$\mathbf{a_i^T x} \leq b_i$	$\mathbf{a_i^T x} = b_i$	$\mathbf{a_i^T x} \geq b_i$
Constraint on the dual variable	$y_i \geq 0$	y_i unrestricted	$y_i \leq 0$

Table 1: Restrictions of dual variables.

Observe that the dual problem itself is a linear program, so it makes sense to think about the dual of the dual problem. In the following theorem we characterize the program obtained by applying two sequential duality operations.

Theorem 2.13. *A primal LP problem is polynomially equivalent to the dual of its dual.*

Proof. Suppose we are given a pair of primal and dual LP problems, as in (2.2) and (2.3), respectively. The dual problem can be equivalently written in the following form:

$$\begin{array}{rrcl} -\min & -\mathbf{b^T y} & & \\ \text{subject to} & \mathbf{A^T y} & \leq & \mathbf{c} \\ & -\mathbf{A^T y} & \leq & -\mathbf{c} \\ & -\mathbf{y} & \leq & \mathbf{0}. \end{array}$$

The dual of the above problem can be derived using Definition 2.12 and is equal to:

$$\begin{array}{rlcl} -\max & \left[\mathbf{c^T}\ -\mathbf{c^T}\ \mathbf{0^T}\right] \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \\ \mathbf{z_3} \end{bmatrix} & & \\ \text{subject to} & [\mathbf{A}\ -\mathbf{A}\ -\mathbf{I}] \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \\ \mathbf{z_3} \end{bmatrix} & = & \mathbf{b} \\ & \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3} & \geq & \mathbf{0}. \end{array} \tag{2.4}$$

The above problem represents the dual of the dual of the primal problem. Note that with minor changes it can be written in the equivalent form of inequalities with no nonnegativity constraints:

$$\begin{array}{rrcl} -\max & \mathbf{c^T}(\mathbf{z_1} - \mathbf{z_2}) & & \\ \text{subject to} & \mathbf{A}(\mathbf{z_1} - \mathbf{z_2}) - \mathbf{z_3} & = & -\mathbf{b} \\ & \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3} & \geq & \mathbf{0}. \end{array} \tag{2.5}$$

Note that it holds that $-\max \mathbf{c^T}(\mathbf{z_1} - \mathbf{z_2}) = \min -\mathbf{c^T}(\mathbf{z_1} - \mathbf{z_2}) = \min \mathbf{c^T}(\mathbf{z_2} - \mathbf{z_1})$. Now we get equivalent form of the problem (2.5):

$$\begin{array}{rrcl} \min & \mathbf{c^T}(\mathbf{z_2} - \mathbf{z_1}) & & \\ \text{subject to} & \mathbf{A}(\mathbf{z_2} - \mathbf{z_1}) + \mathbf{z_3} & = & \mathbf{b} \\ & \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3} & \geq & \mathbf{0}. \end{array} \tag{2.6}$$

Setting the $\mathbf{x} = \mathbf{z_2} - \mathbf{z_1}$ and using the proof of Lemma 2.8 we get that the above problem is equivalent to the initial primal problem, that is $\min\{\mathbf{c^T x} \mid \mathbf{Ax} \leq \mathbf{b}\}$. □

By Theorem 2.13, the duality operators is in some sense an involution. This explains its name.

Theorem 2.14 (The LP Duality Theorem)**.** *If a linear program has a finite optimal value, then so does its dual and the optimal values are equal. If the problem is unbounded, then its dual is infeasible.*

Proof. Let the primal LP problem be given in standard form, $\min\{\mathbf{c^T x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Its dual is $\max\{\mathbf{b^T y} \mid \mathbf{A^T y} \leq \mathbf{c}\}$. For every pair of feasible solutions $\mathbf{x}, \mathbf{y}$ of the primal and dual problems, respectively, we have

$$\mathbf{c^T x} \geq \mathbf{y^T A x} = \mathbf{y^T b} \tag{2.7}$$

so the optimal value of the dual problem is bounded from above by the optimal value of the primal problem. We construct a new LP problem consisting of the combined constraints of the primal and dual problems and requiring additionally that $\mathbf{c^T x} \leq \mathbf{b^T y}$. This LP problem is feasible if and only if equality in the inequality (2.7) is reached. The objective function of the new LP has no influence to the existence of a feasible solution (it could be a constant), so here we only list the constraints. Vectors $\mathbf{x}$ and $\mathbf{y}$ represent vectors of variables. The constraints are:

$$\begin{array}{rcl} \mathbf{Ax} & = & \mathbf{b} \\ \mathbf{A^T y} & \leq & \mathbf{c} \\ \mathbf{c^T x} & \leq & \mathbf{b^T y} \\ -\mathbf{x} & \leq & \mathbf{0}. \end{array}$$

Putting these constraints into matrix form gives

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A^T} \\ -\mathbf{I} & \mathbf{0} \\ \mathbf{c^T} & -\mathbf{b^T} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{c} \\ \mathbf{0} \\ 0 \end{bmatrix}. \tag{2.8}$$

From Theorem 2.10 it follows that a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ satisfying the above system of inequalities exist if and only if for an arbitrary nonnegative vector $\mathbf{z}$ equality $\mathbf{\bar{A}^T z} = \mathbf{0}$ implies $\mathbf{\bar{b}^T z} \geq 0$, where $\mathbf{\bar{A}\bar{x}} \leq \mathbf{\bar{b}}$ represents the system (2.8).
Let $\mathbf{z} = \left(\mathbf{s^T}, \mathbf{t^T}, \mathbf{u^T}, \mathbf{v^T}, w\right)^T$ be a nonnegative vector, where $\mathbf{s}, \mathbf{t} \in \mathbb{R}^m$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, w \in \mathbb{R}$. Based on the above corollary, we have to prove the following statement: if $\mathbf{Au} - w\mathbf{b} = 0$ and $\mathbf{A^T s} - \mathbf{A^T t} + \mathbf{v} + w\mathbf{c} = \mathbf{0}$, then $\mathbf{b^T s} - \mathbf{b^T t} + \mathbf{c^T u} \geq 0$. We consider two cases.
First, let $w > 0$. Then

$$\mathbf{c^T u} = w^{-1}\left(w\mathbf{c^T}\right)\mathbf{u} = w^{-1}\left(\mathbf{v^T} - \mathbf{s^T A} + \mathbf{t^T A}\right)\mathbf{u} = w^{-1}\mathbf{v^T u} - w^{-1}(\mathbf{s} - \mathbf{t})^{\mathbf{T}}\mathbf{Au} =$$
$$= w^{-1}\mathbf{v^T u} - w^{-1}(\mathbf{s} - \mathbf{t})^{\mathbf{T}} w\mathbf{b} = w^{-1}\mathbf{v^T u} - \mathbf{b^T}(\mathbf{s} - \mathbf{t}).$$

Thus, $\mathbf{c^T u} + \mathbf{b^T}(\mathbf{s}-\mathbf{t}) = w^{-1}\mathbf{v^T u} \geq \mathbf{0}$. All equalities follow directly from the assumptions.
If $w = 0$, the assumptions becomes equal to $\mathbf{Au} = \mathbf{0}$ and $\mathbf{A^T s} - \mathbf{A^T t} - \mathbf{v} = \mathbf{0}$. Suppose that $\mathbf{x_0}$ and $\mathbf{y_0}$ are feasible solutions of the primal and the dual problem, respectively: $\mathbf{Ax_0} = \mathbf{b}$, $\mathbf{x_0} \geq \mathbf{0}$, $\mathbf{A^T y_0} \leq \mathbf{c}$. It follows that $\mathbf{c^T u} \geq \mathbf{y_0^T A u} = \mathbf{0}$ and, on the other hand, $\mathbf{x_0^T A^T}(\mathbf{s}-\mathbf{t}) = \mathbf{b^T}(\mathbf{s}-\mathbf{t})$. These two expressions together give the desired result:

$$\mathbf{b^T}(\mathbf{s}-\mathbf{t}) + \mathbf{c^T u} = \mathbf{x_0^T A^T}(\mathbf{s}-\mathbf{t}) + \mathbf{c^T u} \geq \mathbf{x_0^T v} + \mathbf{y_0^T A u} = \mathbf{x_0 v} \geq \mathbf{0}.$$

If the dual problem is feasible, then the objective function value $\mathbf{b^T y_0}$ where $\mathbf{y_0}$ is an arbitrary feasible solution (of the dual), gives a lower bound for the optimal value of the primal. Thus, if the dual is feasible, then the primal cannon be unbounded. This proves the last statement of the theorem. □

In general, Linear Programming is an area with a lot of theoretical results that is also successfully used in applications. One of the main tools for solving LP problems is the simplex method, developed by Dantzig in 1947 [14]. Algorithms based on the simplex method are very efficient from the practical point of view, although the same does not hold for their theoretical aspects. Recall that if an optimal solution of an LP in standard form exists, then there is always an extreme point $\tilde{\mathbf{x}}$ of the corresponding polyhedron so that the objective function value at $\tilde{\mathbf{x}}$ is optimal. The simplex method is based on that result. The algorithm runs over vertices of polyhedron P and tries to find a vertex with minimal objective value. Starting with some vertex, the simplex method constructs a path of consecutively adjacent vertices of P, until finding an optimal vertex. The order in which the vertices are visited is crucial for the performance of simplex method, since it is responsible to prevent cycling in a set of vertices of equal objective function value, and may enhance the speed of the search.

A step of the simplex algorithm where a way of choosing the next visited vertex is defined is referred to in the literature as a *pivoting step* and any method describing the pivoting steps is called a *pivoting rule*. The most known pivoting rules are: Dantzig's rule (also called the nonbasic gradient method), which was part of the original definition of simplex method due to Dantzig [14], Bland's pivoting rule [6], which is the most used rule in the description of simplex method in literature, the steepest edge rule, defined by Dickson and Frederick [19], Borgwardts' pivoting rule [7], etc. A common property of these rules is that none of them can guarantee to find an optimal solution in polynomial time in the worst case. It is not known if a pivoting rule for which the simplex method would work in polynomial time can be constructed, but for known pivoting rules this is not the case [33]. Feasible regions of known instances of LP problems on which the simplex method runs exponentially long are deformations of the n-dimensional cube. However, a simplex method gives good results, since it is proved to find an optimal solution in polynomial time on average. In particular, for the improved Borgwardt's pivoting rule it is proved that the average number of pivot steps is linear in the size of input data (see, e.g., [51]).

Observe that nonpolynomial running time of the simplex method does not imply nonpolynomial time complexity of linear programming. In fact, there are other methods for solving LP problems, which were proved to be polynomial, even if they are not so efficient in practical use. One of them is the *ellipsoid method.* The ellipsoid method was first developed for minimization of convex function and then applied to minimization of a linear function over a polyhedron. Later it was used by Khachiyan to prove polynomial-time solvability of LP [32]. The method is based on iterative construction of n-dimensional ellipsoids, with strictly decreasing volumes in each step of the iteration. Each ellipsoid contains a polyhedron P that represents a set of feasible solutions of a problem. Thus, if in the some step of iteration we get the sufficiently small ellipsoid, we conclude that there corresponding problem is infeasible. If E_1 is the initial ellipsoid, containing polyhedron P, with center in $\mathbf{x_1}$, the method checks if $\mathbf{x_1}$ is a feasible solution. If yes, the method stops and outputs the feasible solution $\mathbf{x_1}$. Otherwise, the method removes a set S of points, which do not satisfy some of the constraints defining P. A new, smaller ellipsoid containing the set $E_1 \setminus S$ is constructed. Clearly, removed points are not contained in the set P, so $P \subseteq E_1$. If at some step of iteration we get the ellipsoid containg no points, we can conclude that the set P is empty and the problem is infeasible. A detailed description of the ellipsoid method is beyond the scope of this work and can be found e.g., in [51]. The method is polynomial in the worst case and its running time does not depend on the number of rows of input data (equivalently: it does not depend on the number of constraints of the LP), just on coefficient values and number of variables.

Thus, the method is mainly used for finding the feasible solution of the given LP problem, if one exists. If there is a finite optimal solution, then by The LP Duality Theorem (Theorem 2.14), there also exists an optimum of a dual program. In that case the method can be used to find a feasible solution of the LP defined as in equation (2.8), since it corresponds to an optimal solution of the initial LP problem.

While the ellipsoid method has important theoretical meaning, in practical use it is not efficient. In higher dimensions it is not numerically stable even for problems of small size. Unlike the ellipsoid method, the method referred to as the interior point method, developed by Karmarkar [30], is good at both aspects: in theory it solves LP problems in polynomial time and it is also efficiently used in practice (see, e.g., [5]).

2.3 Integer Linear Programming

Integer linear programming (abbreviated ILP) is an extension of linear programming and a useful tool for solving *NP*-hard combinatorial optimization problems. Indeed, many such problems ask for either the existence of a subset of a given set satisfying certain requirements (e.g., a Hamiltonian cycle in a graph) or for a subset that optimizes a given linear weight function (e.g., a maximum weight independent set in a graph). This naturally leads to the introduction of binary variables, one for each element of the ground set, which model the choice of a subset of the set. Additional properties that every feasible solution should possess are modelled with linear constraints.

ILP can be defined in a few equivalent ways. One of the most commonly used definitions is the following.

Definition 2.15. An *integer linear program* is the following optimization problem:

$$\begin{array}{rrcl} \text{minimize} & \mathbf{c^T x} & & \\ \text{subject to} & \mathbf{Ax} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \\ & \mathbf{x} & \in & \mathbb{Z}^n, \end{array} \tag{2.9}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{x}$ is a vector of integer variables, $\mathbf{x} = (x_1, \dots x_n)$.

The parameters $\mathbf{A}, \mathbf{b}, \mathbf{c}$ are usually assumed to have integer components. Note that equivalent definitions of ILP can be obtained either by putting the equalities instead of inequalities or by discarding a nonnegativity constraint. The *LP relaxation* of the above ILP problem is the LP problem obtained from it by dropping the integrality constraint, that is, $\min\{\mathbf{c^T x} \mid \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Clearly, the optimal value of the LP relaxation of given ILP problem is at least as good as the optimal value of the ILP problem. Let Π_1 denote the ILP maximization problem, let Π_{LP_1} denote its LP relaxation and let Π_{LP_2} be the dual LP of the Π_{LP_1}. Then, if Π_2 denotes the ILP obtained by adding integrality constraint to all variables of Π_{LP_2}, using the LP Duality Theorem (Theorem 2.14) we get:

$$\max(\Pi_1) \leq \max(\Pi_{LP_1}) = \min(\Pi_{LP_2}) \leq \min(\Pi_2), \tag{2.10}$$

provided Π_{LP_1} has a finite optimal value. The inequalities in (2.10) can be strict, that is, equality of LP relaxed optima for the primal and dual problems does not imply equality in ILP version. Thus, for integer linear programming problems there is no duality theorem analogous to Theorem 2.14. ILP seems to be more difficult than LP. In order to make this statement more precise, let us define the decision version of the problem.

INTEGER LINEAR PROGRAMMING
Instance: A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, a vector $\mathbf{b} \in \mathbb{R}^m$.
Question: Does the inequality $\mathbf{Ax} \leq \mathbf{b}$ have an integral solution $\mathbf{x}$?

The intuition is confirmed by the following theorem.

Theorem 2.16. INTEGER LINEAR PROGRAMMING *is NP-complete.*

INTEGER LINEAR PROGRAMMING represents one of the 21 Karp's *NP*-complete problems [31]. Since it is one of the basic *NP*-complete problems, few distinct ideas of proofs of its *NP*-completeness can be found in the literature. Here we refer the reader to the most basic one, based on a polynomial reduction from SATISFIABILITY, which can be found in [51]. In a geometric interpretation, an ILP problem represents optimizating a given linear function over the set of integer lattice points in a given polyhedron. Let us define the integer hull P' of a polyhedron P to be the smallest convex set containing the integral vectors of P. Obviously, $P' \subseteq P$ and an integer linear program over P is equivalent to the corresponding linear program over P'. A polyhedron for which $P = P'$ is called an *integral polyhedron.* Thus an integer linear programming problem over an integral polyhedron P is equivalent to its LP relaxation and so can be solved in polynomial time.

One more result considering a polynomial instance of integer linear programming problem is known as *Lenstra's theorem* (see [37]). In 1983 Lenstra gave an algorithm that solves integer linear programming problem with fixed number of variables in time bounded by a polynomial function of the input. It means that integer linear programming problems for which number n of variables does not depend on the input can be solved in polynomial time. The degree of the polynomial is an exponential function of the number of variables, that is, of n.

A variant of ILP where only a subset of the set of variables is restricted to have integer values is called *mixed integer linear programming* (MILP). The presence of variables that are supposed to have integer value makes MILP problems *NP*-complete. Since the theoretical aspects of MILP and ILP are rather similar, in the following we focus on ILP. Although ILP problems in general are very difficult to solve, several methods were developed that can be helpful for solving such problems.

Cutting-plane methods

A cutting plane method is an iterative method based on successive reductions of the feasible region until an optimal integer solution is found. This method can be used for convex continuous optimization, as well as for nonlinear optimization problems. Suppose that we have an ILP minimization problem, with polyhedron P representing the feasible region of its LP relaxation. A cutting plane method starts by finding an optimum of the LP relaxation, say $\bar{\mathbf{x}}$. We assume that $\bar{\mathbf{x}}$ is a vertex of polyhedron P. The method checks if $\bar{\mathbf{x}}$ is integral and if this is the case, it terminates. If this is not the case, then there exists some hyperplane $H = \{\mathbf{x} \mid \mathbf{a^T x} = \delta\}$ that separates $\bar{x}$ from all the integer feasible solutions in P. Algebraically, this means that the inequality $\mathbf{a^T x} \leq \delta$ is satisfied by all the integral vectors in P, but not by the vector $\bar{\mathbf{x}}$.

A hyperplane H as above is called a *cut*, and is added to the list of hyperplanes defining P, so that in the next iteration the feasible region is reduced with respect to the new hyperplane. A cut actually represents an inequality that becomes one more constraint of the initial ILP problem, called a *cut constraint*. The process of finding such an inequality is called the separation problem. The choice of a new constraint can have a big influence on the result computed by the method and on the time it takes to solve the initial ILP problem.

Observe that no integer feasible solution is removed from the feasible region during the execution of the method. The first method with a solution to the separation problem was the Gomory cutting plane method [26]. The method starts with an ILP problem, and solves its LP relaxation using the simplex method. If the solution is not integral, the method uses one of the parameters, which are the result of the simlex method (a row in the simplex tableau, see e.g., [43]). A new constraint defined that way is satisfied by all integer feasible solutions, and is violated by the computed optimal solution of the LP relaxation.

Example 2.17. Consider the LP problem from Example 2.6. Suppose we would like to find an optimal solution of the corresponding ILP problem. In Figure 3 we can see the feasible region and the optimal solution of the LP problem. We can reduce the feasible region by adding a cut constraint. One of the possible choices for that is constraint $5x_1 + x_2 \leq 25$, drawn with red line. Initial feasible solution (point A) violates the cut constraint, while the constraint is satisfied for any of the integer points in feasible region.

The branch-and-bound method

The branch-and-bound method divides an ILP problem into smaller subproblems and solves each of them separately. Suppose that the ILP problem to be solved is

$$\min\{\mathbf{c^T x} \mid \mathbf{Ax} \leq \mathbf{b}, \mathbf{x}\,\text{integral}\},$$

with feasible region defined by polyhedron P. The method solves the LP relaxation and gets an optimal solution $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is integral, the method terminates. Otherwise, it

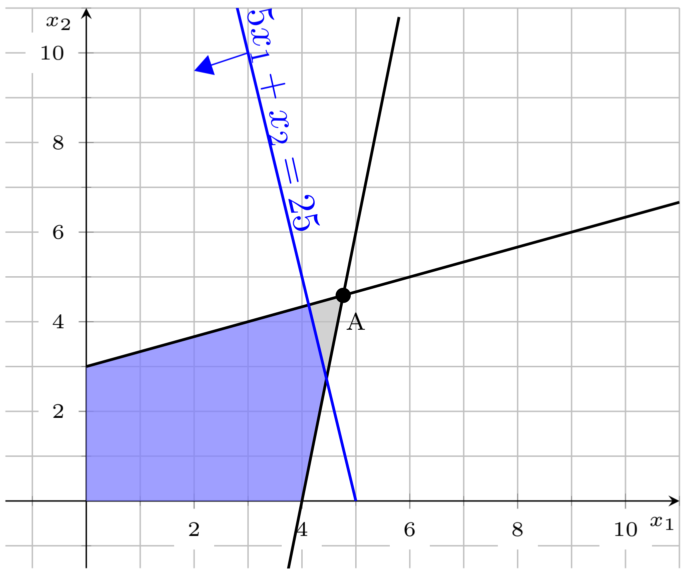


Figure 3: Feasible region, optimal solution of LP relaxed version of problem (point A) and a cut constraint (blue line) of Example 2.17.

takes a nonintegral component of $\bar{\mathbf{x}}$, say $\bar{x}_i$, and separates the initial problem into two subproblems:

- $\Pi_1: \quad \min\{\mathbf{c}^\mathbf{T}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}, x_i \leq \lfloor\delta\rfloor\}$,
- $\Pi_2: \quad \min\{\mathbf{c}^\mathbf{T}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}, x_i \geq \lceil\delta\rceil\}$,

where δ is the value of component $\bar{x}_i$. Observe that the feasible regions of these two problems are disjoint and that all integral vectors from P are contained in their union. The method constructs a tree, with root of the tree corresponding to the initial problem and branches corresponding to the problems based on new, separated polyhedra. The leaves of the tree are either infeasible problems or problems having optimal integral solutions. The best one among optimal solutions on all the leaves is an optimal solution of the initial ILP problem.

In this method the main step is breaking a problem into subproblems, that is, defining which variable the method will branch upon. Once the tree of subproblems is constructed, it should be searched for an optimal solution. If some node, say A, gives us an integral optimal value better than the optimal value of some other node, say B (optimal solution at B is not necessarily integral), we cut the node B, as well as its branches, since in the subtree corresponding to the node B for sure an optimal solution cannot be found. The objective function value of subproblems increases (resp. decreases) for minimization (resp. maximization) problems, with every step going deeper within the tree.

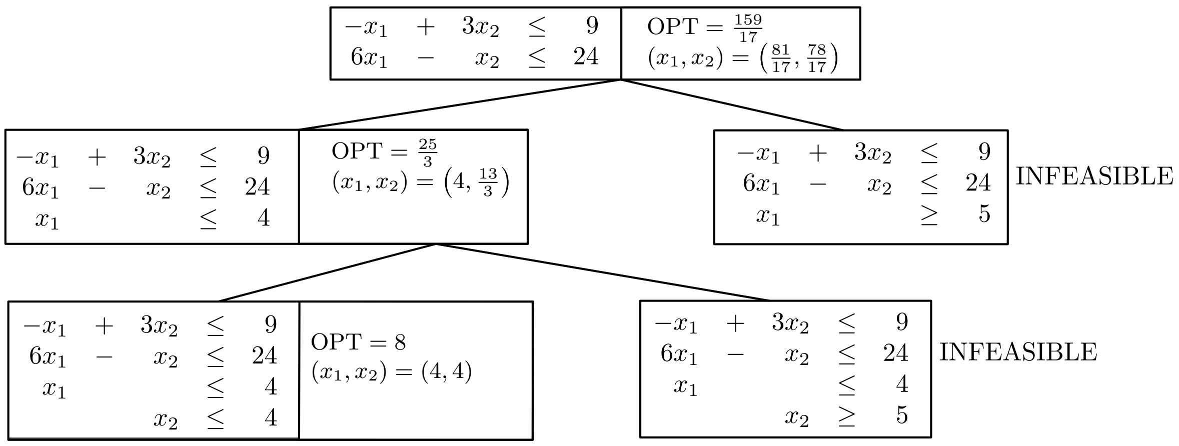


Figure 4: A tree representing branch-and-bound nodes.

Example 2.18. Consider again the problem from Example 2.6. We use the branch-and-bound method in order to find its optimal integer solution. The unique optimal solution of the relaxed problem is $(x_1, x_2) = (\frac{81}{17}, \frac{78}{17})$, with objective function value $\frac{159}{17}$. Since both variables are non integer, we can choose any one of them for the branching step. We choose x_1. Then we get two new LP problems, similar to the initial ones, with added constraints $x_1 \leq 4$ and $x_1 \geq 5$, respectively. An optimal solution of the first one is $(x_1, x_2) = (4, \frac{13}{3})$ with objective function value $\frac{25}{3}$, while the LP on the other branch is infeasible. We continue with branching on variable x_2, since it is non-integral. The added constraints are $x_2 \leq 4$ and $x_2 \geq 5$, respectively. Finally, we get an optimal solution that is integral, $(x_1, x_2) = (4, 4)$, with objective function value equal to 8. The corresponding tree is displayed on Figure 4.

Note that the example displayed on Figure 4 is not large enough to show the "bound" part of the branch-and-bound method. For details regarding the branch-and-bound method we refer a reader to the paper due to Lawler and Wood, from 1966 [35].

Lagrangean relaxation

Lagrangean relaxation is a method used for determining an upper bound for a maximization ILP problem (or a lower bound for a minimization ILP problem), by separating the set of constraints into two disjoint sets. Let the ILP problem be given as

$$\min\{\mathbf{c^T x} \mid \mathbf{Ax} \leq \mathbf{0}, \mathbf{x}\,\text{integer}\}$$

and let the set of constraints $\mathbf{Ax} \leq \mathbf{b}$ separate into two disjoint sets $\mathbf{A_1x} \leq \mathbf{b_1}$ and $\mathbf{A_2x} \leq \mathbf{b_2}$. Assume that the above problem can be efficiently solved with respect to constraint $\mathbf{A_1x} \leq \mathbf{b_1}$, while adding constraint $\mathbf{A_2x} \leq \mathbf{b_2}$ raises the complexity of the problem. In that case we include the problematic set of constraints into the objective function, so that for violation of any constraint δ_i we add some nonnegative penalty coefficient α_i. If constraint $\mathbf{a_ix} \leq b_i$ is violated, then obviously $\mathbf{a_ix} - b_i > 0$. Consider the following ILP problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{c^Tx} + \boldsymbol{\alpha}^\mathbf{T}(\mathbf{A_2x} - \mathbf{b_2}) \\ \text{subject to} & \mathbf{A_1x} \leq \mathbf{b_1} \\ & \mathbf{x}\,\text{integer} \end{array}, \tag{2.11}$$

where $\boldsymbol{\alpha}$ is a positive vector. The above ILP problem is called the *Lagrangean relaxation* of the initial problem. It has a greater set of feasible solutions and gives a lower bound on the optimal value of the initial problem (see, e.g., [5]).

Theorem 2.19. *The optimal value of the Lagrangean relaxation is a lower bound for the optimal value of the initial ILP problem.*

Proof. Let $\bar{\mathbf{x}}$ be an optimal solution of the Lagrangean relaxation problem, and let $\tilde{\mathbf{x}}$ be an optimal solution of the initial problem. Since $\tilde{\mathbf{x}}$ is a feasible solution of the ILP problem, we have $\mathbf{A_2}\tilde{\mathbf{x}} - \mathbf{b_2} \leq \mathbf{0}$. Then

$$\mathbf{c^T}\tilde{\mathbf{x}} \geq \mathbf{c^T}\tilde{\mathbf{x}} + \boldsymbol{\alpha}(\mathbf{A_2}\tilde{\mathbf{x}} - \mathbf{b_2}) \geq \mathbf{c^T}\bar{\mathbf{x}} + \boldsymbol{\alpha}^\mathbf{T}(\mathbf{A_2}\bar{\mathbf{x}} - \mathbf{b_2}) \tag{2.12}$$

The first inequality holds since $\boldsymbol{\alpha} > \mathbf{0}$ and $\mathbf{A_2x} - \mathbf{b_2} \leq \mathbf{0}$ and the second one is a consequence of optimality of $\bar{\mathbf{x}}$ in the Lagrangean relaxation problem. □

If parameters α_i are defined as dual variables (recall from Section 2.2 that every row of matrix $\mathbf{A}$ has a corresponding dual variable), then Theorem 2.19 is referred to as the *Integer Programming Duality Theorem.* The problem of minimizing the gap between these two optimal solutions is called the *Lagrangean dual problem* [25]. This approach is very useful for problems with nice structure and efficiently computable feasible set $\mathcal{D} = \{\mathbf{x}\,\text{integral}, \mathbf{A_1x} \leq \mathbf{b}\}$.

Dynamic programming

Dynamic programming is a method that breaks a problem into a number of smaller subproblems and solves them sequentially. Optimal solutions calculated for subproblems are combined in order to get a solution for a larger problem. The method does not repeat the same calculations, but uses the previously computed solutions. A dynamic programming approach is used for construction of many efficient algorithms for various problems (such as the shortest path problem in acyclic digraphs [54], the maximum weight independent set problem in interval graphs [28], etc.). In integer programming problems dynamic programming is used mostly in the following way: instead of assigning values to all variables x_i at once, we pick up one variable at a time to be assigned some value. In each step we compute the value of the objective function with respect to an already known assignment from the previous steps. If we try to assign few distinct values to variable x_i, we choose one that optimizes the objective value. This approach cannot be simply used in case of general integer variables, it is more appropriate for binary variables, where there are just two possible assignments for each variable. Using dynamic programming for binary ILP problems, the complexity of the problems can be much improved, although it is still exponential, in the size of the coefficients of the problem [5].

2.3.1 Modelling with ILP

Let Π be a combinatorial optimization problem, that we would like to model using concepts of integer linear programming. To construct a well-defined model, we have to determine the set of variables and their domains, the constraints, and the objective function. For problems where variables model decisions about whether certain events happen or not, binary variables are often used. These have value 1 if the corresponding event happens, and 0 otherwise. In this thesis we model the timetabling problem, which consists of a set of events, so in the following we consider binary variables. The objective function is a linear function of the defined variables. The main part of ILP problems are constraints, or, more precisely, the polyhedron defining the feasible region. Let $x_1, \ldots, x_n$ be a set of binary variables and I some index set. Here we list some basic ideas, which are used for modelling our timetabling problem:

(I) At most δ events from set I happen: $\displaystyle\sum_{i \in I} x_i \leq \delta$.

(II) At least δ events from set I happen: $\displaystyle\sum_{i \in I} x_i \geq \delta$.

(III) Exactly δ events from set I happen: $\displaystyle\sum_{i \in I} x_i = \delta$.

(IV) In optimization problems, it often happens that some requirement consists of two constraints, which must not be violated at the same time, i.e., at least one of them should be satisfied. Let requirement P be represented by constraints $\mathbf{a^T x} \leq b$ and $\mathbf{c^T x} \leq d$, where at least one of them should be satisfied. In order to satisfy requirement P, we introduce a binary variable z_P, which is intended to have value 1 if the first equation of requirement P is satisfied, and 0 if the other one is satisfied. This will ensure that at least one of the constraints is satisfied.

We add the following constraints to initial problem:

$$\begin{aligned} \mathbf{a^T x} &\leq b + B_1 \cdot (1 - z_P)\,, \\ \mathbf{c^T x} &\leq d + B_2 \cdot z_P, \end{aligned} \tag{2.13}$$

where B_1, B_2 are properly selected constants. Observe that it can happen that both of inequalities are satisfied.

(V) Disjoint constraints can also be successfully modelled using concepts of ILP. Let requirement C consists of two constraints, which should not be satisfied at the same time, i.e. at least one of them should be violated. Suppose constraints are the same as above:

$$\mathbf{a^T x} \leq b \quad \text{and} \quad \mathbf{c^T x} \leq d. \tag{2.14}$$

Their complements are constraints $\mathbf{a^T x} > b$ and $\mathbf{c^T x} > d$, which in case of integer input data and integral variables are equivalent to constraints $\mathbf{a^T x} \geq b + 1$ and $\mathbf{c^T x} \geq d + 1$, respectively. Then the requirement that at least one of constraints (2.14) is violated is equivalent to requirement that at least one of their complements is satisfied. We can model it using the point (IV) on complement constraints.

3 Timetabling problems

In this chapter we introduce the timetabling problem in its general form, basic results about its computational complexity, as well as a literature overview, with description of the most common approaches used in modelling timetabling problems.

3.1 The Timetable Design Problem

One of the problems that can be modelled using methods of integer linear programming is the timetabling problem. The timetabling problem consists in assigning resources to a set of timeslots. It appears in many fields of everyday life, so depending of the application, resources could be workers and jobs, students and lectures, sports matches, etc. Since there are various definitions of timetabling problems, we introduce here one of the most basic variants, defined in monograph by Garey and Johnson [24, SS19].

TIMETABLE DESIGN

Instance: A set H of "work periods", a set C of "craftsmen", a set P of "projects", a subset $A(c) \subseteq H$ of "available hours" for each craftsman $c \in C$, a subset $A(p) \subseteq H$ of "available hours" for each project $p \in P$, a number $R(c,p) \in \mathbb{Z}_0^+$ of "required work periods", for all $c \in C$ and $p \in P$.

Question: Is there a timetable for completing all the projects, i.e., a function $f : C \times P \times H \to \{0,1\}$ (where $f(c,p,h) = 1$ means that craftsman c works on project p during period h) such that

1. $f(c,p,h) = 1$ only if $h \in A(c) \cap A(p)$,
2. for each $h \in H$ and $c \in C$ there is at most one $p \in P$ for which $f(c,p,h) = 1$,
3. for each $h \in H$ and $p \in P$ there is at most one $c \in C$ for which $f(c,p,h) = 1$,
4. for each pair $(c,p) \in C \times P$ there are exactly $R(c,p)$ values of h for which $f(c,p,h) = 1$?

This problem is known to be *NP*-complete. A proof, based on a polynomial reduction from SATISFIABILITY, can be found in the paper by Even et al. [21]. The problem remains *NP*-complete even if $R(c,p) \in \{0,1\}$ for all $c \in C$ and $p \in P$ [24]. The TIMETABLE DESIGN problem with this restriction will be referred to as the BINARY TIMETABLE DESIGN (BTD) problem.

A feasible solution of this problem is a timetable for completing all projects, or more precisely, a function $f : C \times P \times H \to \{0,1\}$ that satisfies all listed conditions. The problem can have additional constraints, which are desired to be satisfied, but do not necessarily have to hold. For that reason, constraints are divided into two groups: hard constraints and soft constraints. Hard constraints must be satisfied by a feasible timetable, while soft constraints represent requirements that are desirable to be satisfied, but their violation has no influence to the feasibility of the solution. Every soft constraint has a weight, and in case that the constraint is violated, the weight is added to the objective function. Therefore, an optimal solution has minimal value of the objective function and implicitly satisfies as many soft constraints as possible. For example, in the problem defined above, an additional constraint prohibiting assigning of craftsman c_i at timeslot t_j is a hard constraint, while preference of craftsman c_i to project p_j over project p_k is soft constraint desired to be satisfied, although not necessarily.

When speaking about a university timetabling problem, resources are defined as courses, students groups, and sometimes also rooms. The widely used definition for university course timetabling problem (UCT problem) defines it as scheduling a sequence of teaching sessions involving lecturers and students in a predetermined period of time, normally within a week, while satisfying a set of constraints [49]. Every institution has its own set of contraints that should be satisfied when constructing a timetable, so for that reason a general solution for university timetabling problems does not exist. Some institutions have all lectures of the same length, or timeslots of fixed duration where any course can be assigned to an arbitrary timeslot. Also, there are institutions that have a predetermined assignment of rooms to student groups. Many institutions have elective courses, as well as courses that are common to more than one student group, while for some institutions this is not the case. For some institutions collisions between elective subjects for students of same student group are forbidden, while for some others this is just desirable but not absolutely necessary. These constraints are generally the main reason for big complexity of timetabling problems [9]. Relaxing some of them simplifies a timetabling problem and implicitly generates a problem with smaller complexity. The influence of individual constraints to the complexity of the whole problem is researched in the literature, so few special types of constraints were identified as crucial for increased complexity of a timetabling problem.

On the computational complexity of problem variants

Various definitions of a university timetabling problem lead to variations in its computational complexity. A problem with fewer restrictions and with a small set of resources to be assigned appears simple in comparison with a more involved problem. However, it is not easy to define an exact boundary that would determine when a problem becomes difficult to solve.

In the literature there are studies that consider the problem with some set of constraints and then the same problem with the removal of some constraint. The analysis of the computational complexity of the corresponding problem can thus reveal whether a removed constraint has any influence on the hardness of the problem. One such analysis can be found in [39], where the author defined two versions of the problem and resolved their complexity status. It turned out that timetabling problems concerning just timeslots and courses, involving lecturers, students, and arbitrarily many arbitrarily large classrooms can be solved in polynomial time. A polynomial-time algorithm can be obtained using known algorithms for the maximum flow problem, for example particular the algorithm given due to Ford and Fulkerson [23] with implementation of Edmonds and Karp [20]. This theoretical result does not seem to have big implication for real-life timetabling problems, since almost all university timetabling problems considered in the literature are supposed to also assign courses to classrooms.

The variant of the problem not containing room assignment is similar to highschool timetabling problems, since each element of the resource set (student group, course, or lecturer) is supposed to have its own classroom. Another restriction of timetabling problem that increases the problem complexity, although it is not proved that its removal makes the problem solvable in polynomial time, is related to length of the lectures. There are examples of problems concerning lectures of the same length for which the number of steps needed for solving the problem is significantly reduced in comparison with the same problems having the lectures of distinct length [15]. Also, there are algorithms for solving the university timetabling problems that work in two phases. In the first one constraints requiring consecutive hours of some course are violated and are recovered during the second one. However, even if the length of the lectures is the same for all courses, such a problem is still *NP*-complete [39].

3.2 Literature review

Timetabling problems can be formulated in many different ways, depending on preferences of the institution. Therefore, various models are developed in the literature. Some constraints are common to many institutions, while some constraints are very specific. Regarding these specific constraints, authors often try to find some good enough approach and construct either a new model or further develop the existing one, in order to solve their own problem. Various approaches were used, including concepts from graph theory, or more precisely graph colouring, integer linear programming, metaheuristics, constraint programming, and neural networks.

Graph colouring

Given finite, simple, undirected graph $G = (V, E)$, a proper k-colouring of G is defined as a function $f : V \to \{1, 2, \dots, k\}$, such that for adjacent vertices u and v we have $f(u) \neq f(v)$. A correspondence between graph colourings and timetabling problems can be understood as follows: we construct a graph G where every vertex represents one of the events to be scheduled. If two events are in conflict, then in G there is an edge between corresponding vertices. Thus, every colour in a proper k-colouring of G represents some time interval and the event corresponding to vertex v can be scheduled at time corresponding to colour $f(v)$.

One of the first models for university timetabling problem based on graph theory was presented by Welsh and Powell in 1960s [55]. Graph colouring is a well known *NP*-hard problem and several heuristics for the problem are known (see, e.g., [8, 18, 29]). So if the graph G corresponding to a timetabling problem is sufficiently structured, some of heuristics can be used. There are also approaches that combine few distinct heuristics for graph colouring, in such a way that some new heuristic decides which of the heuristics will be used to colour the vertices of a graph. A description of such a method used for a timetabling problem is given by Burke et al. [11]. Recent graph colouring approaches also include assignment of classrooms in the model, so this can be understood as an advantage of this approach [47]. A disadvantage of the graph colouring approach is the difficulty of modelling lectures that do not all have the same length. With one vertex we cannot specify the duration of a lecture in a simple way, so for institutions where timeslots do not have predefined distinct lengths, it is difficult to model and evaluate timetabling problems using graph colourings.

Integer linear programming

In Section 2.3 we have seen principles of using integer linear programming for modelling a combinatorial optimization problem. As mentioned there, if the problem to be solved consists of some events, binary variables can be used. One of the first ideas for modelling a timetabling problem using integer linear programming was developed for a school timetable during the 1960s by Lawrie [36]. After that the number of papers presenting similar models grew rapidly. Models constructed in that time were too big to be solved in real time, since software developed for that purpose was not fast enough. The corresponding theoretically developed models were not useful in real-life applications, which is the main reason for stagnation of this approach during subsequent years. Nowadays, powerful software is available for solving integer linear programs, so ILP is again one of the main approaches for solving combinatorial optimization problems, as well as timetabling problems. There are various models in literature, depending on constraints and preferences of corresponding institutions. One of them is available in papers by Daskalaki et al. [16], and by Daskalaki and Birbas [15]), where a huge set of constraints is represented using linear inequalities. A nice summary of types of constraints used in the literature is given by Aizam and Caccetta (see [3]), while a good description of elements of objective function is available in a paper by Pereira and Costa (see [44]).

Since assignment of courses and rooms for timetabling problems is mostly done for a period of one week, minor changes in input data for some week may affect the whole timetable. A problem of finding an assignment $\tilde{\mathbf{x}}$ of a set of resources to the set of variables, such that differences between $\tilde{\mathbf{x}}$ and an existing initial assignment are minimized does not seem to be easily solvable using integer linear programming. One of the difficulties when using this approach is a big number of variables and constraints for larger instances of problem. Also, it is often not trivial to model some very specific constraint, which is not part of requirements at other institutions, and sometimes at all to understand the timetable from the obtained solution, which is typically a big-dimensional vector consisting of 0s and 1s.

Metaheuristics

Algorithms that find a solution not relying on any randomness are said to be deterministic, while algorithms that contain some random parameters are said to be stochastic. A topic of main interest in the stochastic approach are metaheuristic algorithms. Concretely, for timetabling problems the most used metaheuristics are genetic algorithms and local search methods such as simulated annealing and tabu search, which we briefly describe here (see, e.g., [4, 38]).

Genetic algorithms are based on evolution of some population, which in case of optimization problems represents the set of feasible solutions of the problem. In order to apply a genetic algorithm to some problem, several parameters have to be determined: the crossover operation, the mutation operation, the threshold for elimination, and the fitness function. Suppose that the initial population consists of distinct timetables, where each timetable is one individual, consisting of chromosomes. A gene is defined as a part of chromosome and can be represented using real numbers, binary values, or alphanumeric characters. In the literature, genes are mostly identified with timeslots

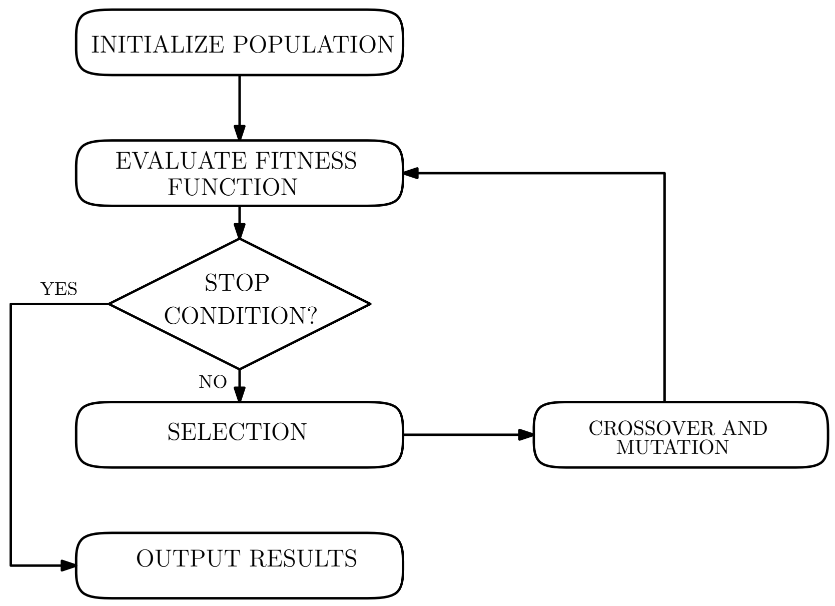


Figure 5: Overview of a genetic algorithm.

and are represented by alphanumeric characters [46]. The crossover operation is defined as a construction of new individuals from the existing ones. Two parent individuals, referred to as a mother and a father, obtain two new individuals so that randomly chosen chromosomes are derived from the father and others from the mother, and vice versa, for construction of one, respectively other child. A mutation operation is the operation of randomly replacing of one of the chromosomes by a new one, in order to get a better solution or to escape from a local optimum. An overview of the algorithm is presented in Figure 5.

Some of the models constructed using genetic algorithms are available in papers by Adewumi et al. [2] and Pongcharoen et al. [46], while some fundamental ideas on this approach can be found in the paper due to Burke [10]. Advantages of using genetic algorithms for solving timetabling problems are the possibility of exact representation of an individual, instead of a vector consisting of 0s ans 1s, as well as the fact that there are no variables supposed to be assigned some value. However, there are difficulties related to finding appropriate parameters so that convergence of the algorithm is fast enough, as well as the complexity of constructing a new population and checking if the individuals are feasible solutions, or, if necessary, repairing them. In general, feasible solutions found by genetic algorithms are not guaranteed to be optimal. Even more holds: a theoretically guaranteed existence of a feasible solution does not imply that the genetic algorithm will find it.

Another metaheuristic-based approach is one based on search methods, with some improvements in order to escape from a local optimum. These algorithms are also based on parameters that have a big influence on the result. The two main parameters are neighbourhood and fitness functions. Given some initial feasible timetable s^*, a search method evaluate the fitness function on the neighbourhood of s^* and chooses the next iteration step with some probability dependent on the value of evaluated function. These methods require a lot of testing of parameters for their enhancement, especially for problems where the space of feasible solutions is only a small part of the set of all solutions (equivalently: of the set of all possible combinations of chromosomes).

Constraint programming

Models based on constraint programming methods contain variables defining events to be assigned, domains for each variable, and constraints representing relations between variables that should be satisfied. Contraints can be understood as properties of variables and this possibility of declarative description of constraints is the biggest advantage of this approach. However, it can happen that the set of feasible1 solutions is too small, of even empty, and in that case constraint programming in its usual definition is not the best method for modelling a problem. Since constraint programming approach contains variables that are assigned stepwise, a solution where none of the variables has assigned a value will be referred to as *empty solution*. Thus, methods for solving a timetabling problem based on a costraint programming approach start with an empty solution and assign variables in some order.

A solution is said to be *complete* if a value is assigned to each variable, and *incomplete* otherwise [48]. Such a method is equivalent to searching over a tree where each node assigns a value to one variable. If it happens that some variable cannot be assigned a valid value, the algorithm backtracks and checks the other subtree of the nearest root. A major advantage of this approach is the possibility of solving the minimal perturbation problem, i.e., of finding a timetable that contains minimal changes in comparison with the previous timetable [40]. Observe that in case of an infeasible problem, the algorithms typically output a solution that has the greatest number of variables assigned, and the values of other variables can be determined manually. In that case a person that do it manually decides if some of the hard constraints can be changed or violated, in order to construct a timetable that is good enough. Methods based on this approach available in the literature (see, e.g., [41]) are mostly made in combination with some heuristics or some local search method, e.g., for ordering a variables (see, e.g., [1]).

Neural networks

Neural networks represent a computational model used for recognizing relations and patterns in some datasets, based on rules developed on the structure of few elements in a set. This approach is based on the biological structure of the brain, so it consists of units called neurons and connections between them, called synapses. Every synapse has some weight, called capacity, and every connection between the elements of a dataset can be represented as a weighted connection between two neurons. Development of neural networks began with a construction of a neural network algorithm that solves the Travelling Salesman Problem, by Hopfield and Tank in 1985 [27]. Since the results obtained by Hopfield and Tank were very sensitive to the values of parameters, many researches believed that optimal selection of these parameters cannot be simply done and that methods based on neural networks cannot bring good results for combinatorial optimization problems [53]. Thus, during the years, neural networks have not been competitive with other approaches when applied to combinatorial optimization problems [53]. Almost one decade after the paper due to Hopfield and Tank was published, concepts of neural networks were successfully applied to a linear programming problems and that was a breaking point in the development of the neural networks [12].

Nowadays, neural networks are a widely used method for solving combinatorial optimization problems. In case of the timetabling problem, algorithms are based on the so-called Hopfield network. As described in the original paper, a Hopfield network consists of a fully interconnected system of n computational elements or neurons [27]. Defining the possible states of neurons and the activation function as a measure that decides the active state of a neuron, this method can be efficiently used in modelling the timetabling problem. An approach based on the Hopfield network is developed by Smith et al. [53] and represents a model for school timetabling problem. Constraints of the model are not modelled as special hard requirements, but are included in the objective function with penalties for each violation. In that case, the objective function corresponds to the parameter of the neural network, called the energy function. A disadvantage of this approach is the number of synapses, since addition of one single neuron to the network increases the number of synapses and implicitly also the complexity of solving the problem. For that reason, algorithms based on neural networks are often developed in combination with some heuristic methods. This can give very efficient results [53].

4 The UP FAMNIT timetabling problem

As already mentioned in previous chapters, there is no general solution for the university course timetabling problem, since every institution has its own requirements and preferences. Thus, in order to model a timetabling problem for some concrete institution, we have to describe rules and requirements of the institution, which are relevant for the timetable. In this chapter we first describe informally and then define formally the timetabling problem for a concrete case of the Faculty of Mathematics, Natural Sciences and Information Technologies, University of Primorska (abbreviated UP FAMNIT). Using the description and the definition from this chapter, in Chapter 6 we introduce an integer linear program modelling the corresponding timetabling problem.

4.1 Description of the teaching process at the institution

The core of a university timetabling problem is assigning events represented by courses to ordered pairs of a time interval and a classroom. In this work we construct a timetable for one week, or more precisely for five working days. Every day consists of a fixed number of small time intervals, called timeslots.

Each course offered by the faculty has a specified type: lectures, tutorials, or a combination of both. If a course is a combination of lectures and tutorials, where these two are not strictly separated, we treat it like a lecture. If there is a clear difference between these two, then lectures and tutorials are separated and we consider them as distinct items, one defined to be of type lectures, and another to be of type tutorials.

Another division of courses is with respect to their electiveness. In every study programme, some courses are mandatory and some are elective. Mandatory courses are relevant just for one particular year of the study programme, while elective courses can often be selected by students of different years and even of different programmes. For our model it is irrelevant if a course is mandatory or elective, since we know for which groups of students the course is available and there should be no overlapping between the courses of the same program in the same year of studies.

There are courses that, for some special reasons, such as the number of available computers or microscopes in a classroom, have a bound on the number of students attending it in one group. In such cases, students are divided in sections, with respect to requirements of the course. The number of sections of one students group can differ from one course to another, so in order to construct a corresponding unit representing students, we use a method of student sectioning described by Schindl in [50]. A given student group s represented by alphabetic order of students, which is sectioned into i sections for one course and into j sections for another one, is divided into m subgroups ($m \geq i, j$) as can be seen in Figure 6. A set of students generated this way will be referred to as a student group in the rest of the text. The number of student groups is now the same for all courses of the corresponding student program. If a course is supposed to be scheduled just for some part of students, then we make it incident with all student groups representing that set of students. A set of student groups who are supposed to attend some course is given as part of input data.

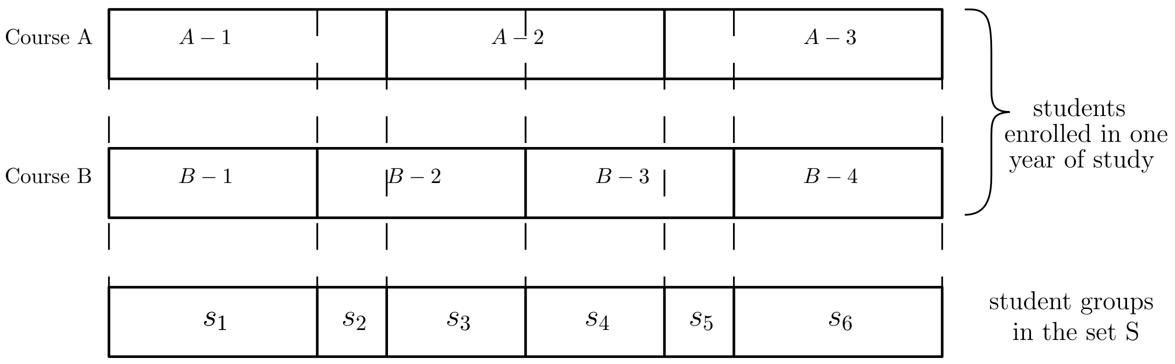

Course A
A − 1
A − 2
A − 3
Course B
B − 1
B − 2
B − 3
B − 4
students enrolled in one year of study
s1
s2
s3
s4
s5
s6
student groups in the set S

Figure 6: Constructing student subgroups.

Part of input related to a course is also the information about the lecturer who is supposed to teach it. Most courses are taught by a single lecturer, but there are also cases when the number of lecturers who teach some course is greater than one. In that case, the arrangement of lecturers within the course during the semester in not part of input data and there is an incidence relation between the corresponding course with each lecturer from the set of lecturers who are supposed to teach that course. Some lecturers have primary employment at some other institution, so they have particular time restrictions, which should be respected in the timetable. Also, lecturers who are not situated in the same municipality as the faculty are supposed not to have lectures at the first and the last timeslots of the day, to allow time for commuting. If some course is taught by two or more lecturers, then we suppose that there exists a set of timeslots for the corresponding course to be scheduled, with respect to availabilities of all involved lecturers.

For every course there is a parameter representing the number of required hours on a weekly level. The corresponding lecturer has the possibility to determine the division of hours in blocks with respect to his/her desire. For example, a course taking four hours per week can be taught either in one block of four hours, or in two blocks of two hours, etc. In this way the number of occurrences of some course on the week level is specified, as well as the number of required consecutive hours for each occurrence. Note that if there is more than one lecturer teaching some course, all lecturers incident with that course are assumed to agree on the desired division of the course.

The next resource to be described are classrooms. Availability of the classrooms is known by the administrative sector of the institution and available as part of input. Some of the classrooms are available without any restriction, while for some other this is not the case. Also, some classrooms are available, but for an additional cost, or are undesired for some other reasons, so their usage should be minimised. Classrooms are of three types: regular classrooms, computer classrooms, and laboratory classrooms. In the later two cases additional information is available, such as number of computers, number of microscopes, etc. Courses having requirements for special classroom equipment should be assigned to an appropriate classroom. Respecting the classroom capacity in the number of students attending some course is also obvious. Note that large classrooms should be occupied most of the time, since just few of available classrooms have large capacities and the number of large student groups is not too small. Not all buildings containing classrooms are located in the same town, so distance between them should be respected when preparing the timetable. We will say that two classrooms are at the same location if both classrooms are in the same town. It is desirable for a student group not to have lectures at more than one location at the same day. One more obvious restriction concerning classrooms is that consecutive hours of one course are supposed to be scheduled into the same classroom.

The requirements described above are more or less common for many similar institutions in the world. There are also some more specific things that have big influence on the final result of the timetable structure. The first of them concerns upper bounds on the number of teaching hours for teaching staff at day level. Parameters representing maximal number of teaching hours are determined with respect to type of course. For example, one lecturer, say ℓ, cannot teach more than $\rho_L(\ell)$ timeslots per day of courses being classified to be of type "lectures". At the same time, the same person cannot teach more that $\rho(\ell)$ hours per day in total, not regarding the type of the course. Upper bounds on the number of teaching hours per day for student groups are also given, but do not represent a strict bound, just a preference.

The second special requirement describes the nature of lectures for master's students of some programmes: since the percentage of employed students enrolled in some programmes is very big, it is desirable to have teaching classes at afternoon hours, since otherwise such students would not be able to attend. Also, there are periodical research seminars for some of the departments at fixed time and place, so in the corresponding timeslots teaching staff from the respective departments should not be assigned teaching. The same holds for student groups who are supposed to attend research seminars. Restrictions of this type also influence availability of some classrooms and this fact should be considered when modelling the problem. The institution offers courses for some study programmes of another institution, which have interdisciplinary character. Teaching sessions relevant to these courses should be scheduled early in the morning, or in the late afternoon, but not both in the same day, concerning one student group.

We define a meeting to be the main unit for timetabling. A meeting m is an order pair consisting of a course c as the first coordinate, and a set of incident student groups as the second one. In such a way a course that has to appear more than once, for distinct sets of student groups, is represented by a few meetings, where each of them is related to some of the corresponding set of student groups. For example, if a course c is supposed to be scheduled for student groups s_1 and s_2 separately, then we consider two meetings: $m_1 = (c, s_1)$ and $m_2 = (c, s_2)$. For every meeting the set of lecturers incident with the meeting and the division into blocks are inherited from the definition of the corresponding course.

Requirements for timetabling – summary

When preparing the timetable, there are requirements that should be respected. As already mentioned, hard constraints must be satisfied, while a violation of soft constraints is allowed, but not desirable. In the following we list all hard and soft constraints relevant for the considered institution.

Hard constraints

(i) **Every meeting has to be assigned to available resources.** The first requirement of this type says that every meeting has to be assigned to an available timeslot. Teachers are not available all the time, so they cannot have teaching sessions at certain timeslots. Another type of time restriction included here represents restrictions related to classrooms. Some classrooms are available all the time, but some others are only available on specified timeslots. The third requirement contained here is one determining acceptable classrooms for each meeting. If a meeting has some specific requirement connected with equipment of classroom, the requirement should be satisfied in order for the meeting to be productive. Classrooms that satisfy requirements of meetings and capacities of student groups are determined by the student sector of the institution.

(ii) **Overlapping is not permitted.** A given student group cannot be assigned to more than one meeting or to more than one classroom at a time. A similar constraint holds for lecturers. Also, every classroom is allowed to be assigned at most one meeting per timeslot.

(iii) **The timetable has to be complete.** All hours of each meeting have to be assigned to the timetable, with respect to division into blocks according to lecturers decision. The hours of one meeting contained in one day have to be scheduled consecutively and in the same classroom.

(iv) **Pre-scheduled meetings.** Some meetings and research seminars have predefined times and classrooms. Students and teaching staff related to such events should have an appropriate schedule allowing them to attend such events.

(v) **Upper bounds on the number of hours per lecturer.** There are parameters that determine the maximum number of hours one lecturer can teach per day, with respect to different types of meetings, as well as an overall bound.

(vi) **Students restrictions.** A student group from another institution can have lectures just in the early morning or in the late afternoon timeslots, with an additional requirement that both options are not allowed to happen at the same time for the same student group. Students belonging to the same student group cannot have meetings at two distinct locations in a day.

Soft constraints

(i) **Minimal use of payable classrooms.** As mentioned above, there are classrooms that can be used for teaching, but are payable. Since the use of such classrooms incurs additional costs, it is desirable to minimize the number of teaching hours assigned to such classrooms.

(ii) **Compact timetable and lunch break.** For each student group, the meetings corresponding to the group should be placed in the timetable without "big holes", as compactly as possible. If possible, it is desirable for students to have a lunch break in the middle of the day.

(iii) **Other student-related requirements.** The amount of meetings should be bounded in the sense that there is an upper bound on the number of teaching hours on day level for every student group. Since many students are from other parts of the country, it is desirable to minimize lectures on Friday afternoon.

There are study programmes that are supposed to have teaching hours scheduled only at afternoon slots.

4.2 Formal definition

In this section we give a formal definition of the timetabling problem for the described institution based on requirements described in the previous section. Since soft constraints are just a measure for the quality of timetable and represent a part of the objective function, they have no influence to the existence of a feasible solution. A problem considered here is referred to as a Famnit Timetable Design, with respect to hard constraints. This problem checks if there exists a feasible solution of the system, i.e., a timetable that satisfies hard constraints. The Famnit Timetable Design problem will be proved *NP*-complete in Chapter 5. For the formalization of hard and soft constraints, see Sections 6.3 and 6.4 respectively.

Definition 4.1. The Famnit Timetable Design problem is defined as follows:
Instance:

- a finite set D of "days";
- a finite set T of "timeslots" - set T is linearly ordered, so that first timeslot in a week is the smallest element of T, and last timeslot in the week is the greatest one; with respect to this ordering we define addition in T so that given a timeslot t and a number $i \in \mathbb{N}$, timeslot $t' = t + i$ is defined as a timeslot being the $(t+i)$-th element of the linear order of set T (clearly, i cannot be arbitrarly large, we discuss this in description of constraints);
- for each $d \in D$, a finite set $T(d) \subseteq T$ of timeslots at day $d \in D$; clearly, the family of sets $T(d)$ defined this way represents a partition of set T;
- a finite set M of "meetings";
- a finite set S of "student groups";
- a finite set L of "lecturers";
- a finite set R of "rooms";
- a finite set K of "locations";
- a subset $T(m) \subseteq T$ of available hours for each meeting $m \in M$ (for every meeting the lecturer is known, so $T(m)$ depends on lecturer availability);

- a subset $T(r) \subseteq T$ of available hours for each room $r \in R$;
- a subset $M(s) \subseteq M$ of meetings incident with each student subgroup $s \in S$;
- a subset $M(\ell) \subseteq M$ of meetings incident with each lecturer $\ell \in L$;
- a subset $R(m) \subseteq R$ of available rooms for each meeting $m \in M$;
- a subset $M(k) \subseteq M$ of meetings that take place at location $k \in K$;
- a multi-set $P(m)$ of natural numbers; $p \in P(m)$ if p hours of meeting m have to be scheduled consecutively some time during the week;
- a number $\rho(\ell) \in \mathbb{N}$ for each lecturer $\ell \in L$, the maximum number of hours ℓ can teach per day;
- a finite set N of parts of a day (e.g. morning, noon, evening...); $T(n) \subseteq T$ is the set of all timeslots at n-th part of day d, over all $d \in D$;
- a set $S' \subseteq S$ of student subgroups that can only have lectures within a single part of a day;
- pre-scheduled things:
  set $G \subseteq M \times T \times R$ of pre-scheduled triples,
  set $F \subseteq M \times T \times R$ of triples that cannot be scheduled (unacceptable triples).

Question:
Is there a timetable that schedules all meetings, that is, a function $f : M \times T \times R \to \{0, 1\}$ (where $f(m, t, r) = 1$ means that meeting m is assigned to timeslot t and room r) that schedules the desired number of hours of all meetings and satisfies the following constraints.

1. Each meeting is scheduled to an acceptable timeslot:
$$\forall m \in M, \forall t \in T, \forall r \in R : \ f(m, t, r) = 1 \Rightarrow t \in T(r) \cap T(m).$$
2. Each meeting is assigned to a classroom that satisfies the requirements:
$$\forall m \in M, \forall t \in T, \forall r \in R : \ f(m, t, r) = 1 \Rightarrow r \in R(m).$$
3. Every meeting is assigned to at most one classroom at a time:
$$\forall m \in M, \forall t \in T, \forall r_1, r_2 \in R : \ f(m, t, r_1) = f(m, t, r_2) = 1 \Rightarrow r_1 = r_2.$$

4. At most one meeting is assigned to every pair of timeslot and classroom:
$$\forall t \in T, \forall m_1, m_2 \in M, \forall r \in R: \; f(m_1,t,r) = f(m_2,t,r) = 1 \Rightarrow m_1 = m_2.$$
5. Every lecturer is assigned to at most one meeting and classroom at a time:
$$\forall \ell \in L, \forall t \in T \, \exists \, \text{at most one pair} \, (m,r) \in M(\ell) \times R \, \text{such that} \, f(m,t,r) = 1.$$
6. Every student group is assigned to at most one meeting and classroom at a time:
$$\forall s \in S, \forall t \in T \, \exists \, \text{at most one pair} \, (m,r) \in M(s) \times R\text{such that} \, f(m,t,r) = 1.$$
7. Each meeting is assigned to a determined number of teaching hours in a week:
$$\forall m \in M \, \text{there are exactly} \, a(m) \, \text{pairs} \, (r,t) \in R \times T \, \text{such that} \, f(m,t,r) = 1,$$
where $a(m)$ is defined as a sum of elements in the multiset $P(m)$.
8. Triples representing forbidden assignments should be satisfied:
$$\forall (m,t,r) \in F : f(m,t,r) = 0.$$
9. Triples representing pre-scheduled assignments should be satisfied:
$$\forall (m,t,r) \in G : f(m,t,r) = 1.$$
10. Each student group s can have lectures just at one location in a day:
$$\forall (s,d) \in S \times D \, \text{there is at most one} \, k \in K \, \text{such that} \, f(m,t,r) = 1,$$
$$\text{for some} \, m \in M(s) \cap M(k), t \in T(d), r \in R.$$
11. Each lecturer can teach at most a given number of hours in a day:
$$\forall \ell \in L, \forall d \in D : \sum_{m \in M(\ell)} \sum_{t \in T(d)} \sum_{r \in R(m)} f(m,t,r) \leq \rho(\ell).$$
12. Hours of one meeting scheduled in one day must be consecutive:
$$\forall m \in M, \forall d \in D, \forall t_1, t_2 \in T(d), \forall r_1, r_2 \in R :$$
$$f(m,t_1,r_1) = 1 \wedge f(m,t_2,r_2) = 1 \Rightarrow \forall t \in T(d), t_1 < t < t_2 : \sum_{r \in R} f(m,t,r) = 1.$$
13. Consecutive hours of one meeting have to take place in the same classroom:
$$\forall m \in M, \forall d \in D, \forall t \in T(d), \forall r_1, r_2 \in R :$$
$$t+1 \in T(d) \wedge f(m,t,r_1) = 1 \wedge f(m,t+1,r_2) = 1 \Rightarrow r_1 = r_2.$$
14. Some student groups can have lectures in at most one part of a day:
$$\forall s \in S', \forall d \in D \, \exists \, \text{at most one} \, n \in N : f(m,t,r) = 1$$
$$\text{for some} \, m \in M(s), t \in T(n) \cap T(d), r \in R.$$

5 Proof of *NP*-completness

In Section 3.1 we mentioned the Timetable Design problem, which is known to be *NP*-complete. In order to prove that Famnit Timetable Design is *NP*-complete, we will apply Theorem 2.4 and show that it belongs to the class *NP* and that some *NP*-complete problem polynomially reduces to it.

5.1 The problem is in NP

To see that the problem is in *NP*, we have to argue that for every yes instance I, there exists a certificate with which the fact that I is a yes instance can be verified in time polynomial in the size of input data. Such a certificate is naturally given by a timetable that schedules all meetings, that is, a function $f : M \times T \times R \to \{0, 1\}$ satisfying the constraints given in Definition 4.1. Thus, we can construct sets $S_1, S_2 \subseteq M \times T \times R$ such that f takes value 1 on set S_1 and 0 on set S_2. Clearly, the verification of feasibility can be done in polynomial time, since we have to check if f satisfies all constraints listed in the definition. In the worst case we have to check all elements of the corresponding index set determining some condition. That can be achieved in time polynomial in the size of input data.

5.2 Reduction from an *NP*-complete problem

We showed that our problem is in *NP*. What remains is to prove that it is at least as hard as some known *NP*-complete problem. The *NP*-complete problem used for this proof is the Binary Timetable Design (BTD) problem defined in Section 3.1. We construct a polynomial reduction $\mathcal{R}$ that reduces an instance I of the BTD problem to an instance $\mathcal{R}(I)$ of Famnit Timetable Design (FTD) problem.

Recall that an instance of the BTD problem consists of a set C of craftsmen, a set P of projects, a set H of work periods, subsets $A(c)$, $A(p)$ of H that represent available work periods for craftsman $c \in C$ and poject $p \in P$, respectively, and a number $R(c, p) \in \{0, 1\}$ of required work periods. In the following steps we construct the reduction $\mathcal{R}$.

i) To every work period $h \in H$ associate a timeslot t_h and define set T to consist of elements t_h: $T = \{t_h \mid h \in H\}$.

ii) To every work period $c \in C$ associate a lecturer ℓ_c and define set L to consist of elements ℓ_c: $L = \{\ell_c \mid c \in C\}$.

iii) To every project $p \in P$ associate a student subgroup s_p and define set S to consist of elements s_p: $S = \{s_p \mid p \in P\}$.

iv) Let set W consist of ordered pairs $(c, p) \in C \times P$ such that $R(c, p) = 1$. Then to each element of the set W associate a meeting $m_{c,p}$ and a room $r_{c,p}$. Construct sets M, R of these elements, respectively: $M = \{m_{c,p} \mid (c, p) \in W\}$, $R = \{r_{c,p} \mid (c, p) \in W\}$. It is obvious that sets W, M, R constructed this way have the same cardinalities, and there is a one to one correspondence between any two of them.

v) Some sets will consist of just one element: $D = \{d_0\}, K = \{k_0\}$, $N = \{n_0\}$. Consequently, let $T(d_0) = T$ and $M(k_0) = M(q_0) = M$.

vi) Let $m_{c,p} \in M$ be arbitrary. This element corresponds to a unique element (c, p) of set W. For each element h of the set $A(c) \cap A(p) \subseteq H$ take the timeslot $t_h \in T$ and define a subset $T(m_{c,p}) \subseteq T$ to consist just of such elements t_h: $T(m_{c,p}) = \{t_h \mid h \in A(c) \cap A(p)\}$. For any $r \in R$ let $T(r) = T$. For any $m_{c,p} \in M$ define the set $R(m_{c,p}) \subseteq R$ as $R(m_{c,p}) = \{r_{c,p}\}$.

vii) For each $r_{c,p} \in R$ define the set set $M(r_{c,p}) \subseteq M$ as $M(r_{c,p}) = \{m_{c,p}\}$.

viii) For each $\ell_c \in L$ define the set $M(\ell_c) \subseteq M$ as $M(\ell_c) = \{m_{c',p'} \in M \mid c = c', p' \in P\}$.

ix) For each $s_p \in S$ define the set $M(s_p) \subseteq M$ as $M(s_p) = \{m_{c',p'} \in M \mid p = p', c' \in C\}$.

x) If $m = m_{c,p} \in M$ is arbitrary, then number $a_m = a_{c,p}$ has value equal to $R(c, p)$. Construction of set M implies that for each element $m \in M$, we have $a_m = 1$. It follows that the set $P(m)$ consist just of one element, which is equal to 1.

xi) Other elements of instance $\mathcal{R}(I)$ of the FTD problem are constructed as follows: for every $\ell \in L$ number $\rho(\ell)$ has some big value, e.g.,the number of all timeslots. Sets S', G, F are constructed as sets with no elements (empty sets).

With the above construction of reduction $\mathcal{R}$, we construct from any instance I of problem BTD instance $\mathcal{R}(I)$ of FTD. This reduction has to have two important properties. One of them is that $\mathcal{R}$ has to be evaluated in polynomial time. The other one is that $\mathcal{R}$ has to preserve answers, or in other words, it must be true that answer to an arbitrary instance I of BTD is positive if and only if answer to the instance $\mathcal{R}(I)$ of FTD is positive.

$\mathcal{R}$ can be computed in polynomial time

Given an instance I for the BTD problem, we construct an instance $\mathcal{R}(I)$ of FTP problem. Thus every element belonging to $\mathcal{R}(I)$ corresponds to some element, or combination of elements from initial set I. This means that for the construction of every element of the set $\mathcal{R}(I)$ we have to look at a set of some n-tuples of elements in I (for an appropriate n) and for each such element construct the corresponding element of $\mathcal{R}(I)$. Clearly, this can be done in polynomial time.

$\mathcal{R}$ preserves answers

Suppose first that instance I of the BTD problem has positive answer, that is, there exists a function $g : C \times P \times H \to \{0, 1\}$, where

$$g(c,p,h) = \begin{cases} 1, & \text{if craftsman } c \text{ works on project } p \text{ during the work period } h; \\ 0, & \text{otherwise,} \end{cases}$$

and g satisfies conditions from Section 3.1. We would like to show that also instance $\mathcal{R}(I)$ has a positive answer by determining a function $f : M \times T \times R \to \{0, 1\}$, where

$$f(m,t,r) = \begin{cases} 1, & \text{if meeting } m \text{ is assigned to timeslot } t \text{ and room } r; \\ 0, & \text{otherwise,} \end{cases}$$

that satisfies conditions 1-14 in Definition 4.1.

From the construction of reduction $\mathcal{R}$ we know that elements $m \in M$ and $r \in R$ can be uniquely represented as $m_{c,p}$ and $r_{c,p}$, respectively, for some $(c, p) \in W$. Similarly, each $t \in T$ can be uniquely represented as t_h for some $h \in H$. Using this correspondence we can write arguments of function f with additional indices. Below is the definition of function f in terms of function g. For all $m_{c,p} \in M, t_h \in T, r_{c',p'} \in R$, set

$$f(m_{c,p}, t_h, r_{c',p'}) = \begin{cases} g(c,p,h), & \text{if } (c,p) = (c',p'); \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, function f is well defined. Function f has to satisfy constraints from the definition of the FTD problem. We check each constraint separately. To simplify notation we will denote ordered pairs (c, p) and (c', p') from set W by letters w and w', respectively.

1. Suppose that $f(m_w, t_h, r_{w'}) = 1$. By definition of function f it follows that $w = w'$ and $g(c, p, h) = 1$. From the first condition for function g in the definition of

Timetable Design problem it follows that $h \in A(c) \cap A(p)$. Reduction $\mathcal{R}$ bijectively maps set T into H and set $T(m_w)$ into $A(c) \cap A(p)$, so it holds that t_h belongs to $T(m_w)$, which is equal to $T(m_w) \cap T(r_w)$, since $T(r_w) = T$.

2. Function $f(m_w, t_h, r_{w'})$ can have value 1 only if $w = w'$. But that means that $r_w \in R(m_w)$, so this constraint is satisfied.

3. Let $m_w \in M$ and $t_h \in T$ and consider two rooms $r_{w'}, r_{w''} \in R$ such that $f(m_w, t_h, r_{w'}) = f(m_w, t_h, r_{w''}) = 1$. By definition of function f, it has to be true that $w = w'$ and $w = w''$, and so $w' = w''$. Hence elements $r_{w'}$ and $r_{w''}$ are equal, or in other words, there is at most one such element.

4. Let $t_h \in T$ and $r_w \in R$ and suppose that there are two meetings $m_{w'}, m_{w''} \in M$ such that $f(m_{w'}, t_h, r_w) = f(m_{w''}, t_h, r_w) = 1$. By definition of function f, it holds that $w'' = w = w'$, so $m_{w'} = m_{w''}$.

5. Let $\ell_c \in L$ and $t_h \in T$ and consider two pairs $(m_{w'}, r_{w'}), (m_{w''}, r_{w''}) \in M \times R$ such that $m_{w'}, m_{w''} \in M(\ell_c)$ and $f(m_{w'}, t_h, r_{w'}) = f(m_{w''}, t_h, r_{w''}) = 1$. Since $m_{w'}, m_{w''} \in M(\ell_c)$, it follows from the construction of $M(\ell_c)$ that $c' = c''$, where $w' = (c', p')$ and $w'' = (c'', p'')$. From the definition of f we have that $g(c, p', h) = 1$ and $g(c, p'', h) = 1$. But from the second constraint of g in the definition of Timetable Design problem it follows that $p' = p''$, and because of that it holds that $w' = w''$.

6. Let $s_p \in S$ and $t_h \in T$ and consider two pairs $(m_{w'}, r_{w'})$, $(m_{w''}, r_{w''}) \in M \times R$ such that $m_{w'}, m_{w''} \in M(s_p)$ and $f(m_{w'}, t_h, r_{w'}) = f(m_{w''}, t_h, r_{w''}) = 1$. Since $m_{w'}, m_{w''} \in M(s_p)$, from the construction of $M(s_p)$ it follows that $p' = p''$, where $w' = (c', p')$ and $w'' = (c'', p'')$. Then from the definition of f we have that $g(c', p, h) = 1$ and $g(c'', p, h) = 1$. But from the third constraint of g in definition of Timetable Design problem it follows that $c' = c''$, and because of that it holds that $w' = w''$.

7. Suppose that for a given meeting $m_{c,p}$ there are not exactly $a(m_{c,p}) = 1$ pairs $(r_{c,p}, t_h)$ such that $f(m_{c,p}, t_h, r_{c,p}) = 1$. Suppose first that there is no such pair. Then f has value 0 for these arguments, and so $g(c, p, h) = 0$. This is in contradiction with the fourth condition of function g in the definition of Timetable Design problem and definition of set M. Now suppose that there is more than one such pair, say $(r_{c,p}, t_{h'})$ and $(r_{c,p}, t_{h''})$, such that $f(m_{c,p}, t_{h'}, r_{c,p}) = 1$ and $f(m_{c,p}, t_{h''}, r_{c,p}) = 1$. What we get is that $g(c, p, h') = g(c, p, h'') = 1$, but this is in contradiction with the fourth condition of function g, since we deal with the binary version of the problem.

8. The condition holds trivially since set F is the empty set.

9. The condition holds trivially since set G is the empty set.

10. The condition holds trivially since set K is a singleton.

11. The condition holds since by the definition of number $\rho(l)$, the limit cannot be exceeded.

12. This condition is trivially true, since the assumption of the condition is never satisfied, that is, for every $m_w \in M$ there is just one triple (m_w, t_h, r_w) such that $f(m_w, t_h, r_w) = 1$.

13. Let $m_w \in M$ and $t_h \in T$. Suppose that $f(m_w, t_h, r_{w'}) = f(m_w, t_h + 1, r_{w''}) = 1$, where by $t_h + 1$ is denoted successor of the element t_h in linearly ordered set T. Because of definition of the function f, it holds that $w' = w'' = w$ and so $r_{w'} = r_{w''}$ even without concerning consecutive timeslots.

14. This condition holds trivially since set S' is the empty set.

Since function f satisfies all conditions, it represents a feasible timetable, so instance $\mathcal{R}(I)$ has positive answer.

Suppose now that the transformed instance $\mathcal{R}(I)$ has positive answer to FTD problem. We would like to derive that instance I also has positive answer to BTD problem. In order to show this, we have to define a function g in terms of function f. This is done as follows: for all $c \in C, p \in P, h \in H$, set

$$g(c,p,h) = \begin{cases} f(m_{c,p}, t_h, r_{c,p}), & \text{if } R(c,p) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly as in the other direction of the proof, we have to show that function g describes a feasible solution for the BTD problem, or more precisely that g satisfies conditions from the definition of the Timetable Design problem.

1. Suppose that $g(c,p,h) = 1$. From the above formulation of g, it follows that $f(m_{c,p}, t_h, r_{c,p}) = R(c,p) = 1$. The first condition of f gives that $t_h \in T(r_{c,p}) \cap T(m_{c,p}) = T(m_{c,p})$, which corresponds to the set $A(c) \cap A(p)$.

2. Let $h \in H$ and $c \in C$ and consider two projects $p', p'' \in P$ such that $g(c,p',h) = g(c,p'',h) = 1$. Then it follows that also $f(m_{c,p'}, t_h, r_{c,p'}) = f(m_{c,p''}, t_h, r_{c,p''}) = 1$. From the construction of set $M(\ell_c)$ it follows that $m_{c,p'} \in M(\ell_c)$, as well as $m_{c,p''} \in M(\ell_c)$. Condition 5 from the definition of FTD ensures that elements $m_{c,p'}$ and $m_{c,p''}$ are equal. Equality of p' and p'' follows.

3. Let $h \in H$ and $p \in P$ and consider two craftsmen $c', c'' \in C$ such that $g(c', p, h) = g(c'', p, h) = 1$. Similarly as in the previous condition, we conclude that $f(m_{c',p}, t_h, r_{c',p}) = f(m_{c'',p}, t_h, r_{c'',p}) = 1$. But it also holds that both elements (c', p), and (c'', p) are contained in set $M(s_p)$, and so condition 7 from the definition of FTP problem ensures that these two elements are equal, and consequently $c' = c''$.

4. If $R(c, t) = 0$, then obviously $g(c, p, h) = 0$ for all elements $h \in H$. Thus, assume that $R(c, t) = 1$. Then also $a(m_{c,p}) = 1$ and condition 7 from the definition of FTD ensures that for each $m_{c,p}$ there is exactly one pair $(r_{c',p'}, t_h)$ so that $f(m_{c,p}, t_h, r_{c',p'}) = 1$. We see that $(c, p) = (c', p')$, so this pair is fixed, and argument t_h determines this pair, so element h is also unique.

6 An integer programming formulation

In this chapter we tie together things described in previous chapters and introduce an integer linear program for the university timetabling problem at UP FAMNIT. Consider the timetabling problem from Definition 4.1. We use the same terminology as in Section 4.1.

6.1 Parameters of the ILP

Here we recall the structural elements introduced in the definition of Famnit Timetable Problem and describe them in more detail. Note that in the rest of this thesis we use simplified notation of some structural elements, so sets $T(d), M(\ell), M(s), M(k), R(m),\ T(\ell), T(r)$ are denoted by $T_d, M_\ell, M_s, M_k, R_m, T_\ell, T_r$, respectively.

- M is defined as the set of meetings to be assigned: every element $m \in M$ is an ordered pair containing a course as the first coordinate, and a set of student subgroups as the second one. All meetings have a type determined: lectures or tutorials. The set of meetings in M labelled as "lectures" or "tutorials" will be denoted by M^{lec} and M^{tut}, respectively. Obviously, M is the disjoint union of these two sets. An example of element $m \in M$ is an ordered pair $(\text{Course}\,1, \{s_1, s_2\})$. For any meeting, lecturer and number of hours of the meeting to be assigned are known, so for meeting m, number of hours per week is a_m. Any meeting m is divided in parts, with respect to lecturers' desire. These parts will be called blocks. So for any meeting m there is a vector p_m, with element $p_m(i) = k$, if a block of duration i of meeting m has to be repeated k times per week. For any meeting m it holds that $\sum_i p_m(i) \cdot i = a_m$. For any meeting m we define a set H_m to be the set of all block lengths appearing in the division of meeting m, i.e., $H_m = \{i \mid p_m(i) \neq 0\}$. Let us introduce a short example: let meeting m with $a_m = 5$ be separated in two blocks, with durations of 2 and 3 hours, respectively. Such a division is denoted as $(2 + 3)$. Then the corresponding vector is $p_m = (0, 1, 1)$ and so $H_m = \{2, 3\}$. Another way of division of meeting m is in two blocks of length 1 and one block of length 3, denoted by $1 + 1 + 3$. In that case we have $p_m = (2, 0, 1)$ and $H_m = \{1, 3\}$.

- L: the set of lecturers.
- D: the set of days in a week, $D = \{1, 2, 3, 4, 5\}$.
- R: the set of classrooms.
- T: the set of all timeslots in the week; each element $t \in T$ is an ordered pair, $t = (d, h)$, where d and h represent day and timeslot within the day, respectively; in this work timeslots are supposed to have length 60 minutes. As already mentioned in Definition 4.1, T is linearly ordered set, with $t < t'$, where $t = (d, h)$, $t' = (d', h')$, if $d < d'$ or if $d = d'$ and $h < h'$. The set of timeslots belonging to day d is denoted by the T_d. The number of timeslots in one day is denoted by a constant τ (hence $h \in \{1, \dots, \tau\}$, so the number of all timeslots is 5τ). Given a timeslot $t = (d, h)$ and a number i, such that $i \leq \tau - h$, we have $t + i := (d, h + i)$.
- K: the set of locations, in our example there are two locations, denoted by k_1 (Izola – University Campus Livade) and k_2 (Koper).
- S: the set of student groups.
- M_ℓ: the set of class meetings given by lecturer $\ell \in L$.
- M_s: the set of class meetings of a group of students $s \in S$.
- M_k: the set of class meetings that have to take place at location $k \in K$, in particular: M_{k_1}, M_{k_2};
- R_m: the set of rooms that are acceptable for meeting $m \in M$. A classroom is acceptable for a particular meeting m if it satisfies requirements connected with equipment of classroom and if it has sufficient capacity. From this set we can construct a set M^R of ordered pairs (m, r) containing all acceptable combinations of meetings m and rooms r: $M^R := \{(m, r) \mid m \in M, r \in R_m\}$.
- T_ℓ: the set of timeslots $t \in T$ in which lecturer $\ell \in L$ can have lectures. Here few conditions have to be included. First, individual requirements of lecturers are reflected in this set. The first and the last timeslots in a day are not acceptable for lecturers not situated in same municipality as the institution. If a lecturer has another job, is a member of some committee, or has some other regular obligations, these constraints are also assumed to included in the set T_ℓ. For instance, every Monday at 10AM teaching staff of the Mathematics department

should have the possibility to attend the Mathematical research seminar. In order to make this possible, they should not have any teaching obligations at that time. Similarly, every Monday at 4PM teaching staff of Department of Information Sciences and Technologies (abbreviated as DIST) should have the possibility to attend the DIST research seminar, so there are no teaching obligations for them at that time.

- T_r: the set of timeslots $t \in T$ in which a given classroom $r \in R$ can be used.
- M_{DIST}: set of the meetings given by lecturers researching in Department of Information Sciences and Technologies – in order to make fewer overlapping with DIST research seminar;
- M_{PM}: the set of meetings that are supposed to be assigned at afternoon timeslots;
- T_{AM}: the set of morning timeslots; denote the number of morning timeslots in a day by τ_{AM};
- T_{PM}: the set of afternoon timeslots; denote the number of afternoon timeslots in a day by τ_{PM};
- T_{MAS}: the set of timeslots for the Mathematical research seminar;
- T_{DIST}: the set of timeslots for the DIST research seminar;
- S_{AK}: the set of student groups consisting of students of external programs – Applied Kinesiology students;

6.2 Variables of the ILP

In the integer linear program all variables are binary. There are three different sets of variables and we list them in the following.

- x-**variables**: For every triple of a meeting $m \in M$, a timeslot $t \in T$, and a room $r \in R_m$ that is acceptable for that meeting, there is one corresponding variable $x_{m,t,r}$. This variable takes value 1 if meeting m is scheduled at timeslot t in classroom r, and 0 otherwise:

$$x_{m,t,r} = \begin{cases} 1, & \text{if meeting } m \text{ is scheduled at timeslot } t \text{ in classroom } r, \\ 0, & \text{otherwise.} \end{cases}$$

- y-**variables**: For every triple of a meeting $m \in M$, a timeslot $t \in T$ and a predefined length $i \in H_m$ of individual blocks of meeting m we define a variable $y_{m,t,i}$.

The variable takes value 1 if timeslot t is the first appearance of i consecutive hours of m, and 0 otherwise:

$$y_{m,t,i} = \begin{cases} 1, & \text{if timeslot } t \text{ is first appearance of } i \text{ consecutive hours of meeting } m, \\ 0, & \text{otherwise.} \end{cases}$$

- z**-variables**: In the last set of variables are so called z-variables, auxiliary variables for modelling some hard and soft constraints. Given a constraint of type p and the corresponding index set I_p, we define a variable $z_{p,i}$ for every $i \in I_p$. Values of such variables will be determined by description of corresponding constraint. These variables will appear in the modelling of hard constraints of type F_1 (Section 6.3) and soft constraints of types S_2 and S_3 (Section 6.4).

6.3 Constraints of the ILP

A) Every meeting has to be assigned to available resources:

A_1) Lecturers cannot have lectures at unacceptable timeslots:

$$\sum_{m \in M_\ell} \sum_{t \in T \setminus T_\ell} \sum_{r \in R_m} x_{m,t,r} = 0 \quad \forall \ell \in L.$$

A_2) Classrooms can only be used at specified timeslots:

$$\sum_{(m,r) \in M^R} \sum_{t \in T \setminus T_r} x_{m,t,r} = 0, \quad \forall r \in R.$$

A_3) Every meeting has to take place in an acceptable classroom: this constraint is already satisfied by definition of variables, since $x_{m,t,r}$ variables are defined just for classrooms $r \in R$ that satisfy classroom requirements for the meeting $m \in M$.

B) Overlapping is not permitted

B_1) For every student group at most one meeting and one classroom can be assigned to every teaching period:

$$\sum_{m \in M_s} \sum_{r \in R_m} x_{m,t,r} \leq 1 \quad \forall s \in S, \forall t \in T.$$

B_2) Every member of the teaching staff shall be assigned at most one meeting and one classroom at a time:

$$\sum_{m \in M_\ell} \sum_{r \in R_m} x_{m,t,r} \leq 1 \quad \forall \ell \in L, \forall t \in T.$$

B_3) Every classroom can be assigned to at most one meeting at a time:

$$\sum_{(m,r)\in M^R} x_{m,t,r} \le 1 \quad \forall r \in R, \forall t \in T.$$

C) Timetable has to be complete:

C_1) All meetings in the curriculum of each student subgroup should be in the timetable and in the right amount of teaching periods, with respect to weekly duration:

$$\sum_{t\in T}\sum_{r\in R_m} x_{m,t,r} = a_m \quad \forall m \in M.$$

C_2) A meeting m of duration $i \in H_m$ has to start and finish at the same day, so some variables $y_{m,t,i}$ are defined to have value 0:

$$y_{m,t,i} = 0 \quad \forall m \in M,\, \forall i \in H_m,\, \forall t = (d,h) \in T \,\text{s.t.}\, h > \tau - i + 1.$$

Recall that the parameter τ represents the number of timeslots in a day.

C_3) Given a meeting m, at most one timeslot can be the first appearance of m in a single day:

$$\sum_{i\in H_m}\sum_{t\in T_d} y_{m,t,i} \le 1, \quad \forall m \in M,\, \forall d \in D.$$

C_4) A given meeting m of duration i (where i is the index of a nonzero element of vector p_m) has to appear exactly $p_m(i)$ times per week (i.e. in $p_m(i)$ days). All such indices i are contained in set H_m, so we have:

$$\sum_{t\in T} y_{m,t,i} = p_m(i), \quad \forall m \in M,\, \forall i \in H_m.$$

C_5) If a course m of duration i is assigned at day d, it has to be assigned to exactly i hours:

$$i \cdot \sum_{t\in T_d} y_{m,t,i} \le \sum_{r\in R_m}\sum_{t\in T_d} x_{m,t,r}, \quad \forall m \in M, \forall d \in D, \forall i \in H_m.$$

C_6) Appearances of meeting m of duration i in a single day should be consecutive:

$$y_{m,t,i} \le \sum_{r\in R_m} x_{m,t+j,r}, \quad \forall t = (d,h) \in T, \forall m \in M, \forall i \in H_m, \forall j = 0, \ldots, i-1,$$

where for $t = (d,h)$, such that $h + j \le \tau$ we have $t + j := (d, h + j)$.

C_7) All consecutive hours of one meeting should take place in the same classroom:

$$x_{m,t,r} + x_{m,t+1,r'} \le 1 \quad \forall m \in M, \forall t \in T,\, \forall r, r' \in R_m \,\text{s.t.}\, r \ne r'.$$

D) Pre-scheduled meetings:

D_1) There are activities at the Faculty that do not belong to the course offer. Such activities represent pre-scheduled meetings, which should be scheduled to already determined pairs of timeslots and classrooms. If a meeting is related to some lecturers or students groups, their regular course meetings are undesirable at that time. In particular, Research seminar at Mathematics' department have to be assigned to Monday, 10AM, in one specified room, call it r_1. Hence, that room is unavailable for other lectures at that time. Also, the DIST research seminar has to be assigned to Monday, 4PM, in an already defined room, r_2.[1]

$$\sum_{m\in M}\sum_{t\in T_{MAS}} x_{m,t,r_1} = 0$$
$$\sum_{m\in M}\sum_{t\in T_{CSS}} x_{m,t,r_2} = 0$$

D_2) Meetings appearing as first coordinates of elements in set G have predetermined timeslot and classrooms:

$$x_{m,t,r} = 1, \quad \forall (m,t,r) \in G.$$

E) Upper bounds on number of hours at day level

E_1) It is not desired for lecturer ℓ to have more than ρ_ℓ timeslots of teaching obligations per day:

$$\sum_{t\in T_d}\sum_{m\in M_\ell}\sum_{r\in R_m} x_{m,t,r} \le \rho_\ell, \quad \forall \ell \in L, \quad \forall d \in D.$$

There are also upper bounds with respect to meeting type, here we define it parametrically as ρ_1 and ρ_2 for types "lectures" and "tutorials", respectively.

$$\sum_{t\in T_d}\sum_{m\in M_\ell\cap M^{lec}}\sum_{r\in R_m} x_{m,t,r} \le \rho_1, \quad \forall \ell \in L, \quad \forall d \in D,$$

$$\sum_{t\in T_d}\sum_{m\in M_\ell\cap M^{tut}}\sum_{r\in R_m} x_{m,t,r} \le \rho_2, \quad \forall \ell \in L, \quad \forall d \in D.$$

Also, one more condition can be introduced, specifying that the lecturers shall not have more than δ timeslots of teaching in every block of Δ timeslots per day.

$$\sum_{j=0}^{\Delta-1}\sum_{m\in M_\ell}\sum_{r\in R_m} x_{c,t+j,r} \le \delta \quad \forall \ell \in L, \quad \forall t = (d,h) \in T \,\text{s.t.}\, h \le \tau - \Delta$$

[1] These activities are also related to a subset of teaching staff; however, these requirements are already included in the definition of time availability of lecturers, T_ℓ.

This type of constraints assures, for example, that for every lecturer, every period of Δ timeslots of teaching is preceded and followed by a break of length at least $\Delta - \delta$ timeslots.

F) Student requirements

F_1) Students of external interdisciplinary programmes should have lectures just in the morning, or in the evening, but not both. So one of the following two conditions has to be true, for every day and every student group belonging to external programmes. This constraint checks if some meeting happens in some part of day, so it is enough to check if some of corresponding y-variables has value 1. Thus, these constraints are modelled using y-variables:

$$\sum_{m\in M_s}\sum_{t\in T_d\setminus T_{AM}}\sum_{i\in H_m} y_{m,t,i} \leq 0 \quad \forall d\in D, \quad \forall s\in S_{AK}$$
$$\sum_{m\in M_s}\sum_{t\in T_d\setminus T_{PM}}\sum_{i\in H_m} y_{m,t,i} \leq 0 \quad \forall d\in D, \quad \forall s\in S_{AK}.$$

In such a case at least one of two conditions has to be satisfied and we introduce a variable $z_{p,i}$, $i \in I_p$, as discussed in Section 2.3.1 (IV). Since p represents the type of the constraint, for simplicity we will define p for this constraint to be equal to F_1. I_{F_1} is the corresponding index set, in this case defined as $I = S_{AK} \times D$, so the corresponding variable has indices $z_{F_1,s,d}$, for all $(s,d) \in I_{F_1}$.

Constants B_1 and B_2 are chosen to have sufficiently large values. In this case, we put both of them to be equal to 2, since it holds that all together in one day it is not desired for students of Applied Kinesiology programme to have more than two different meetings. Finally, the constraints are represented by the following inequalities:

$$\sum_{m\in M_s}\sum_{t\in T_d\setminus T_{AM}}\sum_{i\in H_m} y_{m,t,i} \leq 2\cdot z_{F_1,s,d} \quad \forall d\in D, \forall s\in S_{AK},$$
$$\sum_{m\in M_s}\sum_{t\in T_d\setminus T_{AM}}\sum_{i\in H_m} y_{m,t,i} \leq 2\cdot (1 - z_{F_1,s,d}) \quad \forall d\in D, \forall s\in S_{AK}.$$

F_2) It is desired for every student subgroup not to have meetings at two distinct locations in a day:

$$\sum_{i\in H_m} y_{m,t,i} + \sum_{i\in H_{m'}} y_{m',t',i} \leq 1, \quad \forall d\in D, \forall s\in S_{k_1}, \forall\{t,t'\}\in T_d,$$
$$\forall m\in M_{k_1}\cap M_s, \forall m'\in M_{k_2}\cap M_s.$$

6.4 Soft constraints

Among the implicitly generated feasible solutions, we would like to get the best one. In order to do that, we define an objective function, and an optimal solution is one that will give the minimal value to the objective function. If some soft constraint is violated, then the objective function value will grow. Soft constraints will be included in the objective function as the sum of relevant variables, multiplied by the corresponding weights. If a soft constraint of type p cannot be included in the objective function using x- and y- variables, it will be modelled using z-variables, namely $z_{p,i}$, for every $i \in I_p$, where I_p is the index set relevant for constraints of type p. In this section we introduce a representation of each soft constraint in order to construct the corresponding variables. Given a constraint of type p, variable $z_{p,i}$, if exists, has following definition:

$$z_{p,i} = \begin{cases} 1, & \text{if constraint of type } p \text{ is not satisfied for element } i \text{ of the index set } I_p, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

From equation (6.1) it follows that a violation of the constraint of type p, for some $i \in I_p$, can be represented by a penalty term in the objective function, defined as $w_i z_{p,i}$, where w_i is some positive weight. Obviously, a positive penalty increases the value of the objective function, so solutions containing violated soft constraints will have a larger value of the objective function.

The soft constraints are formalized as follows:

S_1) **Minimize use of payable classrooms.** Some classrooms are available for lecturing, but for an additional payment. It is desired to minimize the use of such classrooms. This constraint can be represented using existing variables, by adding element (6.2) to the objective function. Given a classroom $r \in R$ and a timeslot $t \in T$, we denote by $w_{S_1,r,t}$ the "cost" of using classroom at timeslot t. Observe that the weight $w_{S_1,r,t}$ depends on the choice of the room r and the timeslot t. In the case of UP FAMNIT the timeslot t has no influence to the weight value at the time of this writing, although weights defined by both indices are more general and can easily be adopted in case that choice of timeslot becomes important for the cost of classroom. A constraint is represented by:

$$\sum_{r \in R} \sum_{t \in T_r} \sum_{m \in M} w_{S_1,r,t} \cdot x_{m,t,r}. \tag{6.2}$$

S_2) **Compact timetable.** Timetable compactness can have more forms. One of them is from the lecturers' point of view. This constraint is related to grouping of teaching obligations of teaching staff, since it is not desirable for one teacher to have some teaching hours in the morning and then again at the evening, with a long break in between. Since the number of teachers who teach just one course, that is, who are incident just with one meeting, is not too small, here it makes sense to refer just to teachers teaching more than one session. Denote set of corresponding teachers by L_+. Then we introduce binary variable $z_{S_2,\ell,d}$, for every $(\ell, d) \in L_+ \times D$, where S_2 represents the type of constraint. As already mentioned, sets T_{AM} and T_{PM} represent morning and afternoon timeslots, respectively. Constraints representing S_2 are the following:

$$\sum_{m\in M_\ell} \sum_{t\in T_d\cap T_{PM}} \sum_{r\in R_m} x_{m,t,r} \leq 0, \quad \forall d \in D, \forall \ell \in L_+,$$
$$\sum_{m\in M_\ell} \sum_{t\in T_d\cap T_{AM}} \sum_{r\in R_m} x_{m,t,r} \leq 0, \quad \forall d \in D, \forall \ell \in L_+.$$

Using variables $z_{S_2,l,d}$ we get:

$$\sum_{m\in M_l} \sum_{t\in T_d\cap T_{PM}} \sum_{r\in R_m} x_{m,t,r} \leq B_1 z_{S_2,\ell,d}, \quad \forall d \in D, \forall \ell \in L_+,$$
$$\sum_{m\in M_\ell} \sum_{t\in T_d\cap T_{AM}} \sum_{r\in R_m} x_{m,t,r} \geq B_2(z_{S_2,\ell,d} - 1), \quad \forall d \in D, \forall \ell \in L_+.$$

Constants B_1 and B_2 have to have sufficiently large values. In this case it is sufficient for them to be equal to the number of morning and afternoon timeslots in a day, respectively. Thus, we define B_1 and B_2 to have values τ_{PM} and τ_{AM}, respectively.

For every variable $z_{S_2,\ell,d}$ there is also a corresponding weight $w_{S_2,\ell,d}$ used in objective function:

$$\sum_{\ell\in L_G} \sum_{d\in D} w_{S_2,\ell,d} \cdot z_{S_2,\ell,d}. \tag{6.3}$$

S_3) **Requirements related to students.** As mentioned in the description of the teaching process at the institution, there are some Master's study programmes that are supposed to offer lectures just at afternoons timeslots. Meetings relevant to these programmes belong to the set M_{PM}, and number of such meetings scheduled for earlier timeslots should be minimised. Timeslots defined as afternoon timeslots are contained in the set $T_{PM} \subset T$. If it is not possible to put all meetings from M_{PM} in afternoon slots, there can be some measure that decides which of the requirements are preferred to be satisfied. For that reason we introduce weights $w_{S_3,m}$ for every meeting $m \in M_{PM}$, describing the importance of meeting. A greater weight means that the course is more desirable to be scheduled in the afternoon. The number of undesirable assignments is minimized by adding the following element to the objective function:

$$\sum_{m\in M_{PM}} \sum_{t\in T\setminus T_{PM}} \sum_{i\in H_m} w_{S_3,m} \cdot y_{m,t,i}. \tag{6.4}$$

Another constraint related to students' preferences concerns minimization of lectures scheduled at Friday afternoon. Most students go home during the weekend so such lectures are undesirable. Timeslots contained here can be determined by $T_5 \cap T_{PM}$. Since there are some additional properties that can influence the priority of scheduling lectures at described timeslots (e.g., the number of students attending some meeting), here we define the corresponding weights, $w_{S_3,m,t}$, for every meeting $m \in M$ and timeslot $t \in T_5 \cap T_{PM}$. These preferences can be modelled by adding the following sum to the objective function:

$$\sum_{m \in M} \sum_{t \in T_5 \cap T_{PM}} \sum_{r \in R_m} w_{S_3,m,t} \cdot x_{m,t,r}. \tag{6.5}$$

A third constraint in this group of constraints concerns upper bound on number of teaching hours related to one student group in a day. These are parameters, say $\rho_S(s)$, which define the numbers representing the desirable upper bounds. Here we define variables $z_{S_3,i}$, for each $i \in I_{S_3}$, with S_3 representing this constraint, and I_{S_3} being set of pairs $(s, d) \in S \times D$. It means that for every pair representing a student group s and a day d there is a variable $z_{S_3,s,d}$, which determines if constraint of type S_3 is satisfied for (s, d). If constraint is not satisfied, the variable gets value 1, and 0 otherwise. The constraint is originally represented by the inequality

$$\sum_{m \in M_s} \sum_{t \in T_d} \sum_{r \in R_m} x_{m,t,r} \leq \rho_S(s), \quad \forall s \in S, \forall d \in D.$$

From this we evaluate conditions for $z_{S_3,s,d}$, using methods described in Section 2.3.1 (IV):

$$\sum_{m \in M_s} \sum_{t \in T_d} \sum_{r \in R} x_{m,t,r} \leq \rho_S(s) + B_1 z_{S_3,s,d}, \quad \forall s \in S, \forall d \in D,$$

$$\sum_{m \in M_s} \sum_{t \in T_d} \sum_{r \in R} x_{m,t,r} \geq \rho_S(s) + 1 - B_2(1 - z_{S_3,s,t}), \quad \forall s \in S, \forall d \in D.$$

In the above equations, constants B_1 and B_2 can be determined in a few different ways. One possibility is to define them to have values $B_1 = \tau - \rho_S(s)$ and $B_2 = \rho + 1$, so that the corresponding constraint S_3 is satisfied for $s \in S$ and $d \in D$ whenever the variable $z_{S_3,s,d}$ has value 0, and violated whenever the variable $z_{S_3,s,d}$ has value 1. Given a variable $z_{S_3,s,d}$ we define a corresponding weight $w_{S_3,s,d}$, for normalization with other weights of the model. If there is no need for weights, they can be set to have value 1. Minimization of violations is represented by the sum (6.6), which is added to the objective function:

$$\sum_{d \in D} \sum_{s \in S} w_{S_3,s,d} \cdot z_{S_3,s,d}. \tag{6.6}$$

S_4) **Requirements related to lecturers.** In Section 4.2 we defined sets $T(\ell)$, for each $\ell \in L$, representing available timeslots for lecturer ℓ. Even if timeslots are contained in the set $T(\ell)$, there are some of them that might be preferred by the lecturer. For that reason we introduce a soft constraint representing a measure of lecturers' preferences with respect to the timeslots that are assigned to teaching hours. For each pair of lecturer $\ell \in L$ and timeslot $t \in T(\ell)$ we define a weight $w_{S_4,\ell,t}$ representing the measure of preferences. A modelled constraint has the form:

$$\sum_{\ell \in L} \sum_{m \in M(\ell)} \sum_{t \in T} \sum_{r \in R_m} w_{S_4,\ell,t} \cdot x_{m,t,r}.$$

6.5 The objective and the size of the ILP

Putting together the sums described above, we can formulate the objective function of the ILP model as follows:

$$\sum_{t \in T_r} \sum_{m \in M} \sum_{r \in R_m} w_{S_1,r,t} \cdot x_{m,t,r} + \sum_{\ell \in L_+} \sum_{d \in D} w_{S_2,\ell,d} \cdot z_{S_2,\ell,d} +$$
$$\sum_{m \in M_{PM}} \sum_{t \in T \setminus T_{PM}} \sum_{i \in H_m} w_{S_3,m} \cdot y_{m,t,i} + \sum_{m \in M} \sum_{t \in T_5 \cap T_{PM}} \sum_{r \in R_m} w_{S_3,m,t} \cdot x_{m,t,r} +$$
$$\sum_{d \in D} \sum_{s \in S} w_{S_3,s,d} \cdot z_{S_3,s,d} + \sum_{\ell \in L} \sum_{m \in M(\ell)} \sum_{t \in T} \sum_{r \in R_m} w_{S_4,\ell,t} \cdot x_{m,t,r}.$$

For notational clarity, in this section we denote the cardinalities of sets $L, L_+, M, R, S, S_{AK}, T, T_d, G$ by $\lambda, \lambda_+, \mu, \rho, \sigma, \sigma_{AK}, \theta, \tau, \gamma$, respectively. The number of blocks of distinct lengths for each meeting $m \in M$ is denoted by $\omega(m)$, while $\rho(m)$ and $\mu(s)$ denote the number of rooms that are acceptable for meeting m, and the number of meetings incident with student group s, respectively. Note that, as there are 5 days belonging to set D, we have $\theta = 5\tau$. In total, the number of variables of described ILP is equal to

$$\mu\theta\rho + \mu\theta \sum_{m \in M} \omega(m) + 2\delta\sigma + \delta\lambda.$$

The numbers of x-, y- and z-variables is represented in Table 2. Computing the number of constraints is not that trivial. In Table 3 we list the number of constraints grouped by constraint type. The total number of constraints in the ILP is equal to the sum of all terms displayed in the right column of Table 3.

x-variables	$\theta\rho \sum_{m\in M} \rho(m)$
y-variables	$\theta \sum_{m\in M} \omega(m)$
z-variables related to constraint F_1	$\delta\sigma_{AK}$
z-variables related to constraint S_2	$\delta\lambda$
z-variables related to constraint S_3	$\delta\sigma$
Total	$\theta\rho \sum_{m} \rho(m) + \theta \sum_{m\in M} \omega(m) + \delta\sigma + \delta\sigma_{AK} + \delta\lambda$

Table 2: Number of variables of the described ILP model.

Constraint name	Number of corresponding constrains
A	$\lambda + \rho$
B	$(\sigma + \lambda + \rho)\,\theta$
C	$\mu\,(1 + \delta + w(m) + \theta + \theta w(m) + \theta\rho(m)(\rho(m) - 1))$
D	$\gamma + 2$
E	$3\delta\lambda$
F	$2\delta\sigma_{AK} + \sigma\delta\tau(\tau - 1)$
S_2	$2\delta\lambda_+$
S_3	$2\sigma\delta$

Table 3: Number of constraints of the ILP.

As we can see, degrees of polynomials representing the size of proposed ILP are not that big. Since the described ILP is supposed to schedule all meetings at the faculty, parameters concerned here (number of meetings, number of student groups) are expected to have relatively large values (in the range of several dozens, possibly over 100). Thus, for real input data we can expect to have an ILP with a large number of variables and constraints.

In the next chapter we describe the results obtained by implementation of the model on real input data.

7 Results

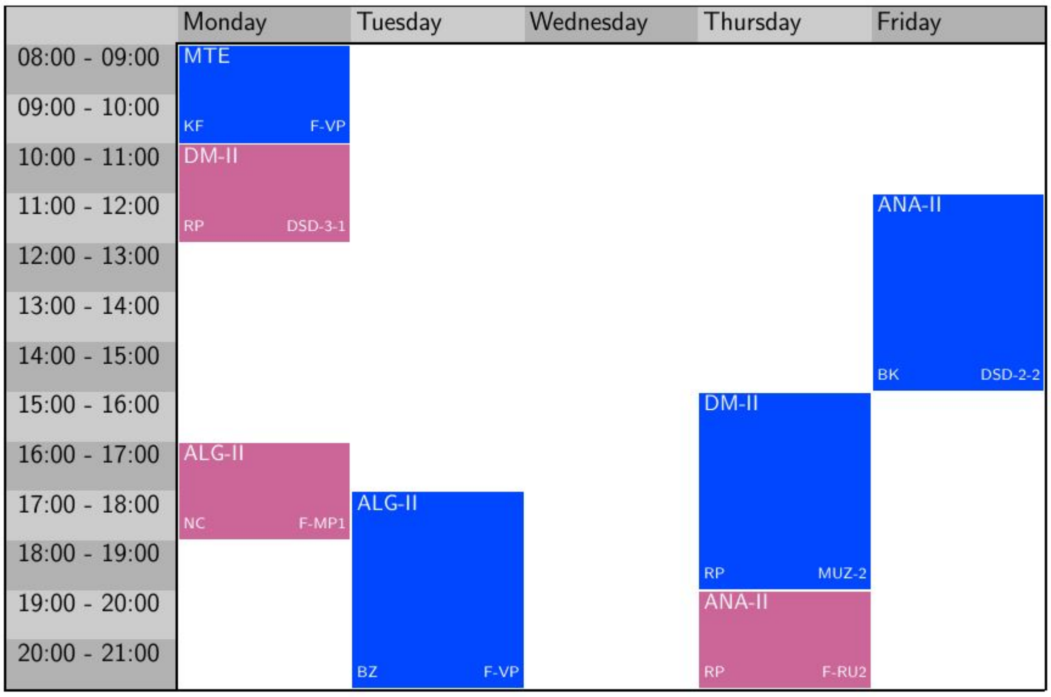
In this chapter we present results obtained using an implementation of the integer linear programming model derived in Chapter 6. The model was implemented using the open source programming software Zimpl, available from the web page http://zimpl.zib.de/, with user manual described in [34], and evaluated using the Gurobi Optimization software, available from the web page www.gurobi.com, with user manual contained in [42]. In what follows the Gurobi Optimization software will be referred to as a solver. The specifications of computer used for the computations are: RAM `32GB DDR3 1800Mhz` and CPU: `Intel i7-3820 3.60GHz`.

In order to find an optimal solution of the proposed model, we used input data for the Spring Semester of the academic year 2016/17 at the Faculty of Mathematics, Natural Sciences and Information Technologies, University of Primorska. Regarding these data, we have 185 meetings, 26 classrooms, and 65 timeslots. At the faculty there are 17 distinct study programmes that in total define 48 student groups, while the number of lecturers is equal to 118.

The timetable is supposed to be prepared for 5 working days and $\tau = 13$ timeslots within a day, each of them of the length equal to 60 minutes. Thus, in total there are 65 timeslots in the week. For the concrete data we define values of parameters introduced in the description of the model. We have: the overall upper bound on number of teaching timeslots of lecturer ℓ, $\rho_L(\ell) = 8$, for every lecturer $\ell \in L$; upper bounds with respect to meeting type, $\rho_1 = 6$, $\rho_2 = 8$. The upper bound regarding the students' requirements, $\rho_S(s)$ is defined to have the value equal to 9, for every student group s.

In Section 6.4 we introduced weights used for the formulation of the terms of objective function. These weights measure various criteria that are desired to be optimized in order to define the quality of the timetable. The number of weights is too large, so we do not list them here. All weights have values between 0 and 5. The objective function is being minimized so a large penalty weight means that we would really like corresponding variable to take value 0, meaning that the constraint is satisfied. This way the objective function value grows when an undesired feature of the schedule is present, since the corresponding weight multiplies by 1 and not by 0.

Using the Zimpl software we generated a `.lp` file representing the proposed model

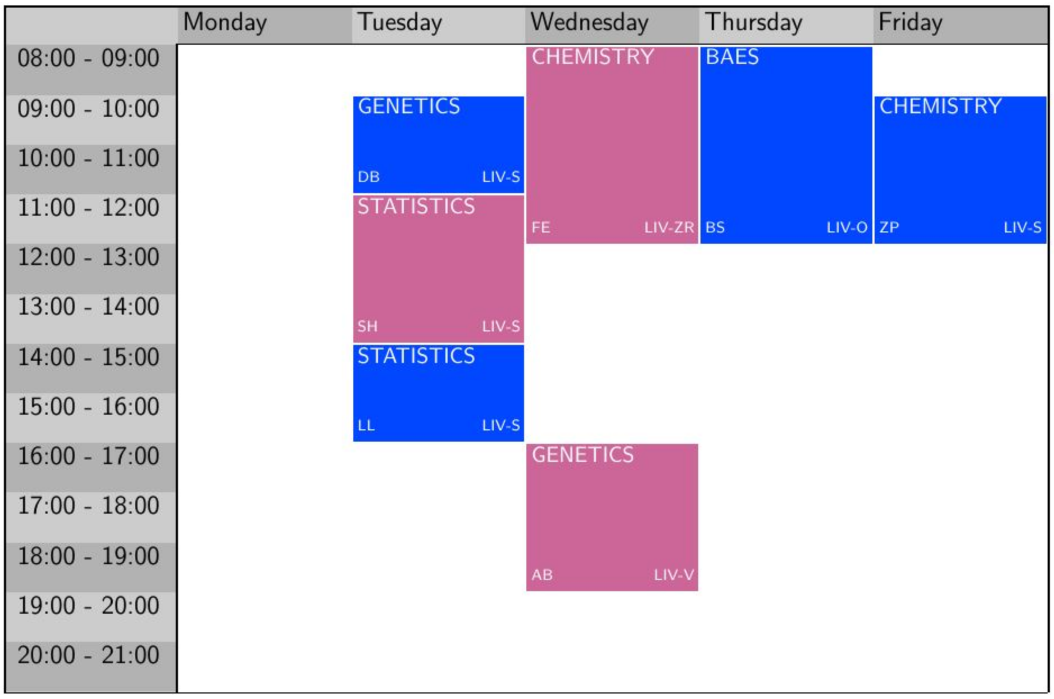


	Monday	Tuesday	Wednesday	Thursday	Friday
08:00 - 09:00	MTE				
09:00 - 10:00	KF F-VP				
10:00 - 11:00	DM-II				
11:00 - 12:00	RP DSD-3-1				ANA-II
12:00 - 13:00					
13:00 - 14:00					
14:00 - 15:00					BK DSD-2-2
15:00 - 16:00				DM-II	
16:00 - 17:00	ALG-II				
17:00 - 18:00	NC F-MP1	ALG-II			
18:00 - 19:00				RP MUZ-2	
19:00 - 20:00				ANA-II	
20:00 - 21:00		BZ F-VP		RP F-RU2	

Figure 7: The timetable obtained from the model for 1st year students of the study programme Mathematics.

for real data in ILP standard form, containing 171, 455 variables and 2, 752, 376 constraints, where 7, 780, 635 entries of corresponding matrix are nonzero. The resulting `.lp` file represents the input for the solver. As expected, since all variables of the ILP are constrained to be integral (binary), the complexity of the problem is very large. For that reason finding an optimal solution of the problem was a time-consuming process; we stopped the computation after 48 hours. During the 48 hours of computing, no feasible solution was found. Here let us mention that in the initial implementation we defined timeslots to have lengths equal to 30 minutes (as was the case in the manually computed timetable). However, such an implementation resulted in an almost doubled number of variables (more than 300, 000), which made even computing a feasible solution prohibitive. We thus decided to define the timeslots of length 60 minutes. In fact, in both cases (the complete model, timeslots of lengths first 60 and later 30 minutes) we were testing distinct parameters that can be set using Gurobi Optimization (Heuristics, MIPFocus), but this did not result in finding a feasible solution in a reasonable amount of time.

Our next attempt of solving the problem was to solve a changed model, where we simplified the objective function so that just the first term (the sum related to the constraint S_3, represented by inequality (6.4)) of the objective function remained in the model. We got an optimal solution in about $20,000s$. Since the objective function defined this way was not a proper measure of optimality, the corresponding solution



	Monday	Tuesday	Wednesday	Thursday	Friday
08:00 - 09:00			CHEMISTRY	BAES	
09:00 - 10:00		GENETICS			CHEMISTRY
10:00 - 11:00		DB LIV-S			
11:00 - 12:00		STATISTICS	FE LIV-ZR	BS LIV-O	ZP LIV-S
12:00 - 13:00					
13:00 - 14:00		SH LIV-S			
14:00 - 15:00		STATISTICS			
15:00 - 16:00		LL LIV-S			
16:00 - 17:00			GENETICS		
17:00 - 18:00					
18:00 - 19:00			AB LIV-V		
19:00 - 20:00					
20:00 - 21:00					

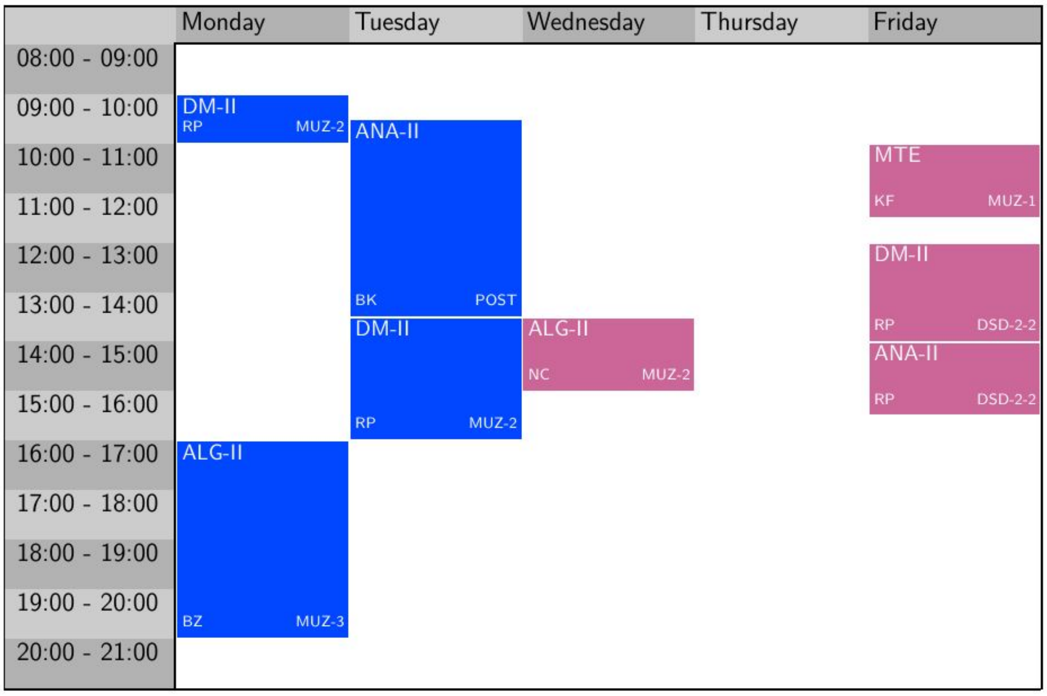
Figure 8: The timetable obtained from the model for 2nd year students of the study programme Biodiversity.

does not model all the desired aspects of a good timetible (since not all soft constraints are taken into account).

It is difficult to display the resulting timetables for all the programmes in this thesis, so we decided to concentrate on a small selection of student groups and to display the timetables representing the result of the implementation just for these student groups. In Figures 7 and 8 we can see the timetables corresponding to the undergraduate study programmes Mathematics of the first year (in the following: MA1) and Biodiversity of the second year (in the following: BI2), respectively. For all timetables displayed in the figures, the blue colour represents meetings of type "lectures" and the purple colour represents meetings of type "tutorials". Also, for every meeting, in the bottom left corner we can see the initials of the involved lecturer, and in the bottom right corner we can see the short name of the classroom in which meeting is supposed to take place. Classrooms' names containing the prefix "LIV" are located in the University Campus Livade – Izola. All the other classrooms are located in Koper.

In order to give the reader some feeling about the quality of the results obtained by the implementation, in Figures 9 and 10 we display the timetables prepared by hand, which were in use in the Spring semester of the academic year 2016/17, for the discussed student groups, MA1 and BI2, respectively.

The solution obtained with the implementation is optimal with respect to simplified objective function. That means that this solution is feasible in general, but with



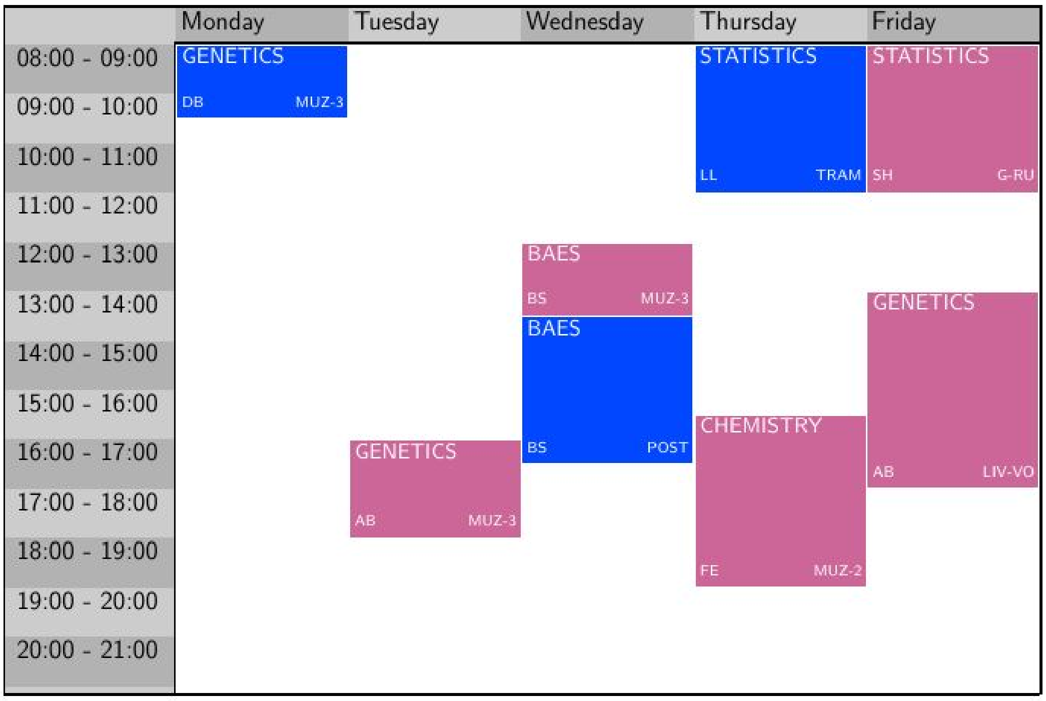
	Monday	Tuesday	Wednesday	Thursday	Friday
08:00 - 09:00					
09:00 - 10:00	DM-II RP MUZ-2	ANA-II BK POST			
10:00 - 11:00					MTE KF MUZ-1
11:00 - 12:00					
12:00 - 13:00					DM-II RP DSD-2-2
13:00 - 14:00		DM-II RP MUZ-2	ALG-II NC MUZ-2		
14:00 - 15:00					ANA-II RP DSD-2-2
15:00 - 16:00					
16:00 - 17:00	ALG-II BZ MUZ-3				
17:00 - 18:00					
18:00 - 19:00					
19:00 - 20:00					
20:00 - 21:00					

Figure 9: The manually prepared timetable for student group MA1.

respect to the whole objective function it can be far from the optimum. Although hard constraints are satisfied, some of the fundamental requirements measuring the quality of the timetable are not satisfied in this solution. Nevertheless, in some aspects the automatially generated timetable seems to be advantageous over the manually prepared one.

For example, if we compare the two timetables for the student group BI1, we can see that the timetable representing the result of implementation seems to be more acceptable for the students. It does not have as many holes during the day as it is the case in the manually prepared timetable. Also, in the model the constraint that an arbitrary student group cannot have lectures at two distinct locations at the same day is satisfied (it is a hard constraint of the model), while for the manually prepared timetable that is not the case, since students of student group BI2 have lectures at both locations on Friday.

On the other hand, a potential disadvantage of the timetable constructed automatically is a rather non-uniform distribution of lectures during the week, from the students' point of view. The term of the objective function representing this requirement is not considered by the solution process, so it has no influence on the objective function. However, as we can see in the figures, automatically prepared timetables are not worse than the manually prepared ones, with respect to the number of holes between the lectures. Also, our model has no constraint representing the required breaks between lectures, related to the students, so breaks between distinct meetings during



	Monday	Tuesday	Wednesday	Thursday	Friday
08:00 - 09:00	GENETICS DB MUZ-3			STATISTICS LL TRAM	STATISTICS SH G-RU
09:00 - 10:00					
10:00 - 11:00					
11:00 - 12:00					
12:00 - 13:00			BAES BS MUZ-3		
13:00 - 14:00			BAES BS POST		GENETICS AB LIV-VO
14:00 - 15:00					
15:00 - 16:00				CHEMISTRY FE MUZ-2	
16:00 - 17:00		GENETICS AB MUZ-3			
17:00 - 18:00					
18:00 - 19:00					
19:00 - 20:00					
20:00 - 21:00					

Figure 10: The manually prepared timetable for student group BI2.

the day are completely random. This can be solved so that we require one timeslot to be free of lectures after every meeting; for now this constraint is not really needed. For example, a meeting that is assigned to 4 consecutive hours of lectures is supposed to be taught for a total of $4 \cdot 45' = 3$h, and so within the 4 hours on the schedule the lecture time may be organized in a way that allows for a $30'$ break for the students before they go to the next class. A similar observation holds for tutorial sessions of $2 \cdot 45' = 90'$, which are often carried out in one go without a break, resulting in another half an hour break for the students.

Now let us consider the use of classrooms. There is one classroom in the model, denoted by "ROT", that can be used, but has the greatest possible weight - 5. That means that the use of that classroom is really undesired, but if there is no other option, we can use that classroom for scheduling. In the results obtained here we have that this classroom is used for 15 timeslots, which represents about the 23% of the all timeslots in the week. In comparison with the use of other classrooms that have smaller weights, we can be satisfied with this result. On the other hand, if we consider the use of computer classrooms, we get that computer classroom denoted as "RU1" is occupied for 43 timeslots in the week, where 41 of them are meetings that are supposed to be scheduled in the computer room. That means that the correct usage of this classroom represents the 95% of the whole usage.

We conclude that, generally, the result obtained here represents a good solution for the timetabling problem (that is, a solution that could be used in the application). The existence of a feasible solution represents a good first step in the process of automating the approach to the timetabling problem at UP FAMNIT, as the number of variables is large for relatively fast solving.[1] A few steps have to be taken care of before a complete automation of the process as until now the timetabling problem at UP FAMNIT was solved manually. One of them is the preparation of the data for the model. Manual timetabling allows to include a human note that breaks hard constraints in some situations in order to achieve a better overall result. However, it is difficult if not even impossible to define the constraints in such a way to enable a computer program to achieve the same results. There is no possibility for computer to think about "importance" of some of the hard constraints. Since the work of this thesis represents the first attempt towards automation of the timetabling problem at UP FAMNIT, we expect that some improvements with the more technical part of the whole process of solving the problem will take place in the near future (priority: a simplified collecting of the data).

The timetable constructed here could be used in the teaching process of the faculty, with possibility of minor changes made by human, in order to make it even better. For now, we will try to make further improvements to the model so that we can get a solution with respect to the whole objective function in a reasonable amount of time. The first task for future research is to reduce the number of binary variables and to implement some terms of the objective function in a different way, so that we can at least get a reasonably good solution for the whole model relatively quickly.

[1]We say that a solution is found relatively fast if it takes less time to find it using the proposed model, than to construct it by hand.

8 Conclusion

In the thesis we considered a university timetabling problem (UTP). The UTP can have many definitions, depending on the resources that are supposed to be scheduled to time intervals. In this work we described the teaching process at the Faculty of Mathematics, Natural Sciences and Information Technologies at University of Primorska (UP FAMNIT). Based on that description, we formalized the definition of the timetabling problem at UP FAMNIT, referred to as the FAMNIT TIMETABLE DESIGN (FTD) problem, and proved that so defined problem is *NP*-complete.

Unless $P \neq NP$, the *NP*-completeness of the problem implies non-existence of an efficient solution for the problem. Among the few approaches for modelling timetabling problems, reviewed in Section 3.2, we decided to model the problem using concepts of integer linear programming. We developed an ILP model representing the FTD problem, consisting of binary variables, constraints representing requirements of the faculty, and an objective function measuring the quality of feasible solutions.

The model proposed in this work is implemented for the real input data using the relevant software. As already discussed, the number of binary variables of the ILP is larger than $150,000$, so it is expected that solving the problem to optimality will take a long time. For the given data we did not manage to get an optimal solution of the complete ILP as described in Chapter 6. When solving the simplified ILP, or more precisely the ILP with the same set of constraints and with the objective function containing just one of the main terms, an optimal solution was found in a relatively short time given the size of the problem. We can conclude that besides the large number of binary variables, the objective function also has a big influence on the amount of time needed to get an optimal solution.

One of the possible directions for the improvement of the ILP model developed in thesis would be a reduction in the number of variables. One of the ideas of how to do that is to discard variables with indices that are supposed to have value 0 (e.g., time availabilities of professors/classrooms could be respected during the initialization of variables). Also, during the testing of developed model, we realized that there are constraints that have a big influence on the amount of time needed to solve the problem, so we believe that separating a problem into two subproblems, containing “weak” and “hard” constraints,, respectively, could give us a good solution approach.

9 Povzetek dela v slovenskem jeziku

Problemi sestavljanja urnikov na univerzah so praktični *NP*-težki problemi, ki jih je težko rešiti do optimalnosti. V literaturi je na voljo veliko različnih modelov za tovrstne probleme, ki temeljijo na metodah teorije grafov, kombinatorične optimizacije, logičnega programiranja z omejitvami (ang. constraint logic programming) in/ali celoštevilskega (ali mešanega celoštevilskega) programiranja. Modeli vodijo do optimalnih eksaktnih algoritmov eksponentne časovne zahtevnosti ali do hevrističnih algoritmov polinomske časovne zahtevnosti. Sestavljanje urnika predstavlja zelo specifičen problem, ki je zelo odvisen od posebnosti pogojev, ki morajo veljati zanj. Zaradi tega ne obstaja splošna rešitev problema in je težko algoritem oziroma model, ki je že narejen v literaturi, uporabiti v drugem konkretnem primeru.

V magistrskem delu je podan pregled pomembne in nedavne literature s področja, povzeti so različni modeli in pristopi za reševanje problema urnika, s prednostimi in pomanjkljivostmi. Podan je pregled nekaterih teoretičnih pojmov in rezultatov, povezanih z računsko zahtevnostjo problemov, kombinatorično optimizacijo, linearnim programiranjem in celoštevilskim linearnim programiranjem. Tako so predstavljeni razredi kompleksnosti problemov, kot so razred *P*, *NP*, ter razred *NP*-polnih problemov. Obstoj *NP*-polnih problemov ni očiten, zato je v nalogi naveden rezultat, ki obstoj tovrstnih problemov zagotavlja, znan kot Cookov izrek. Z uporabo Cookovega izreka in polinomskih prevedb problemov je podana karakterizacija *NP*-polnih problemov, ki bistveno olajša postopek dokazovanja *NP*-polnosti določenega problema.

Glede na to da je v delu problem univerzitetnega urnika modeliran z uporabo celoštevilskega linearnega programiranja, je nekaj strani namenjenih teoriji s tega področja. Podani so osnovni rezultati teorije linearnega programiranja, med katerimi so najbolj znani Farkaseva lema in izrek o dualnosti linearnega programa. Definirano je celoštevilsko linearno programiranje, ki predstavlja enega izmed prvih dokazano *NP*-polnih problemov. Razložena je računska zahtevnost celoštevilskega linearnega programiranja in različni pristopi za reševanje celoštevilskih linearnih programov: ravninski rezi, metoda razveji-in-omeji, dinamično programiranje itn.

S ciljem čim bolje razumeti problem, je v delu podana splošna definicija problema urnika, ter kratka razlaga njegove raunske zahtevnosti, saj je splošni problem urnika dokazano *NP*-poln. Posebna pozornost je namenjena pregledu literature, ki se ukvarja zgolj s problemom določitve urnika predavanj na univerzah. Pogoji, ki morajo veljati za posamezen urnik, namreč lahko pomembno vplivajo na časovno zahtevnost problema. V nalogi so predstavljeni primeri pogojev, pod katerimi je problem še vedno rešljiv v polinomskem času, ter pogoji, ki zelo vplivajo na zvišanje računske zahtevnosti problema. Problem urnika oziroma njegove različice so natančno opisane, vključno z vsemi spremenljivkami in omejitvami. Natančno je definirana posebna vrsta problema, poimenovana Famnit Timetable Design (FTD), ki ustreza problemu urnika na Fakulteti za matematiko, naravoslovje in informacijske tehnologije Univerze na Primorskem (UP FAMNIT). Kot nov rezultat je izpeljan dokaz *NP*-polnosti FTD problema, in sicer s konstrukcijo polinomske prevedbe z že znanega *NP*-polnega problema.

V drugem delu magistrskega dela je poudarek na izpeljavi matematičnega modela problema v obliki celoštevilskega linearnega programa. Najprej so natančno opisane podrobnosti izvajanja študijskega procesa na UP FAMNIT. Predstavljeni so pogoji, ki morajo biti zadoščeni pri pripravi urnika (t.i. trdi pogoji), ter pogoji, ki so zaželjeni, vendar niso nujni za postavitev urnika (t.i. mehki pogoji). V posebnem poglavju je opisan celoštevilski linearen program za problem urnika na UP FAMNIT, s spremenljivkami in pogoji. Program je implementiran z uporabo programskega paketa Zimpl (prosto dostopnim na strani http://zimpl.zib.de/) in rešen na konkretnih primerih z uporabo programskega paketa Gurobi (dostopnim na strani http://www.gurobi.com/).

Model je preizkušen na konkretnih podatkih, in sicer za fakultetni urnik za spomladanski semester študijskega leta 2016/17. Zaradi velikega števila spremenljivk je bilo reševanje problema dolgotrajno. Dobljena rešitev tako ni nujno optimalna, ampak le dovolj blizu optimuma, oziroma predstavlja rešitev, ki ima najboljšo vrednost kriterijske funkcije, dobljena v določenem (omejenem) časovnem intervalu. Dobljena rešitev je primerjana z ročno sestavljenim urnikom, tako da je za nekaj škupin študentov predstavljen dejanski urnik predavanj na fakulteti za obravnavani semester in urnik, dobljen kot rezultat implementacije in reševanja celoštevilskega linearnega programa.

Opisan celoštevilski linearen program vsebuje veliko število binarnih spremenljivk, ki zvišajo čas, potreben za reševanje problema, zato je v zaključku predlaganih nekaj idej za pospešitev reševanja programa, v obliki zmanjšanja števila spremenljivk in poenostavitve kriterijske funkcije.

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