Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 1, 89–128
DOI: 10.1214/13-AIHP576

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Moderate deviations for stabilizing functionals in geometric probability

P. Eichelsbacher[a], M. Raič[b] and T. Schreiber[c,1]

[a]*Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstraße 150, D-44801 Bochum, Germany. E-mail:* peter.eichelsbacher@rub.de
[b]*University of Ljubljana (FMF) and University of Primorska (FAMNIT), Jadranska 21, SI-1000 Ljubljana, Slovenia.*
E-mail: martin.raic@fmf.uni-lj.si
[c]*Faculty of Mathematics and Computer Science, Nicholas Copernicus University, Toruń, Poland*

Received 8 October 2010; revised 28 June 2013; accepted 7 July 2013

Abstract. The purpose of the present paper is to establish explicit upper and lower bounds on moderate deviation probabilities for a rather general class of geometric functionals enjoying the stabilization property, under Poisson input and the assumption of a certain control over the growth of the moments of the functional and its radius of stabilization. Our proof techniques rely on cumulant expansions and cluster measures. In addition, we establish a new criterion for the limiting variance to be non-degenerate. Moreover, our main result provides a new central limit theorem, which, though stated under strong moment assumptions, does not require bounded support of the intensity of the Poisson input. We apply our results to three groups of examples: random packing models, geometric functionals based on Euclidean nearest neighbors and the sphere of influence graphs.

Résumé. L'objectif de cet article est d'établir une majoration et une minoration explicite pour les probabilités des déviations modérées d'une classe assez générale de fonctionnelles géométriques possédant une propriété de stabilisation pour des données de Poisson et sous l'hypothèse d'un contrôle de la croissance des moments de la fonctionnelle et de son rayon de stabilisation. Les techniques utilisées dans les preuves reposent sur des développements de cumulants et des mesures de clusters. En outre, nous proposons un nouveau critère pour que la variance limite soit non-dégénérée. De plus, notre résultat principal fournit un nouveau théorème central limite, qui, bien que formulé sous une hypothèse assez forte sur les moments, ne nécessite pas que l'intensité des données de Poisson ait un support borné. Nous appliquons nos résultats à trois groupes d'exemples: les modèles de pavages aléatoires, les fonctionnelles géométriques dépendantes des voisins les plus proches en distance euclidienne et les graphes des sphères d'influence.

MSC: 60F10; 60D05

Keywords: Stabilizing functionals; Moderate deviations; Explicit bounds; Cumulants; Random packing; Random graphs

1. Introduction, main results

1.1. *Introduction*

Stabilization is an important concept expressing in natural geometric terms mixing properties of a broad class of functionals of point processes arising in geometric probability, see [3,27,28]. Even though these processes are presumably also tractable using more traditional mixing concepts, stabilization-based techniques proved extremely convenient in

[1]Research partially supported by the Polish Minister of Science and Higher Education grant N N201 385234 (2008-2010).
Tomasz Schreiber passed away on 1st December 2010.

studying the asymptotic behavior of large random geometric systems. This is due to the geometric nature of these methods which makes them compatible with many stochastic geometric set-ups in which the target functionals arise. In particular, stabilization is often helpful in establishing a direct connection between the microscopic (local) and macroscopic properties of the processes studied, see ibidem for further details. Stabilization has been successfully used to establish laws of large numbers for many functionals [24,28,29] and it has also been employed in a general setting to establish Gaussian limits for re-normalized functionals as well as re-normalized spatial point measures [3,22,25–27]. The functionals to which the afore-mentioned theory applies include those defined by percolation models [22], random graphs in computational geometry [3,27], random packing models [2,28], and the germ–grain models [3]. Large deviation principles for stabilizing functionals have also been established, see [34]. Finally, the corresponding moderate deviation principle, interpolating between the central limit theorem and law of large numbers, has been obtained in [1] and [13], for a rather limited sub-class of the above examples though, namely for empirical functionals of random sequential packing, nearest neighbor graphs and germ–grain models.

In this paper, we use stabilization combined with cumulant expansion techniques in the spirit of [3] to prove moderate deviation bounds for three groups of geometric functionals: random sequential packing along with birth–growth models, functionals based on nearest neighbors and sphere of influence graphs. In each case, we consider a much more general class of geometric functionals, based on a newly introduced concept of *confinement*. In particular, in the birth–growth models, we lift the unnatural lower bound on grain sizes. Moreover, our deviation probability estimates are much more explicit than those established in [1]. In addition, we derive moderate deviation principles in the τ-topology, which is stronger than the weak topology, which is used in [1].

We assume Poisson input with a density, which is assumed to be bounded and integrable, but need not have bounded support. This fact adds to existing central limit theorems, which, to the best of our knowledge, all require the latter assumption: see Remark 1.13. In particular, bounded support is not needed in the first group of applications, random packing models. On the other hand, in the other two groups, nearest neighbors and sphere of influence graphs, bounded support is required in order to ensure stabilization.

On the other hand though, for the applications considered in [1], our approach provides a much narrower range of scaling regimes to which moderate deviation results apply. In particular, our results in the current form do not seem to provide moderate deviation principles in the full range. However, in [1], it is assumed that the random measures associated to the functionals and the Poisson point process can be suitably coupled with their exponential (Gibbs) modifications. In the present paper, we assume no such structure; as a result, the application of our main results is much more straightforward. Moreover, future refinements might actually yield full range moderate deviation principles in some cases: see Remark 1.14.

However, it is well known that many natural multidimensional stochastic systems exhibiting various types of exponential mixing often satisfy the Gaussian moderate deviation principle only up to a certain point beyond the CLT scale, whereafter the Gaussian behavior breaks down and gets replaced by phenomena of a different nature. As a spectacular example, consider the phase separation, condensation and droplet creation as established for many statistical mechanical models in phase transition regime, see the seminal monograph [11] as well as the survey [5]. Consequently, we believe that the Gaussian moderate deviation principle may well be violated by geometric stabilizing functionals for ranges far enough from the CLT regime. Even though we are definitely not in a position to claim that the ranges of Gaussian behavior for deviation probabilities established in this paper are optimal, we should most likely not hope to get a full range Gaussian moderate deviation principle at the level of generality considered here.

1.2. *Terminology and notation*

We begin with some common notation. First, denote $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Next, for a set A, denote by $|A|$ its cardinality. Throughout this paper, fix $d \in \mathbb{N}$. For $x \in \mathbb{R}^d$, denote by $\|x\|$ its Euclidean norm and, furthermore, for $A \subseteq \mathbb{R}^d$, denote by $\operatorname{dist}(x, A)$ the Euclidean distance from x to A. Next, denote by $\mathbf{0}$ the origin in $\mathbb{R}^d$.

Unless specified otherwise, the expression 'measurable' will mean 'Borel-measurable' when applied to subsets of $\mathbb{R}^d$. For a measurable set $A \subseteq \mathbb{R}^d$, denote by $\operatorname{vol}(K)$ its Lebesgue measure. Throughout this paper, the letter Ω will denote a measurable subset of $\mathbb{R}^d$, which will be called a *domain*. Next, the letter κ will denote a probability density function on $\mathbb{R}^d$, vanishing on $\mathbb{R}^d \setminus \Omega$. Abusing the notation slightly, κ will sometimes also denote the corresponding probability measure, i.e., $\kappa(x)\,\mathrm{d}x$. In particular, 'κ-almost everywhere' will mean 'for $\kappa(x)\,\mathrm{d}x$-almost all x'.

Furthermore, we let $\langle f, \mu \rangle$ denote the integral with respect to a signed finite variation Borel measure μ of a μ-integrable function f. For a measurable set $W \subseteq \mathbb{R}^d$, we write $\mathcal{B}(W)$ for the collection of bounded measurable $f : W \to \mathbb{R}$. Finally, we shall assume that $0/0 = 0$ (and $a/0 = +\infty$ for $a > 0$ and $-\infty$ for $a < 0$). The essential supremum of f with respect to a measure μ will be denoted by $\operatorname{ess\,sup}_{\mu(\mathrm{d}x)} f(x)$. In particular, for a non-negative function g, $\operatorname{ess\,sup}_{g(x)\,\mathrm{d}x} f(x)$ will denote the essential supremum of f with respect to the Lebesgue measure restricted to the set $\{x; g(x) > 0\}$.

Next, we introduce *marked points*. Let $(\mathcal{M}, \mathcal{F}_{\mathcal{M}}, \mathbb{P}_{\mathcal{M}})$ be a probability space (*mark space*). Marked points will be the elements of $\breve{\mathbb{R}}^d := \mathbb{R}^d \times \mathcal{M}$ and will be usually denoted by a breve accent. We shall use the following convention: if a letter with a breve accent denotes a marked point and the same letter without the accent appears in the same context, both will refer to the same location. More formally, when $\breve{x} \in \breve{\mathbb{R}}^d$ and x appear in the same context, we shall always assume that $\breve{x} = (x, t)$ for some $t \in \mathcal{M}$. Similarly, for a set $\breve{\mathcal{X}} \subseteq \breve{\mathbb{R}}^d$, $\mathcal{X}$ in the same context will denote the set $\{x; (\exists t \in \mathcal{M})\ (x, t) \in \breve{\mathcal{X}}\}$. A subset $\breve{A} \subseteq \breve{\mathbb{R}}^d$ will be called *measurable* if it is measurable with respect to the product of the Borel σ-algebra on $\mathbb{R}^d$ and the σ-algebra $\mathcal{F}_{\mathcal{M}}$.

Common Conventions. *Let $z \in \mathbb{R}^d$ and $a \in \mathbb{R}$. For $\breve{x} = (x, t) \in \breve{\mathbb{R}}^d$, put $\breve{x} + z := (x + z, t)$ and $a\breve{x} := (ax, t)$. For a set $\breve{\mathcal{X}} \subseteq \breve{\mathbb{R}}^d$, put $\breve{\mathcal{X}} + z := \{\breve{x} + z; \breve{x} \in \breve{\mathcal{X}}\}$ and $\lambda\breve{\mathcal{X}} := \{\lambda\breve{x}; \breve{x} \in \breve{\mathcal{X}}\}$. For sets $\breve{A} \subseteq \breve{\mathbb{R}}^d$ and $B \in \mathbb{R}^d$, define $\breve{A} \cap B := \breve{A} \cap (B \times \mathcal{M})$ and $\breve{A} \setminus B := \breve{A} \setminus (B \times \mathcal{M})$.*

We shall often integrate over $\breve{\mathbb{R}}^d$. For that purpose, we extend the meaning of differential to integration with respect to the product of the Lebesgue measure and $\mathbb{P}_{\mathcal{M}}$. More formally, we define:

$$\int_{\breve{\mathbb{R}}^d} f(\breve{x})\,\mathrm{d}\breve{x} := \int_{\mathbb{R}^d} \mathbb{E} f(x, T)\,\mathrm{d}x, \tag{1.1}$$

where T is a generic random mark with distribution $\mathbb{P}_{\mathcal{M}}$.

For $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ denotes the closed Euclidean ball of radius r centered at x. A set $\breve{\mathcal{X}} \subseteq \breve{\mathbb{R}}^d$ will be called *configuration* if it is locally finite with respect to the product of the Euclidean topology on $\mathbb{R}^d$ and the indiscrete topology on $\mathcal{M}$, i.e., if for each $x \in \mathbb{R}^d$, the set $\breve{\mathcal{X}} \cap B_r(x)$ is finite for some $r > 0$ (recall from the Common Conventions that $\breve{\mathcal{X}} \cap B_r(x)$ is interpreted as $\breve{\mathcal{X}} \cap (B_r(x) \times \mathcal{M})$). A configuration which is also a subset of $\breve{\Omega} := \Omega \times \mathcal{M}$ will be called a *configuration on Ω*. We shall keep the meaning of $\breve{\Omega}$ throughout the paper.

On the set of all configurations, we shall consider the standard σ-algebra defined as the smallest σ-algebra, such that the map $\breve{\mathcal{X}} \mapsto |\breve{\mathcal{X}} \cap \breve{A}|$ is measurable for all measurable sets $\breve{A} \subseteq \breve{\mathbb{R}}^d$. Thus, a subset of the set of all configurations will be called measurable if it is measurable with respect to the latter σ-algebra.

Now we turn to our main object, geometric functionals and the associated random measures. We first refine the definition of a geometric functional from [3,29].

Definition 1.1. *A* geometric functional *defined on a measurable class $\mathcal{C}$ of configurations is a measurable map defined for all pairs $(\breve{x}, \breve{\mathcal{X}})$, where $\breve{\mathcal{X}} \in \mathcal{C}$ and $\breve{x} \in \breve{\mathcal{X}}$. If the class $\mathcal{C}$ is not specified, we shall take the class of all* finite *configurations. Another example of a class that we shall frequently consider are all configurations on a domain Ω.*

Although the main object of our paper will be real-valued functionals, we shall also need $[0, \infty]$-valued as well as even set-valued functionals.

Now we turn to *stabilization*, which plays an essential role in all that follows.

Definition 1.2. *A geometric functional ξ* stabilizes *at $\breve{x}$ with respect to a configuration $\breve{\mathcal{X}} \subseteq \breve{\Omega}$ inside a domain Ω, if there exists a finite $\rho \geq 0$, such that:*

$$\xi(\breve{x}, \breve{\mathcal{Y}}) = \xi\big(\breve{x}, \breve{\mathcal{X}} \cap B_\rho(x)\big) \tag{1.2}$$

for all finite configurations $\breve{\mathcal{Y}} \subseteq \breve{\Omega}$ with $\breve{\mathcal{Y}} \cap B_\rho(x) = \breve{\mathcal{X}} \cap B_\rho(x)$. In other words, the interaction between $\breve{x}$ and a point set is unaffected by changes outside $B_\rho(x)$. In particular, for $r \geq \rho$, $\xi(\breve{x}, \breve{\mathcal{X}} \cap B_r(x))$ does not depend on r. If (1.2) holds, we shall say that ξ stabilizes at $\breve{x}$ within radius ρ *with respect to $\breve{\mathcal{X}}$ inside Ω. If Ω is not specified, we shall take $\Omega = \mathbb{R}^d$.*

We shall say that ξ *is* stabilizing *with respect to configuration* $\check{\mathcal{X}}$ *if it stabilizes with respect to* $\check{\mathcal{X}}$ *at each* $\check{x} \in \check{\mathcal{X}}$ (*all inside a domain* Ω).

A radius of stabilization *for a geometric functional* ξ *inside* Ω *is a* $[0, \infty]$-*valued geometric functional* R *defined on configurations on* $\check{\Omega}$, *such that for all suitable* $\check{\mathcal{X}}$ *and all* $\check{x} \in \check{\mathcal{X}}$, ξ *stabilizes at* $\check{x}$ *within radius* $R(\check{x}, \check{\mathcal{X}})$, *provided that* $R(\check{x}, \check{\mathcal{X}})$ *is finite*.

Geometric functionals, which are initially defined on finite configurations, can be extended to all configurations (i.e., locally finite sets) with respect to which they stabilize. More precisely, if a geometric functional ξ stabilizes at $\check{x}$ with respect to configuration $\check{\mathcal{X}}$, we can define:

$$\xi(\check{x}, \check{\mathcal{X}}) := \lim_{r \to \infty} \xi\bigl(\check{x}, \check{\mathcal{X}} \cap B_r(x)\bigr). \tag{1.3}$$

Clearly, the extended functional stabilizes within the same radius.

Among others, this extension allows us to consider geometric functionals on Poisson point processes. For a locally integrable function $f : \mathbb{R}^d \to [0, \infty)$, denote by $\check{\mathcal{P}}_f$ a Poisson point process with intensity $f \otimes \mathbb{P}_{\mathcal{M}}$ (sometimes, we shall abuse the notation slightly and identify f with the corresponding measure $f(x)\,\mathrm{d}x$). By locally integrable function, we mean that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, i.e., for each $x \in \mathbb{R}^d$, there exists some neighborhood U, such that $f \in L^1(U)$.

Remark 1.1. *It would also be desirable extend the results to the case when the Poisson input is replaced by binomial* (*i.e.*, *a point process consisting of independent and identically distributed random points*) *– so-called* de-Poissonization. *However, this is beyond the scope of the present paper. In particular, the crucial construction* (3.16) *seems not to work. Instead, one can either attempt to find a more sophisticated construction or refer to suitable convergence of binomial point processes to Poisson. In this context, coupling can be used to advantage*: *see Lemma* 4.2 *of* [27] *and Lemma* 3.2 *of* [24].

Throughout this paper, the letter ξ will be reserved for geometric functionals (but we shall also consider geometric functionals denoted by other letters). For a geometric functional denoted by ξ, define its scaled versions by:

$$\xi_\lambda(\check{x}, \check{\mathcal{X}}) = \xi\bigl(\lambda^{1/d}\check{x}, \lambda^{1/d}\check{\mathcal{X}}\bigr)$$

(recalling that the $\lambda^{1/d}$ scaling only modifies the spatial component, not the mark). Our principal objects of interest are the following random point measures on $\mathbb{R}^d$:

$$\mu_\lambda := \sum_{\check{x} \in \check{\mathcal{P}}_{\lambda\kappa}} \xi_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\kappa})\delta_x; \quad \lambda > 0, \tag{1.4}$$

or, equivalently,

$$\langle f, \mu_\lambda \rangle := \int_{\check{\mathbb{R}}^d} f(x)\xi_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\kappa})\check{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\check{x}),$$

along with $\bar{\mu}_\lambda := \mu_\lambda - \mathbb{E}\mu_\lambda$, the centered versions of μ_λ. Thus, in μ_λ and $\bar{\mu}_\lambda$, we shall always refer to an $\mathbb{R}$-valued geometric functional denoted by ξ and a probability density function on $\mathbb{R}^d$ denoted by κ, which will be suppressed in the notation.

Throughout this paper, unless specified otherwise, the letter R will denote a radius of stabilization for a geometric functional denoted by ξ. Moreover, for $\lambda > 0$, denote by R_λ a non-negative geometric functional defined on locally finite sets, such that $\lambda^{-1/d}R_\lambda$ is a radius of stabilization for ξ_λ inside a domain denoted by Ω, or, equivalently, such that the functional $(\check{x}, \check{\mathcal{X}}) \mapsto R_\lambda(\lambda^{-1/d}\check{x}, \lambda^{-1/d}\check{\mathcal{X}})$ is a radius of stabilization for ξ inside $\lambda^{1/d}\Omega$. Of course, if $\Omega = \mathbb{R}^d$, one can simply put $R_\lambda(x, \mathcal{X}) := R(\lambda^{1/d}\check{x}, \lambda^{1/d}\check{\mathcal{X}})$.

In most of our results, we restrict attention to translation invariant functionals.

Definition 1.3. *A geometric functional* ξ *is* translation invariant *if* $\xi(\check{x} + z, \check{\mathcal{X}} + z) = \xi(\check{x}, \check{\mathcal{X}})$ *for all* $\check{x} \in \check{\mathbb{R}}^d$, *all finite configurations* $\check{\mathcal{X}} \subset \check{\mathbb{R}}^d$ *and all* $z \in \check{\mathbb{R}}^d$.

Remark 1.2. *Though most of our main results are formulated under the assumption of translation invariance, the latter is not crucial. In particular, it is not required in our key Lemma 3.1. However, our main results are derived by combining Lemma 3.1 with (1.7), which is known to hold under either translation invariance or more complicated assumptions (see [25]). Anyway, translation invariance is the case in all our applications.*

1.3. *Known results*

From [25] (see also earlier papers [3,29]), we know that under relatively mild assumptions (compared to those stated in Section 1.5), the one and two point correlation functions for $\xi_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\kappa})$ converge in the large λ limit, which establishes the corresponding asymptotics for integrals $\mathbb{E}\langle f, \mu_\lambda\rangle$ and $\sigma_\lambda^2[f] := \mathrm{Var}(\langle f, \mu_\lambda\rangle)$ (*we shall always take* $\sigma_\lambda[f] \geq 0$). Moreover, under additional assumption that κ has bounded support, it is known that the limit of the re-normalized measures $\lambda^{-1/2}\bar{\mu}_\lambda$ is a generalized mean zero Gaussian field in the sense that the finite-dimensional distributions of $\lambda^{-1/2}\bar{\mu}_\lambda$ over $f \in \mathcal{B}(\mathbb{R}^d)$ converge to those of a Gaussian field (for test functions, we may take all $f \in \mathcal{B}(\mathbb{R}^d)$, but only the ones with support coinciding with the support of κ really matter).

To state the results in formal terms, we introduce various assumptions. The first one will be imposed on the density κ.

Assumption D ***(Density).*** *κ is bounded and Lebesgue-almost everywhere continuous.*

Next, we list two assumptions imposed on a family $(g_\lambda)_{\lambda>\lambda_0}$ of geometric functionals either taking values in $\mathbb{R}$ or in $[0, \infty]$.

Assumption $\mathrm{M}(p,\kappa)$ ***(pth Moment).***

$$\limsup_{\lambda\to\infty} \operatorname*{ess\,sup}_{\kappa(x)\,\mathrm{d}\check{x}} \mathbb{E}\bigl|g_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\kappa})\bigr|^p < \infty.$$

Assumption $\mathrm{M1}(p,\kappa)$ ***(pth Moment with One additional point).*** *$(g_\lambda)_\lambda$ satisfies Assumption $\mathrm{M}(p,\kappa)$ and*

$$\limsup_{\lambda\to\infty} \operatorname*{ess\,sup}_{\kappa(x)\,\mathrm{d}\check{x}\otimes\kappa(y)\,\mathrm{d}\check{y}} \mathbb{E}\bigl|g_\lambda\bigl(\check{x}, \check{\mathcal{P}}_{\lambda\kappa} \cup \{\check{y}\}\bigr)\bigr|^p < \infty.$$

Now recall our conventions on R and R_λ from the preceding subsection. In most of our results, we shall need that R satisfies Assumption $\mathrm{M1}(p,\kappa)$ for some q. This is analogous, but not entirely equivalent to the *power-law stabilization of order q* as defined in [23,25,30].

Below we state an assumption formally imposed on R, a radius of stabilization for ξ. Throughout this subsection, let T denote a generic random mark with distribution $\mathbb{P}_{\mathcal{M}}$, independent of all other random objects we consider.

Assumption $\mathrm{FH}(\tau)$ ***(Finiteness with respect to Homogeneous process).*** *R is translation invariant and $\mathbb{P}(R((\mathbf{0},T), \check{\mathcal{P}}_\tau) < \infty) = 1$. Here, τ denotes a positive real number, not a function. Notice that if R is not a priori translation invariant, it can always be replaced by an appropriate translation invariant radius of stabilization.*

Remark 1.3. *Under translation invariance, Assumption $\mathrm{FH}(\tau)$ is essentially equivalent to what is called τ-*homogeneous stabilization *in* [23,25]. *More precisely, a translation invariant geometric functional ξ is τ-homogeneously stabilizing if*:

$$\mathbb{P}\bigl(R\bigl((\mathbf{0},T), \check{\mathcal{P}}_\tau\bigr) < \infty\bigr) = \mathbb{P}\bigl(R\bigl((\mathbf{0},T), \check{\mathcal{P}}_\tau \cup \{\check{x}\}\bigr) < \infty\bigr) = 1 \tag{1.5}$$

for all $\check{x} \in \check{\mathbb{R}}^d$. However, if the latter only holds for almost *all $\check{x}$, say, for all $\check{x} \in \check{\mathbb{R}}^d \setminus (\{\mathbf{0}\})$ outside a set $\check{N} \subset \check{\mathbb{R}}^d \setminus (\{\mathbf{0}\})$ of Lebesgue measure zero, one can modify $\xi((\mathbf{0},t), \check{\mathcal{X}})$ to $\xi((\mathbf{0},t), \check{\mathcal{X}} \setminus \check{N})$ and $R((\mathbf{0},t), \check{\mathcal{X}})$ to $R((\mathbf{0},t), \check{\mathcal{X}} \setminus \check{N})$, keeping translation invariance. Replacing $\check{\mathcal{X}}$ with a marked Poisson point process $\check{\mathcal{P}}_f$, ξ and R have then only been modified on a null event.*

Once relaxed to the 'almost everywhere' condition, the additional point $\breve{x}$ can now be left out. This is because for each $r > 0$, a Poisson point process can be represented as a union of a set $\breve{\mathcal{U}}_{N,r}$ and a Poisson point process on the complement of $\breve{B}_r(\mathbf{0}) := B_r(\mathbf{0}) \times \mathcal{M}$, where N is a Poisson random variable and where for each $n \in \mathbb{N}_0$, $\breve{\mathcal{U}}_{n,r}$ denotes a set of n independent points in $\breve{B}_r(\mathbf{0})$. Saying that the first probability in (1.5) equals one is equivalent to saying that $\mathbb{P}(R((\mathbf{0}, T), \breve{\mathcal{U}}_{n,r} \cup (\breve{\mathcal{P}}_\tau \setminus \breve{B}_r(\mathbf{0}))) < \infty) = 1$ for all $r > 0$ and all $n \in \mathbb{N}_0$, while for the second probability, this reduces to the statement that the latter holds for all $r > 0$ and all $n \in \mathbb{N}$. Therefore, the second probability in (1.5) is in fact redundant. To summarize, any statement on the distributions of the random measures μ_λ which holds under (1.5) still holds under Assumption FH(τ) only.

Definition 1.4. *A family $(g_\lambda)_{\lambda>\lambda_0}$ of geometric functionals enjoys κ-*almost exponential decay *if there exist $a \ge 0$ and $b > 0$, such that* $\operatorname{ess\,sup}_{\kappa(x)\,\mathrm{d}\breve{x}} \mathbb{P}(|g_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})| > t) \le a\mathrm{e}^{-bt}$ *for all $t \ge 0$. If this is satisfied for $g_\lambda = R_\lambda$, where, by our convention, $\lambda^{-1/d} R_\lambda$ is a radius of stabilization for ξ_λ, we shall say that ξ is κ-*almost exponentially stabilizing inside the upscaled domain Ω. *If Ω is not specified, we shall take $\Omega = \mathbb{R}^d$.*

Next, we list three further assumptions imposed on ξ. Recall that $\xi_\lambda(\breve{x}, \breve{\mathcal{X}}) = \xi(\lambda^{1/d}\breve{x}, \lambda^{1/d}\breve{\mathcal{X}})$.

Assumption CWLLN(p, κ) ***(Convergence in the sense of WLLN).*** *κ satisfies Assumption D, ξ is translation invariant and the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption M1(p', κ) for some $p' > p$. Moreover, there exists a radius of stabilization R for ξ satisfying Assumption FH$(\kappa(x))$ for κ-almost all $x \in \mathbb{R}^d$.*

Assumption CV(κ) ***(Convergence of Variance).*** *κ satisfies Assumption D and ξ is translation invariant. Moreover, there exists a radius of stabilization R for ξ satisfying Assumption FH$(\kappa(x))$ for κ-almost all $x \in \mathbb{R}^d$. Finally, there exist $p, q > 0$ with $2/p + d/q < 1$, such that the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption M1(p, κ) and such that there exists a family $(R_\lambda)_{\lambda>\lambda_0}$ of appropriate radii of stabilization satisfying Assumption M(q, κ).*

Assumption CCLT(κ) ***(Convergence in the sense of multivariate CLT).*** *κ satisfies Assumption D and has bounded support, and ξ is translation invariant. Moreover, there exists a radius of stabilization R for ξ satisfying Assumption FH$(\kappa(x))$ for κ-almost all $x \in \mathbb{R}^d$. Finally, there exists a family $(R_\lambda)_{\lambda>\lambda_0}$ of appropriate radii of stabilization, such that either $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption M1(p, κ) for some $p > 2$ and $(R_\lambda)_{\lambda>\lambda_0}$ enjoys almost uniform exponential decay for κ, or such that $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption M1(p, κ) for some $p > 3$ and $(R_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption M(q, κ) for some $q > d(150 + 6/p)$.*

Now we are ready to list the following known results:

- Take $p = 1$ or $p = 2$. If ξ satisfies Assumption CWLLN(p, κ), we have for all $f \in \mathcal{B}(\mathbb{R}^d)$:

$$\frac{\langle f, \mu_\lambda \rangle}{\lambda} \xrightarrow[\lambda\to\infty]{L^p} \int_{\mathbb{R}^d} f(x)\mathbb{E}\bigl[\xi\bigl((\mathbf{0}, T), \breve{\mathcal{P}}_{\kappa(x)}\bigr)\bigr]\kappa(x)\,\mathrm{d}x, \tag{1.6}$$

  where T is a generic random mark with distribution $\mathbb{P}_{\mathcal{M}}$, independent of $\breve{\mathcal{P}}_{\kappa(x)}$.
- If ξ satisfies Assumption CV(κ), there exists a measurable function $V : [0, \infty) \to [0, \infty)$, depending on ξ but not on κ, such that for all $f \in \mathcal{B}(\mathbb{R}^d)$, the variance $\sigma_\lambda^2[f] = \operatorname{Var}(\langle f, \mu_\lambda \rangle)$ satisfies:

$$\lim_{\lambda\to\infty} \frac{\sigma_\lambda^2[f]}{\lambda} = \sigma^2[f] := \int_{\mathbb{R}^d} f^2(x) V\bigl(\kappa(x)\bigr)\kappa(x)\,\mathrm{d}x \tag{1.7}$$

  (like for $\sigma_\lambda[f]$, we shall always take $\sigma[f] \ge 0$). The function V is defined by:

$$\begin{aligned} V(\tau) &:= \mathbb{E}\bigl[\xi\bigl((\mathbf{0}, T), \breve{\mathcal{P}}_\tau\bigr)^2\bigr] \\ &\quad + \tau \int_{\mathbb{R}^d} \bigl\{\mathbb{E}\bigl[\xi\bigl((\mathbf{0}, T), \breve{\mathcal{P}}_\tau \cup \bigl\{(z, T')\bigr\}\bigr)\xi\bigl((z, T'), \breve{\mathcal{P}}_\tau \cup \bigl\{(\mathbf{0}, T)\bigr\}\bigr)\bigr] - \bigl[\mathbb{E}\xi\bigl((\mathbf{0}, T), \breve{\mathcal{P}}_\tau\bigr)\bigr]^2\bigr\}\,\mathrm{d}z \end{aligned} \tag{1.8}$$

  for $\tau > 0$; for convenience, we set $V(0) := 0$. Similarly as before, T and T' are generic random marks with distribution $\mathbb{P}_{\mathcal{M}}$, independent of each other as well as of $\breve{\mathcal{P}}_\tau$.

- If ξ satisfies Assumption CCLT(κ), the finite-dimensional distributions of $\lambda^{-1/2}\bar{\mu}_\lambda$ converge in distribution as $\lambda \to \infty$ to those of a generalized mean-zero Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \int_{\mathbb{R}^d} f_1(x) f_2(x) V\big(\kappa(x)\big)\kappa(x)\,\mathrm{d}x. \tag{1.9}$$

The above results capture the weak law of large numbers and the Gaussian limit behavior of the re-normalized measures $\lambda^{-1/2}\bar{\mu}_\lambda$. For a special case (special mainly with respect to test functions), which is actually a good starting point, they are all proved in [3] (Theorem 2.1). For the general case (in fact, much more general that we consider), see [23] (Theorems 2.1–2.3), noting that we may relax the supremum over $\lambda \geq 1$ along with the supremum over the measure-theoretic support of κ stated in the conditions given ibidem to $\limsup_{\lambda\to\infty}$ and the κ-essential supremum. Furthermore, we can relax τ-homogeneous stabilization to Assumption FH(τ): see Remark 1.3. The convergence of the variance (1.7) and the central limit theorem are also proved in [25] (Theorems 2.1 and 2.2, but see also the remarks below Theorem 2.3).

The above-mentioned central limit theorem is based on Theorems 2.3 and 2.5 of [30], where also the curious condition $q > d(150 + 6/p)$ arises from. The latter results also provide explicit univariate bounds. An extension to multivariate bounds is derived in [26]. Under a different set-up, a multivatiate CLT is also proved in [22].

As to Theorem 2.2 of [23] and its counterpart, Theorem 2.1 of [25], it is worth mentioning that a factor d has been accidentally dropped from their statement, so that there should be $q > dp/(p-2)$ rather than $q > p/(p-2)$, just like in Assumption CV(κ): see Lemma 5.2 of [23] and Lemma 4.2 of [25].

1.4. *Non-degeneracy of the limiting variance*

In view of (1.7), it is important to distinguish between degenerate and non-degenerate limiting variance, i.e., $\sigma[f] = 0$ and $\sigma[f] > 0$. This issue is heavily discussed in [3,27,28]. A further fruitful general result is derived in [26]. However, the verification of the conditions guaranteeing non-degeneracy given in the latter paper might be somewhat involved. Therefore, we take one step forward and establish a new criterion for the non-degeneracy, which seems to be easier to verify. Moreover, the result of [26] is not supported by an example where earlier results do not apply. We here provide one: see Example 2.2.

Like the earlier result from [27], Theorem 2.2 of [26] is based on the *add-one cost* of the total mass functional. The *total mass functional* of a geometric functional ξ is defined by $H(\breve{\mathcal{X}}) := \sum_{\breve{x}\in\breve{\mathcal{X}}} \xi(\breve{x}, \breve{\mathcal{X}})$, while its add-one cost is defined by $\Delta(\breve{x}, \breve{\mathcal{X}}) := H(\breve{\mathcal{X}}) - H(\breve{\mathcal{X}} \setminus \{\breve{x}\})$, provided that $\breve{x} \in \breve{\mathcal{X}}$. Notice that Δ is also a geometric functional. Extending it by our convention from Section 1.2, we have $\Delta(\breve{x}, \breve{\mathcal{X}}) = H(\breve{\mathcal{X}} \cup \{\breve{x}\}) - H(\breve{\mathcal{X}})$ for $\breve{x} \notin \breve{\mathcal{X}}$. The name ‘add-one cost’ is due to the latter case.

In order to derive non-degeneracy, a new concept of stabilization, called *external stabilization*, is introduced in [26]. We here rewrite the corresponding definition for marked configurations and, in addition, introduce the concept of *basic external stabilization*.

Definition 1.5. *Let $\rho > 0$. A configuration $\breve{\mathcal{X}}$ is said to be* basically ρ-externally stable *at a point $\breve{x} \in \breve{\mathbb{R}}^d$ with respect to a geometric functional ξ if*

$$\xi\big(\breve{z}, \breve{\mathcal{Y}} \cup \{\breve{x}\}\big) = \xi\big(\breve{z}, \breve{\mathcal{Y}} \setminus \{\breve{x}\}\big) \tag{1.10}$$

for all finite $\breve{\mathcal{Y}}$ with $\breve{\mathcal{Y}} \cap B_\rho(\breve{x}) = \breve{\mathcal{X}} \cap B_\rho(\breve{x})$ and all $\breve{z} \in \breve{\mathcal{Y}} \setminus B_\rho(\breve{x})$.

The configuration $\breve{\mathcal{X}}$ is said to be ρ-externally stable *at $\breve{x}$ with respect to ξ if the following three conditions hold: first, $\breve{\mathcal{X}}$ is ρ-externally stable at $\breve{x}$ with respect to ξ. Second, ξ stabilizes at $\breve{x}$ within radius ρ with respect to $\breve{\mathcal{X}}$. Third,*

$$\xi\big(\breve{y}, \breve{\mathcal{Y}} \cup \{\breve{x}\}\big) - \xi\big(\breve{y}, \breve{\mathcal{Y}} \setminus \{\breve{x}\}\big) = \xi\big(\breve{y}, \big(\breve{\mathcal{X}} \cap B_\rho(\breve{x})\big) \cup \{\breve{x}\}\big) - \xi\big(\breve{y}, \big(\breve{\mathcal{X}} \cap B_\rho(\breve{x})\big) \setminus \{\breve{x}\}\big) \tag{1.11}$$

for all finite $\breve{\mathcal{Y}}$ with $\breve{\mathcal{Y}} \cap B_\rho(\breve{x}) = \breve{\mathcal{X}} \cap B_\rho(\breve{x})$ and all $\breve{y} \in \breve{\mathcal{Y}} \cap B_\rho(\breve{x})$.

We shall say that $\breve{\mathcal{X}}$ is externally stable *at $\breve{x}$ with respect to ξ if it is ρ-externally stable at $\breve{x}$ for some finite ρ.*

Remark 1.4. *If $\breve{\mathcal{X}}$ is ρ-externally stable at $\breve{x}$ with respect to ξ, then Δ stabilizes at $\breve{x}$ within radius ρ.*

Remark 1.5. *If $\check{\mathcal{X}}$ is ρ-externally stable and $r \geq \rho$, then $\check{\mathcal{X}}$ is also r-externally stable.*

Remark 1.6. *If $\check{\mathcal{X}}$ is ρ-externally stable at $\check{x}$ and $\check{\mathcal{Y}} \cap B_\rho(\check{x}) = \check{\mathcal{X}} \cap B_\rho(\check{x})$, then $\check{\mathcal{Y}}$ is also ρ-externally stable at $\check{x}$.*

Remark 1.7. *Condition (1.11) holds provided that ξ stabilizes at $\check{y}$ with respect to $\check{\mathcal{X}} \cup \{\check{x}\}$ and to $\check{\mathcal{X}} \cup \{\check{x}\}$ within radius $\rho - \|y - x\|$. On the other hand, condition (1.10) cannot be simply derived from the usual radius of stabilization. Therefore it is referred to as basic external stabilization.*

In view of the preceding remark, the following construction of the external radius of stabilization is now immediate.

Proposition 1.1. *Let ξ be a geometric functional with radius of stabilization R. Take $\rho > 0$, $\check{x} \in \check{\mathbb{R}}^d$ and a configuration $\check{\mathcal{X}}$. If $\check{\mathcal{X}}$ is basically ρ-externally stable at $\check{x}$ with respect to ξ, it is also r-externally stable at $\check{x}$ with respect to ξ, where*:

$$r := \rho + \max\bigl\{R\bigl(\check{y}, \check{\mathcal{X}} \cup \{\check{x}\}\bigr), R\bigl(\check{y}, \check{\mathcal{X}} \setminus \{\check{x}\}\bigr); \check{y} \in \bigl(\check{\mathcal{X}} \cup \{\check{x}\}\bigr) \cap B_\rho(\check{x})\bigr\}.$$

In particular, if $\check{\mathcal{X}}$ is basically ρ-externally stable at $\check{x}$ with respect to ξ and ξ stabilizes with respect to $\check{\mathcal{X}} \cup \{\check{x}\}$ and $\check{\mathcal{X}} \setminus \{\check{x}\}$ at all $\check{y} \in (\check{\mathcal{X}} \cup \{\check{x}\}) \cap B_\rho(\check{x})$, then $\check{\mathcal{X}}$ is externally stable at $\check{x}$ with respect to ξ.

In the rest of this subsection as well as in Section 3.5, we fix a translation invariant geometric functional ξ, let H be the total mass functional of ξ and let Δ be its add-one cost. We also recall the convention $\xi_\lambda(\check{x}, \check{\mathcal{X}}) := \xi_\lambda(\lambda^{1/d} x, \lambda^{1/d} \mathcal{X})$; set also $\Delta_\lambda(\check{x}, \check{\mathcal{X}}) := \Delta_\lambda(\lambda^{1/d} x, \lambda^{1/d} \mathcal{X})$. Let R denote a radius of stabilization for ξ and let $\lambda^{-1/d} R_\lambda$ be a radius of stabilization for ξ_λ. Finally, let T denote a generic random mark with distribution $\mathbb{P}_{\mathcal{M}}$, independent of all other random variables.

Now we state the version of Theorem 2.2 of [26] for marked configurations. One can easily check that the proof given ibidem still carries through.

Theorem 1.1. *Suppose that ξ is translation invariant and that there exists $\rho > 0$, such that with strictly positive probability, the marked homogeneous Poisson process $\check{\mathcal{P}}_1$ is ρ-externally stable at $(\mathbf{0}, T)$ and $\Delta((\mathbf{0}, T), \check{\mathcal{P}}_1 \cap B_\rho(\mathbf{0})) \neq 0$. Next, take a probability density function κ on $\mathbb{R}^d$ and a function $f \in \mathcal{B}(\mathbb{R}^d)$, which is Lebesgue-almost everywhere continuous. Suppose that $f\kappa$ is not Lebesgue-almost everywhere zero. Let X be a random variable with density κ. Suppose that for some $s > 2$, we have*:

$$\limsup_{\lambda \to \infty} \mathbb{E} \Biggl| \sum_{\check{x} \in \check{\mathcal{P}}_{\lambda\kappa} \cup \{(X,T)\}} f(x) \xi\bigl(\check{x}, \check{\mathcal{P}}_{\lambda\kappa} \cup \bigl\{(X,T)\bigr\}\bigr) - \sum_{\check{x} \in \check{\mathcal{P}}_{\lambda\kappa} \setminus \{(X,T)\}} f(x) \xi\bigl(\check{x}, \check{\mathcal{P}}_{\lambda\kappa} \setminus \bigl\{(X,T)\bigr\}\bigr) \Biggr|^s < \infty. \tag{1.12}$$

In addition, suppose either that f is an indicator function of a measurable subset of $\mathbb{R}^d$ with vanishing boundary (i.e., the Lebesgue measure of the boundary equals zero) or that for all $K > 0$ and κ-almost all x, we have:

$$\int_{B_K(\mathbf{0})} \mathbb{E} \bigl| \xi\bigl(\check{y}, \check{\mathcal{P}}_{\kappa(x)} \cup \bigl\{(\mathbf{0}, T)\bigr\}\bigr) - \xi(\check{y}, \check{\mathcal{P}}_{\kappa(x)}) \bigr| \, \mathrm{d}\check{y} < \infty. \tag{1.13}$$

Then $\liminf_{\lambda \to \infty} \sigma_\lambda^2[f]/\lambda > 0$.

As already mentioned, the verification of the conditions required in the preceding result might be somewhat involved. In particular, this holds for the moment conditions (1.12) and (1.13). The main goal of this subsection is to show that these conditions can be simply replaced by Assumption $\mathrm{CV}(\kappa)$ introduced in Section 1.3, taking κ to be the uniform density on a suitable domain and restricting attention to the case $f \equiv 1$. Observe first that in this case, the moment condition (1.13) can be left out, while the moment condition (1.12) reduces to:

$$\limsup_{\lambda \to \infty} \mathbb{E} \bigl| \Delta_\lambda\bigl((X, T), \check{\mathcal{P}}_{\lambda\kappa}\bigr) \bigr|^s < \infty. \tag{1.14}$$

Considering only the case $f \equiv 1$ is *not* a big restriction: under Assumption CV(κ), formula (1.7) allows us to derive the limiting variance from the function V; notice that for each $\tau > 0$, $V(\tau)$ is precisely the limiting variance for $f \equiv 1$, taking κ to be a suitable uniform density (see also Remark 1.8).

For convenience, we list two additional assumptions imposed on a family $(g_\lambda)_{\lambda>\lambda_0}$ of geometric functionals. Below we show that they are essentially equivalent to Assumptions M and M1; moreover, with proper parameters and one additional assumption, they are also equivalent to Assumption CV(κ).

Assumption MH(p,τ,Ω) ***(pth Moment with respect to Homogeneous process restricted to Ω).***

$$\limsup_{\lambda\to\infty} \operatorname*{ess\,sup}_{\mathbf{1}(x\in\Omega)\,\mathrm{d}\check{x}} \mathbb{E}\big|g_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\tau}\cap\Omega)\big|^p < \infty.$$

Assumption MH1(p,τ,Ω) ***(pth Moment with respect to Homogeneous process restricted to Ω with One additional point).*** $(g_\lambda)_{\lambda>\lambda_0}$ *satisfies Assumption* MH(p,τ,Ω) *and*

$$\limsup_{\lambda\to\infty} \operatorname*{ess\,sup}_{\mathbf{1}(x,y\in\Omega)\,\mathrm{d}\check{x}\otimes\mathrm{d}\check{y}} \mathbb{E}\big|g_\lambda\big(\check{x}, (\check{\mathcal{P}}_{\lambda\tau}\cap\Omega)\cup\{\check{y}\}\big)\big|^p < \infty.$$

Remark 1.8. *Letting* $v := \mathrm{vol}(\Omega)$, $\Omega^* := (\tau v)^{-1/d}\Omega$, $g^*_{\lambda^*}(\check{x}^*, \check{\mathcal{X}}^*) := g_\lambda((\tau v)^{1/d}\check{x}^*, (\tau v)^{1/d}\check{\mathcal{X}}^*)$, *where* $\lambda = \lambda^*/(\tau v)$, the family $(g_\lambda)_{\lambda\ge\lambda_0}$ satisfies Assumption MH(p,τ,Ω) if and only if the family $(g^*_{\lambda^*})_{\lambda^*\ge\lambda^*_0}$ satisfies Assumption M(p,κ), taking κ to be the uniform density on Ω^* and $\lambda^*_0 := \tau v\lambda_0$. This is true because:

$$\begin{aligned} g_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\tau}\cap\Omega) &= g_{\lambda^*/(\tau v)}\big((\tau v)^{1/d}\check{x}^*, \check{\mathcal{P}}_{\lambda^*/v}\cap(\tau v)^{1/d}\Omega^*\big) = g^*_{\lambda^*}\big(\check{x}^*, (\tau v)^{-1/d}\check{\mathcal{P}}_{\lambda^*/v}\cap\Omega^*\big) \\ &\overset{\mathcal{D}}{=} g^*_{\lambda^*}\big(\check{x}^*, \check{\mathcal{P}}_{\lambda^*\tau}\cap\Omega^*\big) = g^*_{\lambda^*}\big(\check{x}^*, \check{\mathcal{P}}_{\lambda^*\kappa}\big), \end{aligned}$$

where $\check{x}^* := (\tau v)^{-1/d}\check{x}$ *and where* $\overset{\mathcal{D}}{=}$ *denotes equivalence in distribution. Similarly*, the family $(g_\lambda)_{\lambda\ge\lambda_0}$ satisfies Assumption MH1(p,τ,Ω) if and only if the family $(g^*_{\lambda^*})_{\lambda^*\ge\lambda^*_0}$ satisfies Assumption M1(p,κ). *Notice also that* the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption MH(p,τ,Ω), respectively Assumption MH1(p,τ,Ω), if and only if it satisfies Assumption M(p,κ), respectively Assumption M1(p,κ).

Moreover, keeping the relationship between λ *and* λ^*, $\lambda^{-1/d}R_\lambda$ is a radius of stabilization for ξ_λ inside domain Ω if and only if $(\lambda^*)^{-1/d}R^*_{\lambda^*}$ is a radius of stabilization for ξ_{λ^*} inside Ω^*, *where* $R^*_{\lambda^*}(\check{x}^*, \check{\mathcal{X}}^*) := R_\lambda((\tau v)^{1/d}\check{x}^*, (\tau v)^{1/d}\check{\mathcal{X}}^*)$. *As a result*, if the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption MH1(p,τ,Ω) and if there exists a radius R of stabilization for ξ satisfying Assumption FH(τ) and a suitable family $(R_\lambda)_{\lambda>\lambda_0}$ of scaled radii of stabilization satisfying Assumption MH(p,τ,Ω), where $2/p + d/q < 1$, then ξ satisfies Assumption CV(κ).

As mentioned above, the main goal of this subsection is to show that the moment condition (1.14) can be replaced by Assumption CV(κ). In addition, we also simplify the condition on non-vanishing Δ. In order to do this, we state the following definition.

Definition 1.6. *A predicate* P *defined on pairs* $(t, \check{\mathcal{X}})$, *where* $t \in \mathcal{M}$ *and* $\check{\mathcal{X}}$ *is a finite configuration, is said to hold* for notably many *pairs* $(t, \check{\mathcal{X}})$ *if there exists* $n \in \mathbb{N}_0$ *and a* $\mathbb{P}_{\mathcal{M}}(\mathrm{d}t)\otimes\mathrm{d}\check{x}_1\otimes\cdots\otimes\mathrm{d}\check{x}_n$*-nonnull set, such that* $P(t, \{\check{x}_1,\dots,\check{x}_n\})$ *holds for all* $(t, \check{x}_1,\dots,\check{x}_n)$ *in that set.*

Remark 1.9. *In particular, for* $P(t,\check{\mathcal{X}})$ *to hold for notably many pairs* $(t,\check{\mathcal{X}})$, *it suffices that* $P(t,\varnothing)$ *holds on a* $\mathbb{P}_{\mathcal{M}}(\mathrm{d}t)$*-nonnull set.*

Now we are ready to formulate our result on non-degeneracy. We defer the proof to Section 3.5.

Theorem 1.2. *Let* ξ *be translation invariant, let* $0 < \tau < \infty$ *and suppose that there exists a radius of stabilization* R *for* ξ *satisfying Assumption* FH(τ). *Let* Ω *be a domain with* $0 < \mathrm{vol}(\Omega) < \infty$ *and with* $\mathrm{vol}(\partial\Omega) = 0$. *Take* $p, q > 0$ *with* $2/p + d/q < 1$ *and* $\lambda_0 > 0$. *Suppose that the family* $(\xi_\lambda)_{\lambda>\lambda_0}$ *satisfies Assumption* MH1(p,τ,Ω) *and that there* *exists a family* $(R_\lambda)_{\lambda>\lambda_0}$ *satisfying Assumption* MH(q,τ,Ω), *such that for each* $\lambda>\lambda_0$, $\lambda^{-1/d}R_\lambda$ *is a radius of stabilization for* ξ_λ. *Finally, suppose that for notably many pairs* $(t,\check{\mathcal{X}})$, $\Delta((\mathbf{0},t),\check{\mathcal{X}})\neq 0$ *and* $\check{\mathcal{X}}$ *is externally stable at* $(\mathbf{0},t)$ *with respect to* ξ. *With* V *as in* (1.8), *we then have* $V(\tau)>0$.

1.5. *Estimates on deviation probabilities*

Starting from the known results it is natural to investigate the asymptotics of *deviation probabilities* on a scale larger than that of the central limit theorem. To this end we consider a fixed test-function $f\in\mathcal{B}(\mathbb{R}^d)$ and strive to get precise information on bounds of the relative error

$$\frac{\mathbb{P}(\langle f,\bar\mu_\lambda\rangle\geq x)}{1-\Phi(x/\sigma_\lambda[f])},\quad\text{as well as}\quad\frac{\mathbb{P}(\langle f,\bar\mu_\lambda\rangle\leq -x)}{\Phi(-x/\sigma_\lambda[f])},\quad x>0,\tag{1.15}$$

where, as in the preceding Section 1.3, $\sigma_\lambda^2[f]$ denotes the variance and where, as usual,

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-t^2/2}\,\mathrm{d}t$$

is the distribution function of the standard normal. In particular, we are interested in conditions under which the relative error (1.15) converges to 1 uniformly in the interval $0\leq x\leq F(\lambda)$, where $F(\lambda)$ is a nondecreasing function such that $F(\lambda)\to\infty$. Of course $F(\lambda)$ will depend not only on λ but also on other characteristics of our models, in particular their dimensionality. For the sake of readability, the dependence on other quantities is suppressed in all of our notation.

Since we will refine the cumulant expansion method of [3] to establish more precise rates of growth on the cumulants, in both their scale parameter and their order, we will be able to apply a powerful and general lemma on deviation probabilities due to Rudzkis, Saulis and Statulevičius [32], whose version specialized for our purposes is stated as Lemma 3.8 in the sequel for the convenience of the reader.

Before formulating the results, we list the following two key assumptions imposed on a family $(g_\lambda)_{\lambda\geq\lambda_0}$ of geometric functionals:

Assumption MGP(α,κ), $\alpha\geq 0$ ***(Moment Growth with additional Points).*** *There exist* $A\geq 0$ *and* $q>0$, *such that for all* $\lambda\geq\lambda_0$, *all* $k\in\mathbb{N}$ *and all* $r\leq qk$,

$$\operatorname*{ess\,sup}_{\kappa(x_1)\,\mathrm{d}\check{x}_1\otimes\cdots\otimes\kappa(x_r)\,\mathrm{d}\check{x}_r}\mathbb{E}\big|g_\lambda\big(\check{x}_1,\check{\mathcal{P}}_{\lambda\kappa}\cup\{x_1,\ldots,x_r\}\big)\big|^k\kappa(x_1)\,\mathrm{d}\check{x}_1\leq A^k(k!)^\alpha\tag{1.16}$$

(*recall that* $\operatorname{ess\,sup}_{f(x)\,\mathrm{d}x}g(x)$ *denotes the essential supremum of g with respect to the Lebesgue measure restricted to the set* $\{x;\,f(x)>0\}$).

Assumption MGI(α,κ), $\alpha\geq 0$ ***(Moment Growth with respect to Integral).*** *There exists* $A\geq 0$, *such that for all* $\lambda\geq\lambda_0$ *and all* $k\in\mathbb{N}$,

$$\int_{\mathbb{R}^d}\mathbb{E}\big|g_\lambda(\check{x},\check{\mathcal{P}}_{\lambda\kappa})\big|^k\kappa(x)\,\mathrm{d}\check{x}\leq A^k(k!)^\alpha.\tag{1.17}$$

Remark 1.10. *Clearly, Assumption* MGI(α,κ) *is weaker than Assumption* MGP(α,κ).

Remark 1.11. *If there exists a family* $(R_\lambda)_{\lambda\geq\lambda_0}$ *according to our convention from Section* 1.2, *satisfying Assumption* MGI(α,κ) *for some* $\alpha\geq 0$, *then* ξ *is almost surely stabilizing with respect to* $\mathcal{P}_{\lambda\kappa}$ *for all* $\lambda\geq\lambda_0$. *As a result, the random measures* μ_λ *are almost surely defined.*

It is worth to point out two special cases. First, observe that if the functionals $|g_\lambda|$ are uniformly bounded, then the corresponding family satisfies Assumptions MGP$(0,\kappa)$ and MGI$(0,\kappa)$. When this is true for $g_\lambda=R_\lambda$, we shall say that ξ is *uniformly stabilizing inside the upscaled domain Ω*. The second special case is when the family enjoys κ-almost exponential decay (Definition 1.4). In this case, we can estimate:

$$\mathbb{E}\left|g_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\right|^k = \int_0^\infty kt^{k-1}\mathbb{P}\big(\left|g_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\right| > t\big)\,\mathrm{d}t \le \frac{ak!}{b^k}. \tag{1.18}$$

As a result, the family $(g_\lambda)_{\lambda\ge\lambda_0}$ satisfies Assumption MGI$(1,\kappa)$.

Now we state our central assumption, which is imposed on and will be used exclusively for ξ:

Assumption G(γ,κ) ***(General conditions for our main results).*** *κ satisfies Assumption* D *and there exist $\alpha, \beta \ge 0$, $\lambda_0 > 0$ and a family $(R_\lambda)_{\lambda\ge\lambda_0}$ according to our convention from Section* 1.2, *satisfying Assumption* MGI(β,κ) *along with at least one of the following two conditions fulfilled*:

- $(\xi_\lambda)_{\lambda\ge\lambda_0}$ *satisfies Assumption* MGI(α,κ) *and* $1+\alpha+\beta d = \gamma$.
- $(\xi_\lambda)_{\lambda\ge\lambda_0}$ *satisfies Assumption* MGP(α,κ) *and* $\max\{\alpha, 1\}+\beta d = \gamma$.

The following result concerns deviation probabilities.

Theorem 1.3. *Suppose that ξ satisfies Assumption* G(γ,κ) *and take $f\in\mathcal{B}(\mathbb{R}^d)$. Let $\sigma_-[f] := \liminf_{\lambda\to\infty}\lambda^{-1/2}\times\sigma_\lambda[f]$.*

(1) *Suppose that, in addition, $\sigma_-[f] > 0$. Then, for all $\lambda\ge\lambda_1$ and $0\le x\le C_1\sigma_-[f]\lambda^{(1+\gamma)/(1+2\gamma)}$, we have*:

$$\left|\log\frac{\mathbb{P}(\langle f,\bar{\mu}_\lambda\rangle \ge x)}{1-\Phi(x/\sigma_\lambda[f])}\right| \le C_2\left(\frac{1}{\lambda^{1/(2+4\gamma)}} + \frac{x^3}{\lambda^{(2+3\gamma)/(1+2\gamma)}\sigma_-^3[f]}\right), \tag{1.19}$$

$$\left|\log\frac{\mathbb{P}(\langle f,\bar{\mu}_\lambda\rangle \le -x)}{\Phi(-x/\sigma_\lambda[f])}\right| \le C_2\left(\frac{1}{\lambda^{1/(2+4\gamma)}} + \frac{x^3}{\lambda^{(2+3\gamma)/(1+2\gamma)}\sigma_-^3[f]}\right), \tag{1.20}$$

where λ_1 only depends on f, κ, ξ and R, whereas C_1 and C_2 only depend on the ratio $\|f\|_\infty/\sigma_-[f]$ *along with κ, ξ and R.*

(2) *Suppose that $0<\sigma_\lambda[f]\le C_3\lambda^{1/2}$. Then, for all $x\ge 0$, we have*:

$$\mathbb{P}\big(\pm\langle f,\bar{\mu}_\lambda\rangle \ge x\big) \le \exp\left(-\min\left\{C_4\frac{x^2}{\sigma_\lambda^2[f]}, C_5x^{1/(1+\gamma)}, C_6\left(\frac{x^3}{\lambda}\right)^{1/(2+\gamma)}\right\}\right), \tag{1.21}$$

where C_3–C_6 only depend on f, κ, ξ and R.

Remark 1.12. *The second part of the theorem above is especially useful for degenerate cases, i.e., $\sigma[f]=0$. As an example, one can consider the total number of edges in the Voronoi graph*: *see Section* 8.2 *of* [27].

Remark 1.13. *In particular, under Assumption* G(γ,κ) *and provided that $\sigma_-[f]>0$, Theorem* 1.3 *provides a central limit theorem. Comparing to Assumption* CCLT(κ), *none of them implies the others. Assumption* G(γ,κ) *roughly include much stronger moments conditions, but do not require boundedness of the support of κ. Similarly, to the best of our knowledge, none of the existing central limit theorems has been proved under conditions weaker than Assumption* G(γ,κ). *Thus, Theorem* 1.3 *also adds to existing CLT's.*

Remark 1.14. *At least in certain cases, there appears scope for improvement. Some related results indicate that if ξ satisfies Assumption* MGI$(0,\kappa)$ *and R satisfies Assumption* MGP$(0,\kappa)$ (*i.e., both are almost surely uniformly bounded*), *Theorem* 1.3 *should actually hold for $\gamma=0$* (*full range large deviation principles, see next subsection*) *rather than $\gamma=1$. Results leading to full range large deviation principles are derived in* [14] *for sums of locally dependent random variables* (*provided that the random variables as well as the vertex degrees in the dependence graph are uniformly bounded*), *in* [16] *for germ–grain models and in* [17] *for a more general case, where germs are affine subspaces instead of points. Notice that in the latter case, the corresponding geometric functional is even not* *stabilizing in the sense of the present paper. However, it is uniformly stabilizing if we replace balls by direct sums of* $(d-k)$*-dimensional balls and* k*-dimensional subspaces. This is due to different behavior of the variance.*

Indeed, clever modification and application of Lemma 1 *of* [14] *might relax the expression* $\max\{\alpha, 1\} + \beta d$ *in Assumption* $\mathrm{G}(\gamma, \kappa)$ *to some continuous function* $\psi(\alpha, \beta)$ *with* $\psi(0,0) = 0$ *and* $\lim_{\alpha,\beta\to\infty}(\alpha + \beta d - \psi(\alpha, \beta)) = 0$. *Details may appear in forthcoming work.*

1.6. *Moderate deviation principles*

It is natural to investigate the asymptotics of $(\bar{\mu}_\lambda)_\lambda$ on intermediate scales between those appearing in Gaussian and law of large numbers behavior. This leads us to moderate deviation principles (MDPs). In this paper we are able to deduce moderate deviation principles from Theorem 1.3 for a typically *partial* intermediate regime for stabilizing ξ (for the full scale, we have to assume $\alpha = \beta = 0$; see Theorem 1.4 below for a formal statement). We remark that in [1], moderate deviation principles were obtained for an essentially smaller set of examples, including the prototypical random sequential packing and some spatial birth-and-growth models as well as for empirical functionals of nearest neighbor graphs, but they were obtained on *every* intermediate scale.

We say that a family of probability measures $(\nu_\lambda)_\lambda$ on $\mathcal{T}$, which is a measurable as well as a topological space, obeys a large deviation principle (LDP) with speed a_λ and good rate function $I(\cdot) : \mathcal{T} \to [0, \infty]$ as $\lambda \to \lambda_0$ if

- I is lower semi-continuous and has compact level sets $N_L := \{x \in \mathcal{T} : I(x) \le L\}$, for every $L \in [0, \infty)$.
- For every measurable set Γ, we have:

$$-\inf_{x\in\mathring{\Gamma}} I(x) \le \liminf_{\lambda\to\lambda_0} \frac{1}{a_\lambda} \log \nu_\lambda(\Gamma) \le \limsup_{\lambda\to\lambda_0} \frac{1}{a_\lambda} \log \nu_\lambda(\Gamma) \le -\inf_{x\in\overline{\Gamma}} I(x), \tag{1.22}$$

where $\mathring{\Gamma}$ denotes the topological interior of Γ and $\overline{\Gamma}$ denotes its closure.

Notice that we do not assume that the measures are Borel. In other words, open sets are not necessarily measurable.

Similarly we will say that a family of $\mathcal{T}$-valued random variables $(Y_\lambda)_\lambda$ obeys a large deviation principle with speed a_λ and good rate function $I(\cdot) : \mathcal{T} \to [0, \infty]$ if the sequence of their distributions does. Formally a moderate deviation principle is nothing but an LDP. However, we will speak about a moderate deviation principle for a sequence of random variables whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers and that of a Central Limit Theorem.

Take $\gamma \ge 0$ (arising from Assumption $\mathrm{G}(\gamma, \kappa)$) and consider $\lambda \in (0, \infty)$, $\lambda \to \infty$. Let $(a_\lambda)_{\lambda>0}$ be such that

$$\lim_{\lambda\to\infty} a_\lambda = \infty \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{a_\lambda}{\lambda^{1/(2+4\gamma)}} = 0. \tag{1.23}$$

Under these assumptions, we first state the following MDP for $\bar{\mu}_\lambda$:

Theorem 1.4. *Suppose that* ξ *satisfies Assumptions* $\mathrm{G}(\gamma, \kappa)$ *and* $\mathrm{CV}(\kappa)$, *and take* $f \in \mathcal{B}(\mathbb{R}^d)$. *Then, for each* $(a_\lambda)_{\lambda>0}$ *satifying* (1.23), *the family of random variables* $(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle)_\lambda$ *satisfies on* $\mathbb{R}$ *the moderate deviation principle with speed* a_λ^2 *and good rate function*

$$I_f(t) := \frac{t^2}{2\sigma^2[f]}, \tag{1.24}$$

where σ *is as in* (1.7) *and where possible division by zero is handled according to our convention at the end of Section* 1.2.

The next result is a MDP on the level of *measures*. Denote by $\mathrm{Meas}(\mathbb{R}^d)$ the real vector space of finite signed measures on $\mathbb{R}^d$. Equip $\mathrm{Meas}(\mathbb{R}^d)$ with the τ*-topology* generated by the sets:

$$U_{f,x,\delta} := \{\nu \in \mathrm{Meas}(\mathbb{R}^d); |\langle f, \nu\rangle - x| < \delta\},$$

where $f \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}$ and $\delta > 0$. It is well known that since the collection of linear functionals $\{\nu \mapsto \langle f, \nu\rangle;\ f \in \mathcal{B}(\mathbb{R}^d)\}$ is separating in $\mathrm{Meas}(\mathbb{R}^d)$, this topology makes $\mathrm{Meas}(\mathbb{R}^d)$ into a locally convex, Hausdorff topological vector space, whose topological dual is the preceding collection, hereafter identified with $\mathcal{B}(\mathbb{R}^d)$. With this notation we establish the following *measure-level* MDP for $\bar{\mu}_\lambda$:

Theorem 1.5. *Suppose that ξ satisfies Assumptions G(γ, κ) and CV(κ). Then for any family $(a_\lambda)_{\lambda>0}$ satisfying (1.23), the family $(a_\lambda^{-1}\lambda^{-1/2}\bar{\mu}_\lambda)_\lambda$ satisfies the MDP on* $\mathrm{Meas}(\mathbb{R}^d)$, *endowed with the τ-topology, with speed a_λ^2 and the convex, good rate function given by*

$$I(\nu) := \frac{1}{2}\sigma^2\left[\frac{\mathrm{d}\nu}{V(\kappa(x))\kappa(x)\,\mathrm{d}x}\right] \tag{1.25}$$

if $\nu \in$ $\mathrm{Meas}(\mathbb{R}^d)$ *is absolutely continuous with respect to $V(\kappa(x))\kappa(x)\,\mathrm{d}x$, and by $I(\nu) := +\infty$ otherwise. Again, σ is as in* (1.7).

Remark 1.15. *Theorem* 1.5 *provides a MDP with respect to the τ-topology, which is based on* measurable *bounded test functions. Therefore, this result has a stronger nature than the corresponding Theorem* 2.2 *of* [1], *which is stated in the weak topology, based on* continuous *bounded test functions.*

2. Applications

We here provide three groups of applications of our deviation bounds and moderate deviation principles: models related to random sequential packing, functionals related to k nearest neighbors and sphere of influence graphs. These applications have been considered in detail in the context of central limit theorems [3,27,30] and in the context of laws of large numbers in [28,29]. In the context of moderate deviation principles, packing and nearest neighbors were considered in [1]. For all groups of applications, we establish results of a relatively universal nature: to the best of our knowledge, they are more general than those stated in the literature. We show where our large deviation results improve and generalize over [1].

To set up the framework under which our results are stated, we here introduce a new concept, which we shall call *confinement*. Similarly as in stabilization, the idea is that the value of a functional at $\check{x}$ depends only on some 'neighborhood' of $\check{x}$. The concept of stabilization is based on metric neighborhoods, while the concept of confinement is entirely based on sets. In precise terms, it goes as follows.

Definition 2.1. *Let h be a set-valued geometric functional, such that $h(\check{x}, \check{\mathcal{X}}) \subseteq \check{\mathcal{X}}$ for all $\check{\mathcal{X}}$ and all $\check{x} \in \check{\mathcal{X}}$. A geometric functional ξ is* confined to h *if $\xi(\check{x}, \check{\mathcal{X}}) = \xi(\check{x}, h(\check{x}, \check{\mathcal{X}}))$ for all $\check{\mathcal{X}}$ and all $\check{x} \in \check{\mathcal{X}}$.*

Remark 2.1. *Let ξ be confined to h. Then any radius of stabilization for h is also a radius of stabilization for ξ. Similarly, if $\check{\mathcal{X}}$ is basically ρ-externally stable at $\check{x}$ with respect to h, it is also basically ρ-externally stable at $\check{x}$ with respect to ξ.*

Remark 2.2. *Let h_1 and h_2 are set-valued geometric functionals, such that $h_1(\check{x}, \check{\mathcal{X}}) \subseteq h_2(\check{x}, \check{\mathcal{X}}) \subseteq \check{\mathcal{X}}$ for all relevant $\check{x}$ and $\check{\mathcal{X}}$. Suppose that h_1 is stable in the sense that $h_1(\check{x}, \check{\mathcal{Y}}) = h_1(\check{x}, \check{\mathcal{X}})$ if $h_1(\check{x}, \check{\mathcal{X}}) \subseteq \check{\mathcal{Y}} \subseteq \check{\mathcal{X}}$. Then any geometric functional confined to h_1 is also confined to h_2.*

Now we are ready to focus on each group of applications separately.

2.1. *Random sequential packing and related models*

The following prototypical random sequential packing/adsorption (RSA) model arises in diverse disciplines, including physical, chemical, and biological processes. See [28] for a discussion of the many applications, the many references, and also a discussion of previous mathematical analysis. In one dimension, this model is often referred to as the Rényi car parking model [31].

Consider a finite set $\mathcal{X} \subset \mathbb{R}^d$ and to each $x \in \mathcal{X}$ attach a ball with some fixed diameter ρ centered at x. Moreover, to all points in $\mathcal{X}$ attach i.i.d. uniform time marks taking values in some finite time interval, say, [0, 1]. This establishes a *chronological* order on the points of $\mathcal{X}$. As usual, denote by $\check{\mathcal{X}}$ the configuration of points of $\mathcal{X}$ along with their time marks. Declare the first point in the chronological ordering *accepted* and proceed recursively, each time accepting the next point if the ball it carries does not overlap the previously accepted (packed) balls and rejecting it otherwise. The functional $\xi(\check{x}, \check{\mathcal{X}})$ is defined to be 1 if the ball centered at x has been accepted and 0 otherwise. This defines the prototypical random sequential packing/adsorption (RSA) process.

One can also consider infinite periods of packing, i.e., input point processes on $\check{\mathbb{R}}^d = \mathbb{R}^d \times [0, \infty)$. Take a Poisson point process with density $\kappa(x)\,\mathrm{d}x \otimes \mathrm{d}t$, where κ is a probability density function with bounded support. Then, clearly, only finitely many points can be accepted, so that all points that appear after a certain time are rejected. Moreover, almost surely, there is actually no more available space for packing. This is called *jamming*: see [20]. This setting allows to define the random measures μ_λ in just the same way as in (1.4), although the measure on the mark space is infinite. However, the latter fact prevents us from applying our results directly. Although one might use truncation of time, jamming will not be considered in this paper.

The RSA model can be extended in numerous other ways: see [25,28]. In particular, the decision whether to accept or reject a particle can depend on additional characteristics attached to the particle (e.g., mass), it can depend on time (in particular, after a certain time, a particle may be desorbed) and it can even be random. As an example, we consider the *spatial birth–growth model*: the balls attached to subsequent independently time-marked points, i.e., particles, are allowed to have their initial radii bounded random i.i.d. rather than fixed. Moreover, at the moment of its birth each particle begins to *grow* radially with constant speed v until it hits another particle or reaches a certain maximal admissible size ρ – in both these cases it stops growing. In analogy to the basic RSA, a particle is accepted if it does not overlap any previously accepted one and is rejected otherwise.

The mark of a point now consists of the time stamp plus the initial radius of the corresponding ball. The functional of interest is again given by $\xi(\check{x}, \check{\mathcal{X}}) = 1$ if the particle centered at x has been accepted and 0 otherwise. This model, going also under the name of the *Johnson–Mehl growth process* in the particular case where the initial radii are 0, has attracted a lot of interest in the literature, see [3,28] and the references therein.

Let $\check{\mathcal{X}}$ be a configuration of marked particles, where a random mark consists of a pair (t, s), where t is the time stamp and where s is some additional feature of the point. As suggested in [28], consider an oriented graph with vertex set $\check{\mathcal{X}}$, where an edge from $\check{x}$ to $\check{y}$ exists if the particle x has arrived before y and if $\|x - y\| \le \rho$. Given $\check{x} \in \check{\mathcal{X}}$, denote by $\check{A}^{\mathrm{in}}_\rho(\check{x}, \check{\mathcal{X}})$ the set of all particles in $\check{\mathcal{X}}$ from which $\check{x}$ can be reached by a directed path in this graph, along with $\check{x}$ itself.

Some thought shows that the functional ξ considered in the basic RSA model as well as in the spatial birth–growth model is confined to the functional $\check{A}^{\mathrm{in}}_\rho$ according to Definition 2.1. Moreover, this is true for all examples considered in [28]: the key point is that particles are only influenced by the configuration at their arrival, but not by the particles arriving later.

Now let ξ be any geometric functional confined to $\check{A}^{\mathrm{in}}_\rho$. Denoting $D(\check{x}, \check{\mathcal{X}}) := \sup\{\|y - x\|\}; y \in \mathcal{X}\}$, observe that the functional:

$$R(\check{x}, \check{\mathcal{X}}) := D\big(\check{x}, \check{A}^{\mathrm{in}}_\rho(\check{x}, \check{\mathcal{X}})\big) + \rho$$

is a radius of stabilization for $\check{A}^{\mathrm{in}}_\rho$ and, according to Remark 2.1, also for ξ. Moreover, letting $R_\lambda(\check{x}, \check{\mathcal{X}}) := R(\lambda^{1/d}\check{x}, \lambda^{1/d}\check{\mathcal{X}})$, observe that $\lambda^{-1/d} R_\lambda$ is a radius of stabilization for ξ_λ.

Now let $\check{\mathcal{P}}_f$ be a marked Poisson process with bounded intensity f; for the random marks (i.e., the probability measure $\mathbb{P}_{\mathcal{M}}$), assume that the time stamp is continuously distributed (without loss of generality, we may then assume that it is uniform over [0, 1]). Percolation estimates (Section 4 of [28]) then yield the bound:

$$\mathbb{P}\big(D\big(\check{x}, \check{A}^{\mathrm{in}}_\rho(\check{x}, \check{\mathcal{P}}_f)\big) \ge t\big) \le a\mathrm{e}^{-bt} \tag{2.1}$$

for all $\check{x} \in \check{\mathbb{R}}^d$ and all $t \ge 0$, where the constants $a \ge 0$ and $b > 0$ depend only on $\|f\|$ and ρ. As a result, ξ is κ-almost exponentially stabilizing for any bounded density κ.

This puts us into the position to formulate the following result:

Theorem 2.1. *Let $\rho > 0$ and let ξ be a geometric functional on marked points, where the marks along with $\mathbb{P}_{\mathcal{M}}$ are as described above. Suppose that ξ is confined to $\breve{A}^{\mathrm{in}}_{\rho}$ and that κ satisfies Assumption* D. *Take $\lambda_0 > 0$.*

(1) *If the family $(\xi_\lambda)_{\lambda > \lambda_0}$ satisfies Assumption* M1(p, κ) *for some $p > 2$, then ξ satisfies Assumption* CV(κ). *Consequently,* (1.7) *holds.*
(2) *Let $0 < \tau < \infty$. Suppose that the family $(\xi_\lambda)_{\lambda > \lambda_0}$ satisfies Assumption* MH1(p, τ, Ω) *for some $p > 2$ and some convex domain Ω with $0 < \mathrm{vol}(\Omega) < \infty$. Next, suppose that for notably many triples $(t, s, \breve{\mathcal{X}})$, we have $\xi((\mathbf{0}, t, s), \breve{\mathcal{X}}) \neq 0$ and $t > \max\{t'; (x', t', s') \in \breve{\mathcal{X}}\}$. Then, with V as in* (1.8), *we have $V(\tau) > 0$.*
(3) *Let $\alpha \geq 0$. If ξ satisfies Assumption* MGI(α, κ), *then it satisfies Assumption* G$(1 + \alpha + d, \kappa)$; *if ξ satisfies Assumption* MGP(α, κ), *it satisfies Assumption* G$(\max\{\alpha, 1\} + d, \kappa)$. *Consequently, the conclusions of Theorems* 1.3, 1.4 *and* 1.5 *hold with suitable γ.*

Proof.

(1) From the exponential bound (2.1), it follows that the family $(R_\lambda)_{\lambda > \lambda_0}$ satisfies Assumption M(q, κ) for all $q \geq 0$. Similarly, R satisfies Assumption FH(τ) for all $\tau > 0$. As a result, ξ satisfies Assumption CV(κ).
(2) Again from bound (2.1), it follows that the family $(R_\lambda)_{\lambda > \lambda_0}$ satisfies Assumption MH(q, τ, Ω) for all $q \geq 0$. Thus, by Theorem 1.2, it suffices to show that $\Delta((\mathbf{0}, t, s), \breve{\mathcal{X}}) \neq 0$ and that $\breve{\mathcal{X}}$ is externally stable at $(\mathbf{0}, t, s)$ with respect to ξ provided that $\breve{\mathcal{X}}$ is finite, $\xi((\mathbf{0}, t, s), \breve{\mathcal{X}}) \neq 0$, $t > \max\{t'; (x', t', s') \in \breve{\mathcal{X}}\}$ and, in addition, without loss of generality, $(\mathbf{0}, t, s) \notin \breve{\mathcal{X}}$. Since the time stamp t is the largest of all, we have $\breve{A}^{\mathrm{in}}_{\rho}(\breve{x}, \breve{\mathcal{X}}) = \breve{A}^{\mathrm{in}}_{\rho}(\breve{x}, \breve{\mathcal{X}} \cup \{(\mathbf{0}, t, s)\})$. As ξ is confined to $\breve{A}^{\mathrm{in}}_{\rho}$, we also have $\xi(\breve{x}, \breve{\mathcal{X}}) = \xi(\breve{x}, \breve{\mathcal{X}} \cup \{(\mathbf{0}, t, s)\})$ for all $\breve{x} \in \breve{\mathcal{X}}$, so that $\Delta((\mathbf{0}, t, s), \breve{\mathcal{X}}) = \xi((\mathbf{0}, t, s), \breve{\mathcal{X}}) \neq 0$. Moreover, letting $r := \max_{\breve{x} \in \breve{\mathcal{X}}} \|x\| + \rho$, observe that $\breve{\mathcal{X}}$ is r-externally stable at $(\mathbf{0}, t, s)$ with respect to ξ. This proves the desired assertion.
(3) It suffices to observe that κ-uniform exponential decay of the family $(R_\lambda)_{\lambda > \lambda_0}$ implies Assumption MGI(d, κ). □

Now we return to our two examples, the RSA and the spatial birth–growth model. As ξ is then bounded, the family $(\xi_\lambda)_{\lambda > \lambda_0}$ satisfies Assumptions M1(p, κ) and MH1(p, τ, Ω) for all $p \geq 0$, all $0 < \tau < \infty$ and all suitable domains Ω. Since the first particle is always accepted, we have $\xi((\mathbf{0}, s, t), \varnothing) = 1 \neq 0$. As a result, the limiting variance is non-degenerate, i.e., $V(\tau) > 0$ for all $\tau > 0$. Finally, the family $(\xi_\lambda)_{\lambda \geq \lambda_0}$ satisfies Assumption MGP$(0, \kappa)$ and therefore Assumption G$(d + 1)$. Thus, the conclusions of Theorems 1.3, 1.4 and 1.5 hold with $\gamma = d + 1$.

Compared to the results in [1], there are three general novelties: first, we provide more explicit bounds in Theorem 1.3. Second, we consider a much more general class of geometric functionals. Third, we consider a broader class of intensities κ: in particular, they need not have bounded support (in contrast to Theorem 2.2 where bounded support is required because the density has to be bounded away from zero). Thus, in the basic RSA model, our present results add to existing central limit theorems [2,3,12,28], weak laws of large numbers [7,28,29] and large deviations [1,34] for random packing functionals.

Regarding the spatial birth–growth model, note that the paper [1] only succeeds to treat this model under an unnatural positive lower bound for initial particle sizes, which excludes for instance the crucial Johnson–Mehl set-up. Here this condition is no longer required. Our present results add to existing central limit theorems [3,6,25,28] as well as to the large deviation principle [34].

2.2. *Nearest neighbors*

Let $\breve{\mathcal{X}}$ be a locally finite point configuration in $\breve{\mathbb{R}}^d$. Take $\breve{x} \in \breve{\mathcal{X}}$ and $k \in \mathbb{N}$. We define the set of k nearest neighbors of $\breve{x}$ in $\breve{\mathcal{X}}$ to be the set of all $\breve{y} \in \breve{\mathcal{X}} \setminus \{\breve{x}\}$, such that $\|z - x\| < \|y - x\|$ for strictly less than k points $\breve{z} \in \breve{\mathcal{X}} \setminus \{\breve{x}\}$. Thus, if $\breve{\mathcal{X}}$ consists of a point $\breve{x}$, a point $\breve{y}$ with $\|y - x\| = 1$, two more points $\breve{z}, \breve{w}$ with $\|z - x\| = \|w - x\| = 2$ and possibly some more points with the distance to x strictly larger than 2, the set of two nearest neighbors of $\breve{x}$ in $\breve{\mathcal{X}}$ actually consists of three points: $\breve{y}$, $\breve{z}$ and $\breve{w}$.

Let $k \in \mathbb{N}$. Define the two set-valued geometric functional $\mathrm{NN}^{k,\rightarrow}$ and NN^{k} as follows: let $\mathrm{NN}^{k,\rightarrow}(\breve{x}, \breve{\mathcal{X}})$ be the set consisting of $\breve{x}$ and the set of k nearest points of $\breve{x}$ in $\breve{\mathcal{X}}$; let $\mathrm{NN}^{k}(\breve{x}, \breve{\mathcal{X}})$ be the union of $\mathrm{NN}^{k,\rightarrow}(\breve{x}, \breve{\mathcal{X}})$ plus the set of all $\breve{y} \in \breve{\mathcal{X}}$, such that $\breve{x}$ is among the k nearest neighbors of $\breve{y}$ in $\breve{\mathcal{X}}$. We shall consider geometric functionals confined to $\mathrm{NN}^{k,\rightarrow}$ or NN^{k} according to Definition 2.1. Notice that by Remark 2.2, any geometric functional confined to $\mathrm{NN}^{k,\rightarrow}$ is also confined to NN^{k}.

Fix a domain $\Omega \subseteq \mathbb{R}^d$. The construction of a radius of stabilization for NN^k inside Ω is well-known. Following [25], consider a collection $C_1, \dots, C_s$ of infinite open cones with angular radius $\pi/12$ and apex at $\mathbf{0}$, with union $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Let C_i^+ be the open cone concentric to C_i and with angular radius $\pi/6$. For a configuration $\breve{\mathcal{X}} \subset \breve{\Omega} = \Omega \times \mathcal{M}$ and $\breve{x} \in \breve{\mathcal{X}}$, define $R^{k,\Omega,i}(\breve{x}, \breve{\mathcal{X}})$ to be the distance from x to its kth nearest point in $\breve{\mathcal{X}} \cap (C_i^+ + x)$ if such a point exists and this distance is less than $\operatorname{diam}((C_i + x) \cap \Omega)$; otherwise, set $R^{k,\Omega,i}(\breve{x}, \breve{\mathcal{X}}) := \operatorname{diam}((C_i + x) \cap \Omega)$.

Let $R^{k,\Omega}(\breve{x}, \breve{\mathcal{X}}) := \max_i R^{k,\Omega,i}(\breve{x}, \breve{\mathcal{X}})$. From elementary geometry (see [25]), it follows that if $\breve{\mathcal{X}} \subset \breve{\Omega}$, $\breve{x} \in \breve{\mathcal{X}}$ and $\|y - x\| > R^{k,\Omega}(\breve{x}, \breve{\mathcal{X}})$, then neither $\breve{y} \in \mathrm{NN}^{k,\to}(\breve{x}, \breve{\mathcal{X}})$ nor $\breve{x} \in \mathrm{NN}^{k,\to}(\breve{y}, \breve{\mathcal{X}})$. As a result, $R^{k,\Omega}$ is a radius of stabilization inside Ω for NN^k as well as for any geometric functional confined to NN^k.

If $0 < \tau < \infty$, then the homogeneous Poisson process $\mathcal{P}_\tau$ almost surely contains infinitely many points in every cone C_i. Therefore, for $\Omega = \mathbb{R}^d$, $R^{k,\Omega}$ satisfies Assumption FH(τ).

Now assume that Ω is bounded and convex with $\operatorname{vol}(\Omega) > 0$. Take $\lambda > 0$ and let κ be a probability density function vanishing outside Ω, but with $\inf_{x\in\Omega} \kappa(x) > 0$. We will show that $R^{k,\Omega}$ enjoys super-exponential tail decay. Basically, we follow [25], but it turns out that one has to be a bit more careful. Take $\breve{x} \in \breve{\Omega}$ and $i = 1, \dots, s$. It is easy to see that if $R^{k,\Omega,i}(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > \rho$, then there are less than k points in $\breve{\mathcal{P}}_{\lambda\kappa} \cap B_\rho(x) \cap (C_i^+ + x)$, but also at least one point $y \in \Omega \cap (C_i + x)$ with $\|y - x\| > \rho$. By convexity, there also exists a point $z \in \Omega \cap (C_i + x)$ with $\|z - x\| = \rho/2$. Setting $\eta := \sin \frac{\pi}{12}$, we have $B_{\eta\rho/2}(z) \subseteq B_\rho(x) \cap (C_i^+ + x)$.

The continuation of the argument in [25] works provided that $B_{\eta\rho/2}(z) \subseteq \Omega$, but this is not necessarily true. However, letting $D := \operatorname{diam}(\Omega)$, we have $\Omega' := \{(1 - \frac{\eta\rho}{2D})z + \frac{\eta\rho}{2D} w;\ w \in \Omega\} \subseteq \Omega$ as well as $\Omega' \subseteq B_{\eta\rho/2}(z) \subseteq B_\rho(x) \cap (C_i^+ + x)$. Therefore, the set $\Omega' \times \mathcal{M}$ contains less than k points in $\breve{\mathcal{P}}_{\lambda\kappa}$. Since $\operatorname{vol}(\Omega') = (\frac{\eta\rho}{2D})^d \operatorname{vol}(\Omega)$ and $\Omega' \subseteq \Omega$, the probability that $\Omega' \times \mathcal{M}$ contains less than k points in $\breve{\mathcal{P}}_{\lambda\kappa}$ is bounded from above by:

$$\sum_{l=0}^{k-1} \frac{1}{l!} \left[\lambda m \left(\frac{\eta\rho}{2D}\right)^d \operatorname{vol}(\Omega)\right]^l \exp\left[-\lambda m \left(\frac{\eta\rho}{2D}\right)^d \operatorname{vol}(\Omega)\right],$$

where $m = \inf_{x\in\Omega} \kappa(x)$. This is also an upper bound on $\mathbb{P}(R^{k,\Omega,i}(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > \rho)$. Consequently, there exist $a \ge 0$ and $b > 0$ depending only on Ω, κ and k, such that $\mathbb{P}(R^{k,\Omega}(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) \ge \rho) \le a\mathrm{e}^{-b\lambda\rho^d}$ for all $\rho \ge 0$, all $\lambda > 0$ and all $\breve{x} \in \breve{\Omega}$.

Recalling that $\xi_\lambda(\breve{x}, \breve{\mathcal{X}}) = \xi(\lambda^{1/d}\breve{x}, \lambda^{1/d}\breve{\mathcal{X}})$, observe that if ξ is confined to NN^k, then ξ_λ is also confined to NN^k. Therefore, $R^{k,\Omega}$ is a radius of stabilization for ξ_λ inside Ω, so that we can set $R_\lambda(\breve{x}, \breve{\mathcal{X}}) := \lambda^{1/d} R^{k,\Omega}(\breve{x}, \breve{\mathcal{X}})$. Then we have $\mathbb{P}(R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > \rho) \le a\mathrm{e}^{-b\rho^d}$, with a and b uniform in $\breve{x}$, λ and ρ. Consequently,

$$\begin{aligned}\mathbb{E}\big(R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\big)^j &= \int_0^\infty j\rho^{j-1}\mathbb{P}\big(R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > \rho\big)\,\mathrm{d}\rho \le aj\int_0^\infty \rho^{j-1}\mathrm{e}^{-b\rho^d}\,\mathrm{d}\rho \\ &= \frac{aj}{d}\int_0^\infty t^{j/d-1}\mathrm{e}^{-bt}\,\mathrm{d}t = \frac{aj}{b^{j/d}d}\Gamma\left(\frac{j}{d}\right) \le B^j(j!)^{1/d} \end{aligned} \tag{2.2}$$

for some B depending only on a and b. Thus, the family $(R_\lambda)_{\lambda>0}$ satisfies Assumption MGI($1/d, \kappa$) (and, since $R_\lambda(\breve{x}, \breve{\mathcal{Y}}) \le R_\lambda(\breve{x}, \breve{\mathcal{X}})$ for $\breve{\mathcal{Y}} \supseteq \breve{\mathcal{X}}$, even Assumption MGP($1/d, \kappa$)). This puts us into the position to formulate the following result:

Theorem 2.2. *Let $k \in \mathbb{N}$ and let ξ be a geometric functional confined to NN^k. Take a convex bounded domain Ω and κ satisfying Assumption D and with $\inf_{x\in\Omega} \kappa(x) > 0$. Let $\lambda_0 > 0$ and let the cones C_i be as above.*

(1) *If the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption M1(p, κ) for some $p > 2$, then ξ satisfies Assumption CV(κ). Consequently, (1.7) holds.*
(2) *Let $0 < \tau < \infty$. Suppose that the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH1(p, τ, Ω) for some $p > 2$. Next, suppose that notably many pairs $(t, \breve{\mathcal{X}})$ satisfy the following two conditions: first, $\Delta((\mathbf{0}, t), \breve{\mathcal{X}}) \ne 0$ (recalling the definition of the add-one cost from Section 1.4); second, there exists $\rho > 0$, such that for each cone C_i, each of the sets $\breve{\mathcal{X}} \cap C_i \cap B_\rho(\mathbf{0})$ and $(\breve{\mathcal{X}} \cap C_i) \setminus B_{\rho/\eta}(\mathbf{0})$ contains at least k points; here, $\eta = \sin \frac{\pi}{12}$. Then, with V as in (1.8), we have $V(\tau) > 0$.*
(3) *Let $0 < \tau < \infty$. Suppose that ξ is confined to $\mathrm{NN}^{k,\to}$ and that the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH1(p, τ, Ω) for some $p > 2$. Next, suppose that notably many pairs $(t, \breve{\mathcal{X}})$ satisfy the following two*

conditions: first, $\xi((\mathbf{0},t),\check{\mathcal{X}})\neq 0$*; second, there exists* $\rho>0$*, such that* $\check{\mathcal{X}}\cap B_\rho(\mathbf{0})=\varnothing$ *and such that each intersection* $\check{\mathcal{X}}\cap B_{\rho\sqrt{3}}(\mathbf{0})\cap C_i$ *contains at least* $k+1$ *points. Then we have* $V(\tau)>0$.

(4) *Let* $\alpha\geq 0$*. If* ξ *satisfies Assumption* MGI(α)*, then it satisfies Assumption* G($2+\alpha$)*; if* ξ *satisfies Assumption* MGP(α)*, it satisfies Assumption* G($\max\{\alpha,1\}+1$)*. Consequently, the conclusions of Theorems* 1.3*,* 1.4 *and* 1.5 *hold with suitable* γ*.*

Proof.

(1) From (2.2), it follows that the family $(R_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption M(q,κ) for all $q\geq 0$. As a result, ξ satisfies Assumption CV(κ).

(2) Similarly as before, we find that the family $(R_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(q,τ,Ω) for all $q\geq 0$. By Theorem 1.2, it remains to show that for each $t\in\mathcal{M}$, any finite configuration $\check{\mathcal{X}}\subset\check{\mathbb{R}}^d\setminus\{(\mathbf{0},t)\}$ satisfying the specified conditions is externally stable at $(\mathbf{0},t)$ with respect to ξ. First, take a finite configuration $\check{\mathcal{Y}}$ with $\check{\mathcal{Y}}\cap B_\rho(\mathbf{0})=\check{\mathcal{X}}\cap B_\rho(\mathbf{0})$ and observe first that no point in $\check{\mathcal{Y}}\setminus B_\rho(\mathbf{0})$ is among the k nearest neighbors of $(\mathbf{0},t)$ in $\check{\mathcal{Y}}\cup\{(\mathbf{0},t)\}$; similarly, $(\mathbf{0},t)$ is not among the k nearest neighbors of any point in $\check{\mathcal{Y}}\setminus B_\rho(\mathbf{0})$. Therefore, $\check{\mathcal{X}}$ is basically ρ-externally stable at $(\mathbf{0},t)$ with respect to ξ.

Next, let $x\in B_\rho(\mathbf{0})$, $y\in C_i\setminus B_{\rho/\eta}(\mathbf{0})$ and take u on the axis of C_i, inside C_i. Then the angle between y and u is smaller than $\pi/12$. By elementary geometry, the angle between y and $y-x$ is also smaller than $\pi/12$, for $\|x\|/\|y\|<\eta$. Consequently, the angle between $y-x$ and u is smaller than $\pi/6$. As a result, we have $C_i\setminus B_{\rho/\eta}(\mathbf{0})\subseteq(C_i^+ + x)$ for all $x\in C_i\cap B_\rho(\mathbf{0})$. Thus, taking $\check{x}\in(\check{\mathcal{X}}\cap B_\rho(\mathbf{0}))\cup\{(\mathbf{0},t)\}$, each of the sets $(C_i^+ + x)\cap\check{\mathcal{X}}$ contains at least k points, so that ξ stabilizes with respect to $\check{\mathcal{X}}$ and $\check{\mathcal{X}}\cup\{(\mathbf{0},t)\}$ at all $\check{x}\in(\check{\mathcal{X}}\cap B_\rho(\mathbf{0}))\cup\{(\mathbf{0},t)\}$. By Proposition 1.1, $\check{\mathcal{X}}$ is externally stable at $(\mathbf{0},t)$ with respect to ξ. This proves the desired assertion.

(3) Set $A:=B_{\rho\sqrt{3}}(\mathbf{0})\setminus B_\rho(\mathbf{0})$. Similarly as in the preceding point, it suffices to show that for each pair $(t,\check{\mathcal{X}})$ satisfying the specified conditions, we have $\Delta((0,t),\check{\mathcal{X}})\neq 0$ and $\check{\mathcal{X}}$ is externally stable at $(\mathbf{0},t)$ with respect to ξ. First, by elementary geometry, we have $\operatorname{diam}(C_i\cap A)=\rho$. Therefore, $(\mathbf{0},t)\notin\mathrm{NN}^{k,\to}(\check{y},\check{\mathcal{X}}\cup\{(\mathbf{0},t)\})$ for all $\check{y}\in(C_i\cap A)\times\mathcal{M}$ and therefore for all $\check{y}\in A\times\mathcal{M}$. However, this is also true if $\check{y}\notin B_{\rho\sqrt{3}}(\mathbf{0})\times\mathcal{M}$ and therefore for all $\check{y}\in\check{\mathcal{X}}$. Since ξ is confined to $\mathrm{NN}^{k,\to}$, we then have $\xi(\check{y},\check{\mathcal{X}}\cup\{(\mathbf{0},t)\})=\xi(\check{y},\check{\mathcal{X}})$. As a result, $\Delta((\mathbf{0},t),\check{\mathcal{X}})=\xi((\mathbf{0},t),\check{\mathcal{X}})\neq 0$.

Now take a finite configuration $\check{\mathcal{Y}}$ with $\check{\mathcal{Y}}\cap B_{\rho\sqrt{3}}(\mathbf{0})=\check{\mathcal{X}}\cap B_{\rho\sqrt{3}}(\mathbf{0})$. Similarly as above, we find that for all $\check{z}\in\check{\mathcal{Y}}\setminus B_{\rho\sqrt{3}}(\mathbf{0})$, we have $(\mathbf{0},t)\notin\mathrm{NN}^{k,\to}(\check{y},\check{\mathcal{Y}}\cup\{(\mathbf{0},t)\})$. Therefore, $\check{\mathcal{X}}$ is basically $\rho\sqrt{3}$-externally stable at $(\mathbf{0},t)$ with respect to ξ. Moreover, since $C_i\cap A\cap\check{\mathcal{X}}$ contains at least $k+1$ points and since $\operatorname{diam}(C_i\cap A)=\rho$ for all i, we have $\mathrm{NN}^{k,\to}(\check{y},\check{\mathcal{X}}\cup\{(\mathbf{0},t)\})\subseteq B_\rho(y)\times\mathcal{M}$ for all $\check{y}\in\check{\mathcal{X}}\cap B_{\rho\sqrt{3}}(\mathbf{0})=\check{\mathcal{X}}\cap A$. Therefore, ξ stabilizes at $\check{y}$ within radius ρ with respect to $\check{\mathcal{X}}$ as well as to $\check{\mathcal{X}}\cup\{(\mathbf{0},t)\}$. By Proposition 1.1, $\check{\mathcal{X}}$ is externally stable at $(\mathbf{0},t)$ with respect to ξ. This proves the desired assertion.

(4) This follows immediately from the fact that the family $(R_\lambda)_{\lambda>0}$ satisfies Assumption MGI$(1/d,\kappa)$. □

Theorem 2.2 adds to the existing results on non-degeneracy of the limiting variance (see [27]), central limit theorems (see Chapter 4 of [21] as well as [3]) and, of course, large deviation results. For the latter, observe that the paper [1] was only able to deal with the *empirical functionals of nearest neighbors graphs*, where $\xi(x,\mathcal{X})$ is the indicator of the event that the total edge length exceeds a certain threshold, or of some event involving the degree of the graph at x and possibly also the edge length, such as ‘the total length of edges incident to x exceeds a certain multiplicity of the graph degree of x’ etc. No marks have been considered in [1]. Our result includes a broader class of intensities κ and a much more general collection of geometric functionals. The following example serves as a classical one.

Example 2.1. *Define* $\xi(x,\mathcal{X})$ *to be the sum of the distances from* x *to its* k *nearest neighbors* (*there are no marks*)*. Then* ξ *is obviously confined to* $\mathrm{NN}^{k,\to}$*. Recalling that* Ω *is a bounded convex domain with* $\operatorname{vol}(\Omega)>0$*, that* $\inf_\Omega\kappa>0$ *and that* κ *vanishes outside* Ω*, a similar argument as the one used for the radius of stabilization shows that the family* $(\xi_\lambda)_{\lambda>\lambda_0}$ *satisfies Assumption* MGP$(1/d,\kappa)$*, as well as Assumptions* M1(p,κ) *and* MH1(p,τ,Ω) *for all* $p,\tau>0$*. Therefore,* ξ *satisfies Assumption* CV(κ) *and, more importantly, Assumption* G$(2,\kappa)$*. This means that the range where moderate deviation results apply is independent of the dimension. Finally, it is obvious that for all* *non-empty configurations* $\mathcal{X}$, *we have* $\xi(\mathbf{0}, \mathcal{X}) \neq 0$. *As a result, the limiting variance is non-degenerate, i.e.,* $V(\tau) > 0$ *for all* $\tau > 0$.

With a little extra effort, one can prove the same results for the 'undirected' case, that is, for the sum of all distances from x to the points in $\mathrm{NN}^1(x, \mathcal{X})$ (*this is twice the total edge length of the undirected k nearest neighbors graph*). *However, for the non-degeneracy of the limiting variance, one has to refer to part* (2) *rather than to part* (3) *of Theorem* 2.2. *This requires a bit more involved argument, which we shall not provide here because the non-degeneracy is proved explicitly in* [26]; *it can also be deduced from the earlier paper* [27].

Curiously, the paper [26] provides no explicit application of its general result on non-degeneracy of the limiting variance, i.e., Theorem 2.2 ibidem. For the k nearest neighbors, the limiting variance is computed explicitly. Therefore, we give another example, where we prove non-degeneracy, but the construction from the earlier paper [27] does not work.

Example 2.2. *Define* $\xi(x, \mathcal{X}) := \exp(-\sum_{y \in \mathrm{NN}^k(x,\mathcal{X})} \|y - x\|)$. *Turning first to large deviations, observe that* ξ *is bounded. Therefore, taking* Ω *and* κ *as before,* ξ *satisfies Assumption* CV(κ) *as well as Assumption* G($2, \kappa$). *Thus, we obtain just the same range of moderate deviation results as in the previous example.*

Now we turn to the limiting variance. Consider first a configuration $\mathcal{X}$ *containing some point* $y \neq \mathbf{0}$ *and no point in* $B_{\|y\|}(\mathbf{0}) \setminus \{y\}$. *Letting* $A_\rho := B_{\rho\sqrt{3}}(\mathbf{0}) \setminus B_\rho(\mathbf{0})$, *assume also that each intersection* $\mathcal{X} \cap C_i \cap A_{\|y\|}$ *contains at least two points. Then y is the nearest neighbor of* $\mathbf{0}$. *Moreover, similarly as in the proof of Theorem* 2.2(3), *we find that* $\mathbf{0}$ *is not the nearest neighbor of any point in* $\mathcal{X}$. *Therefore, we have* $\Delta(\mathbf{0}, \mathcal{X}) = \mathrm{e}^{-\|y\|} - (1 - \mathrm{e}^{-\|y\|})\xi(y, \mathcal{X})$.

Now take a finite configuration $\mathcal{X}$, *which satisfies the condition from the preceding paragraph. If, in addition,* $\|y\| < \log 2$, *then, clearly,* $\Delta(\mathbf{0}, \mathcal{X}) > 0$. *Moreover, taking* $r := \|y\|\sqrt{3}/\sin\frac{\pi}{12}$, *assume that each of the sets* $(\mathcal{X} \cap C_i) \setminus B_r(\mathbf{0})$ *is non-empty. One can easily check that notably many configurations* $\mathcal{X}$ *satisfy this condition. From Theorem* 2.2(2), *it then follows that* $V(\tau) > 0$ *for all* $\tau > 0$.

In [27], *the argument used to show non-degeneracy of the limiting variance requires, among others, that* Δ *is stabilizing. A relatively simple construction of a radius of stabilization for* Δ, *much similar to the above-mentioned construction of radius of stabilization for* ξ, *is provided for the total edge length of the nearest neighbor graph* (*considered also in Example* 2.1) *in the plane* ($d = 2$); *a much more complicated construction is used for the number of components. Here, we demonstrate that the construction used in* [27] *for the total edge length does not work here.*

In [27], *the radius of stabilization at the origin is obtained by means of six disjoint equilateral triangles, such that the origin is a vertex of each triangle. If this construction works for some geometric functional, one can also take a covering of* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *by a family of open angles. Clearly, one can assume that their measures are at most* $\pi/6$. *Now take any family* $C_1', \ldots, C_m'$ *of open cones with angular radii at most* $\pi/3$, $k \geq 2$, $a > 1 + \sqrt{3}/3$ *and define* $R^\Delta(x, \mathcal{X})$ *to be a times the minimal* ρ, *such that any set* $\mathcal{X} \cap (C_i' + x)$ *contains at least k points. Below we show that for any* $\tau > 0$, *the probability that* R^Δ *is not a radius of stabilization for* Δ *is strictly positive.*

Take a configuration $\mathcal{X}$ *containing a point* $y \in C_1'$, *no point in* $B_{\|y\|}(\mathbf{0}) \setminus \{y\}$, *some point* $z \in C_1$ *with* $\|z\|/\|y\| > a\sqrt{3}$, *no point in* $B_{\|z-y\|}(z) \setminus \{y\}$ *and no point in* $B_{a\|y\|\sqrt{3}}(\mathbf{0}) \setminus B_{\|y\|\sqrt{3}}(\mathbf{0})$. *Defining* A_ρ *as before, assume also that each of the sets* $\mathcal{X} \cap A_{\|y\|} \cap C_i$ *contains at least* $k + 1$ *points. Some thought shows that such a configuration occurs with non-zero probability in any homogeneous Poisson point process* $\mathcal{P}_\tau$.

Letting $\mathcal{Y} := \mathcal{X} \cap B_{a\|y\|\sqrt{3}}(\mathbf{0})$, *observe that for any point in* $\mathcal{Y}$, *there exists another point in the same set within distance* $\|y\|$. *Moreover, since* $a > 1 + \sqrt{3}/3$ *and since* $\mathcal{X} \cap (B_{a\|y\|\sqrt{3}}(\mathbf{0}) \setminus B_{\|y\|\sqrt{3}}(\mathbf{0})) = \varnothing$, *there exists no point in* $\mathcal{X} \setminus B_{a\|y\|\sqrt{3}}(\mathbf{0})$ *within distance* $\|y\|$. *Therefore, the nearest neighbor in* $\mathcal{X} \cup \{\mathbf{0}\}$ *of any point in* $\mathcal{Y}$ *also lies in* $\mathcal{Y}$. *As a result,* $\mathrm{NN}^1(y, \mathcal{X}) \supseteq \mathrm{NN}^1(y, \mathcal{Y})$. *Moreover, the inclusion is proper because y is the nearest neighbor of z in* $\mathcal{X}$. *Therefore,* $\mathrm{NN}^1(y, \mathcal{X}) < \mathrm{NN}^1(y, \mathcal{Y})$. *Since* $\Delta(\mathbf{0}, \mathcal{X}) = \mathrm{e}^{-\|y\|} - (1 - \mathrm{e}^{-\|y\|})\xi(y, \mathcal{X})$ *and analogously for* $\mathcal{Y}$, *we have* $\Delta(\mathbf{0}, \mathcal{X}) > \Delta(\mathbf{0}, \mathcal{Y})$. *Therefore,* Δ *does not stabilize at* $\mathbf{0}$ *within radius* $a\|y\|\sqrt{3}$, *nor does it stabilize within radius* $R^\Delta(\mathbf{0}, \mathcal{X})$ *because* $R^\Delta(\mathbf{0}, \mathcal{X}) \leq a\|y\|\sqrt{3}$.

Thus, we can conclude that the construction from [27] *does not work in this case. However, this does not mean that* Δ *does not stabilize. In fact, it almost surely stabilizes at* $\mathbf{0}$ *with respect to* $\mathcal{P}_\tau$: *examining the proof of Theorem* 2.2(2) *and using the basic properties of homogeneous Poisson point processes, we find that it provides an explicit construction of an external radius of stabilization, which is almost surely finite. By Remark* 1.4, *this is also a radius of stabilization for* Δ.

2.3. *Sphere of Influence Graphs*

Given a locally finite set $\check{\mathcal{X}} \subset \check{\mathbb{R}}^d$, the *sphere of influence graph* $\mathrm{SIG}(\check{\mathcal{X}})$ is a graph with vertex set $\check{\mathcal{X}}$ constructed as follows: for each $\check{x} \in \check{\mathcal{X}}$, let $B(\check{x}, \check{\mathcal{X}})$ be a ball around x with radius equal to $\min_{\check{y} \in \check{\mathcal{X}} \setminus \{\check{x}\}}\{\|y - x\|\}$ (in particular, the ball is degenerate if two points with different marks share the same location). Then $B(\check{x}, \check{\mathcal{X}})$ is called the *sphere of influence* of $\check{x}$. Draw an edge between $\check{x}$ and $\check{y}$ iff the balls $B(\check{x}, \check{\mathcal{X}})$ and $B(\check{y}, \check{\mathcal{X}})$ overlap. The collection of such edges is the *sphere of influence graph* (SIG) on $\check{\mathcal{X}}$ and is denoted by $\mathrm{SIG}(\check{\mathcal{X}})$.

In [27], non-degeneracy of the limiting variance and central limit theorems are derived for a variety of functionals, i.e., the total number of edges, the total edge length, the number of vertices of fixed degree and, most remarkably, for the number of components. Except for the latter functional, these results are extended in [3] to random measures.

Here we shall only consider functionals confined to the functional NSIG, where $\mathrm{NSIG}(\check{x}, \check{\mathcal{X}})$ denotes the set of points which are adjacent to $\check{x}$ in $\mathrm{SIG}(\check{\mathcal{X}})$ (including $\check{x}$). Notice that the total number of edges, the total edge length and the number of vertices of fixed degree can all be expressed in terms of suitable functionals ξ confined to NSIG, while for the number of components, this seems not to be possible.

First, we turn to stabilization. A construction of a radius of stabilization is given in [27] and is also used in [3]. However, the results ibidem do not entirely fit the concept of stabilization and external stabilization used here. In particular, they do not include a domain Ω. Therefore, we here refine the construction in a similar way as in the case of nearest neighbors. First, we rewrite the stabilization result from p. 1030 of [27].

Proposition 2.1. *Let $\check{\mathcal{X}} \subset \mathbb{R}^d$ be a finite configuration. Take $\check{x} \in \check{\mathcal{X}}$, $\rho > 0$ and an open cone C in $\mathbb{R}^d$ with angular radius $\pi/12$ and apex at x. Assume that the intersection $(\check{\mathcal{X}} \setminus \{\check{x}\}) \cap B_\rho(x)$ is non-empty and that there also exists a marked point $\check{y} \in \check{\mathcal{X}} \cap (C \setminus B_{3\rho}(x))$. Let $r = \|y - x\|$. Then no point in $\check{\mathcal{X}} \cap (C \setminus B_r(x))$ is adjacent to $\check{x}$ in $\mathrm{SIG}(\check{\mathcal{X}})$. Moreover, for any finite configuration $\check{\mathcal{Y}}$ with $\check{\mathcal{Y}} \cap B_{2r}(x) = \check{\mathcal{X}} \cap B_{2r}(x)$, we have $\mathrm{NSIG}(\check{x}, \check{\mathcal{Y}}) \cap C = \mathrm{NSIG}(\check{x}, \check{\mathcal{X}}) \cap C$.*

Proof. Take $\check{z} \in \check{\mathcal{X}} \cap (C \setminus B_r(x))$ and let $z' := x + \frac{r}{\|z-x\|}(z - x)$. Since $y, z' \in C$, we have $\|z' - y\| < 2\eta r$, where $\eta = \sin\frac{\pi}{12}$ (but not necessarily $\|z' - y\| < r/2$, as estimated in display (7.4) of [27]). Therefore, $\|z - y\| \le \|z - z'\| + \|z' - y\| < \|z - x\| - (1 - 2\eta)r$, so that $B(\check{z}, \check{\mathcal{X}})$ does not overlap with $B_{(1-2\eta)r}(\check{x})$. Since $(1 - 2\eta)r > 3(1 - 2\eta)\rho > \rho$, it does not overlap with $B(\check{x}, \check{\mathcal{X}})$ either.

Finally, if $\check{\mathcal{X}} \cap B_{2r}(\check{x}) = \check{\mathcal{Y}} \cap B_{2r}(\check{x})$, then any two points in $\check{\mathcal{X}} \cap B_r(\check{x}) = \check{\mathcal{Y}} \cap B_r(\check{x})$ are adjacent in $\mathrm{SIG}(\check{\mathcal{X}})$ if and only if they are adjacent in $\mathrm{SIG}(\check{\mathcal{Y}})$. Combined with the above, this proves the result. □

This allows us to construct a radius of stabilization inside a domain $\Omega \subseteq \mathbb{R}^d$ in a similar way as in the case of nearest neighbors. Consider a collection $C_1, \dots, C_s$ of infinite open cones with angular radius $\pi/24$ and apex at $\mathbf{0}$, with union $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Let C_i^+ be the open cone concentric to C_i and with angular radius $\pi/12$. Take a configuration $\check{\mathcal{X}} \subset \check{\Omega} = \Omega \times \mathcal{M}$ and $\check{x} \in \check{\mathcal{X}}$. Suppose that $\check{\mathcal{X}}$ contains at least one more point and denote by ρ the distance from x to the nearest neighbor of $\check{x}$ in $\check{\mathcal{X}}$ (which equals zero if there is another marked point at the same location). Next, suppose that the set $(\check{\mathcal{X}} \cap (C_i^+ + x)) \setminus B_{3\rho}(x)$ is non-empty and denote by r the distance from x to its nearest neighbor in $(\check{\mathcal{X}} \cap (C_i^+ + x)) \setminus B_{3\rho}(x)$. Set $R^{\Omega,i}(\check{x}, \check{\mathcal{X}}) := 2r$ if this construction works and $2r < \mathrm{diam}((C_i + x) \cap \Omega)$; otherwise, set $R^{\Omega,i}(\check{x}, \check{\mathcal{X}}) := \mathrm{diam}((C_i + x) \cap \Omega)$. Let $R^\Omega(\check{x}, \check{\mathcal{X}}) := \max_i R^{\Omega,i}(\check{x}, \check{\mathcal{X}})$. Proposition 2.1 and some thought show that R^Ω is a radius of stabilization inside Ω for the functional NSIG.

If $0 < \tau < \infty$, then the homogeneous Poisson process $\mathcal{P}_\tau$ almost surely contains a point in every cone C_i arbitrarily far from the origin. Therefore, for $\Omega = \mathbb{R}^d$, R^Ω satisfies Assumption FH(τ).

Now assume that Ω is bounded and convex with $\mathrm{vol}(\Omega) > 0$. Take $\lambda > 0$ and let κ be a probability density function vanishing outside Ω, but with $\inf_{x \in \Omega} \kappa(x) > 0$. Again, we will show that R^Ω enjoys super-exponential tail decay. Take $\check{x} \in \check{\Omega}$ and $i = 1, \dots, s$. It is easy to see that if $R^{\Omega,i}(\check{x}, \check{\mathcal{P}}_{\lambda\kappa}) > u$, then, first, either the set $(B_{u/9}(x) \cap (C_i^+ + x)) \times \mathcal{M}$ or the set $((B_{u/2}(x) \setminus B_{u/3}(x)) \cap (C_i^+ + x))) \times \mathcal{M}$ contains no point in $\check{\mathcal{P}}_{\lambda\kappa}$, and, second there is at least one point in $y \in \Omega \cap (C_i + x)$ with $\|y - x\| > u$. Now let $0 \le \theta \le 1$. By convexity, there also exists a point $z_\theta \in \Omega \cap (C_i + x)$ with $\|z_\theta - x\| = \theta u$. Setting $\varepsilon := \sin\frac{\pi}{24}$, we have $B_{\theta\varepsilon u}(z_\theta) \subseteq (B_{\theta(1+\varepsilon)u}(x) \setminus \mathring{B}_{\theta(1-\varepsilon)u}(x)) \cap (C_i^+ + x)$, where $\mathring{B}_r(x)$ denotes the open ball of radius r centered at x.

Letting $D := \mathrm{diam}(\Omega)$, we have $\Omega_\theta := \{(1 - \frac{\theta\varepsilon u}{D})z_\theta + \frac{\theta\varepsilon u}{D}w; w \in \Omega\} \subseteq \Omega$ as well as $\Omega_\theta \subseteq B_{\theta\varepsilon u}(z_\theta) \subseteq (B_{\theta(1+\varepsilon)u}(x) \setminus \mathring{B}_{\theta(1-\varepsilon)u}(x)) \cap (C_i^+ + x)$. In particular, routine calculation shows that $\Omega_{1/11} \subseteq B_{u/9}(x)$ and $\Omega_{2/5} \subseteq$

$B_{u/2}(x) \setminus B_{u/3}(x)$. Therefore, either $\Omega_{1/11} \times \mathcal{M}$ or $\Omega_{2/5} \times \mathcal{M}$ contains no points in $\breve{\mathcal{P}}_{\lambda\kappa}$. Since $\mathrm{vol}(\Omega_\theta) = (\frac{\theta\varepsilon u}{D})^d \mathrm{vol}(\Omega)$ and $\Omega_\theta \subseteq \Omega$, the probability that $\Omega_\theta \times \mathcal{M}$ contains no point in $\breve{\mathcal{P}}_{\lambda\kappa}$ is bounded from above by $\exp[-\lambda m(\frac{\theta\varepsilon u}{D})^d \mathrm{vol}(\Omega)]$, where $m = \inf_{x\in\Omega} \kappa(x)$. Consequently,

$$\mathbb{P}\big(R^{\Omega,i}(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > u\big) \le \exp\left[-\lambda m\left(\frac{\varepsilon u}{11D}\right)^d \mathrm{vol}(\Omega)\right] + \exp\left[-\lambda m\left(\frac{2\varepsilon u}{5D}\right)^d \mathrm{vol}(\Omega)\right] \le 2\mathrm{e}^{-b\lambda u^d}, \tag{2.3}$$

where $b = m(\varepsilon/(11D))^d \mathrm{vol}(\Omega)$. Thus, the radius of stabilization satisfies $\mathbb{P}(R^{\Omega}(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > u) \le 2s\mathrm{e}^{-b\lambda u^d}$.

Recalling that $\xi_\lambda(\breve{x}, \breve{\mathcal{X}}) = \xi(\lambda^{1/d}\breve{x}, \lambda^{1/d}\breve{\mathcal{X}})$, observe that if ξ is confined to NSIG, then ξ_λ is also confined to NSIG. Therefore, R^Ω is a radius of stabilization for ξ_λ inside Ω, so that we can set $R_\lambda(\breve{x}, \breve{\mathcal{X}}) := \lambda^{1/d} R^\Omega(\breve{x}, \breve{\mathcal{X}})$. Then we have $\mathbb{P}(R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa}) > u) \le 2s\mathrm{e}^{-bu^d}$. Similarly as in (2.2), it follows that the family $(R_\lambda)_{\lambda>0}$ satisfies Assumption MGP$(1/d, \kappa)$. This puts us into the position to formulate the following result:

Theorem 2.3. *Let $k \in \mathbb{N}$ and let ξ be a geometric functional confined to* NSIG. *Take a convex bounded domain Ω and κ satisfying Assumption* D *and with* $\inf_{x\in\Omega} \kappa(x) > 0$. *Let $\lambda_0 > 0$ and let the cones C_i be as above.*

(1) *If the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption* M1(p, κ) *for some $p > 2$, then ξ satisfies Assumption* CV(κ). *Consequently,* (1.7) *holds.*
(2) *Let $0 < \tau < \infty$ and suppose that the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption* MH1(p, τ, Ω) *for some $p > 2$. Next, suppose that notably many pairs $(t, \breve{\mathcal{X}})$ satisfy the following two conditions: first, $\Delta((\mathbf{0}, t), \breve{\mathcal{X}}) \neq 0$; second, there exist $\rho > 0$ and $r > 4\rho$, such that none of the sets $\breve{\mathcal{X}} \cap C_i \cap B_\rho(\mathbf{0})$, $\breve{\mathcal{X}} \cap C_i \cap (B_r(\mathbf{0}) \setminus B_{4\rho}(\mathbf{0}))$ and $\breve{\mathcal{X}} \cap (C_i \setminus B_{4(r+\rho)}(\mathbf{0}))$ is empty. Then, with V as in* (1.8), *we have $V(\tau) > 0$.*
(3) *Let $\alpha \ge 0$. If ξ satisfies Assumption* MGI(α), *then it satisfies Assumption* G$(2 + \alpha)$; *if ξ satisfies Assumption* MGP(α), *it satisfies Assumption* G$(\max\{\alpha, 1\} + 1)$. *Consequently, the conclusions of Theorems* 1.3, 1.4 *and* 1.5 *hold with suitable γ.*

Proof. Parts (1) and (3) follow exactly in the same way as parts (1) and (4) of Theorem 2.2. Now we turn to part (2). Clearly, the family $(R_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(q, τ, Ω) for all $q \ge 0$. By Theorem 1.2, it remains to show that for each $t \in \mathcal{M}$, any finite configuration $\breve{\mathcal{X}} \subset \breve{\mathbb{R}}^d \setminus \{(\mathbf{0}, t)\}$ satisfying the specified conditions is externally stable at $(\mathbf{0}, t)$ with respect to ξ.

First, we claim that $\breve{\mathcal{X}}$ is basically $(r+\rho)$-externally stable at $(\mathbf{0}, t)$ with respect to NSIG. Take a finite configuration $\breve{\mathcal{Y}}$ with $\breve{\mathcal{Y}} \cap B_{r+\rho}(\mathbf{0}) = \breve{\mathcal{X}} \cap B_{r+\rho}(\mathbf{0})$ and $\breve{z} \in \breve{\mathcal{Y}} \setminus B_{r+\rho}(\mathbf{0})$. What we have to show is that inserting a marked point at the origin into $\mathcal{Y}$ does not affect the set of marked points adjacent to $\breve{z}$ in the sphere of influence graph.

Inserting $(\mathbf{0}, t)$ into $\breve{\mathcal{Y}}$ can affect the set of points adjacent to $\breve{z}$ in two ways: either it can make a new edge between $(\mathbf{0}, t)$ and $\breve{z}$, or it can make some other point $\breve{x}$ no longer adjacent to $\breve{z}$. The latter can happen if $\breve{x}$ is adjacent to $\breve{z}$ in SIG$(\breve{\mathcal{Y}})$ and $\mathbf{0} \in B(\breve{x}, \breve{\mathcal{Y}})$. Therefore, it suffices to show that $\mathbf{0} \in B(\breve{x}, \breve{\mathcal{Y}})$ for no $\breve{x} \in \breve{\mathcal{Y}} \setminus B_\rho(\mathbf{0})$, that no point $\breve{x} \in \breve{\mathcal{Y}} \cap B_\rho(\mathbf{0})$ is adjacent to $\breve{z}$ in SIG$(\breve{\mathcal{Y}})$ and that $(\mathbf{0}, t)$ is not adjacent to $\breve{z}$ in SIG$(\breve{\mathcal{Y}} \cup \{(\mathbf{0}, t)\})$.

Take $\breve{x} \in \breve{\mathcal{Y}} \setminus B_\rho(\mathbf{0})$. Recall that $x \in C_i$ for some i and that $\breve{\mathcal{Y}} \cap C_i \cap B_\rho(\mathbf{0}) = \breve{\mathcal{X}} \cap C_i \cap B_\rho(\mathbf{0})$ contains at least one point, say, $\breve{y}$. By elementary geometry, $\|x - y\| < \|x\|$. Therefore, $\mathbf{0} \notin B(\breve{x}, \breve{\mathcal{Y}})$.

Now take $\breve{x} \in \breve{\mathcal{Y}} \cap (B_\rho(\mathbf{0}) \setminus \{\mathbf{0}\})$. Again, choose i with $x \in C_i$. In addition, choose $v \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, such that the angle between x and v equals $\pi/4$. Clearly, $v \notin C_i$, but $v \in C_j$ for some j. There exists $\breve{w} \in \breve{\mathcal{Y}} \cap C_j \cap B_\rho(\mathbf{0})$. Then $\breve{w} \neq \breve{x}$, but the angle between w and x is less than $\pi/3$, so that $\|w - x\| < \rho$. In other words, the intersection $(\breve{\mathcal{Y}} \setminus \{\breve{x}\}) \cap B_\rho(\breve{x})$ is non-empty. Now choose k, such that $z \in C_k$, and choose $\breve{y} \in \breve{\mathcal{Y}} \cap C_k \cap (B_r(\mathbf{0}) \setminus B_{4\rho}(\mathbf{0}))$. By elementary geometry, we have $C_k \setminus B_{4\rho}(\mathbf{0}) \subseteq C_k^+ + x$, so that $y \in (C_k^+ + x) \setminus B_{3\rho}(x)$. Observe that $z \in C_k^+ + x$, but also $\|z - x\| \ge \|z\| - \|x\| > r \ge \|y - x\|$, so that $z \in (C_k^+ + x) \setminus B_{\|y-x\|}(x)$. By Proposition 2.1, $\breve{x}$ and $\breve{z}$ are not adjacent in SIG$(\breve{\mathcal{Y}})$.

Finally, take $\breve{x} = (\mathbf{0}, t') \in \breve{\mathcal{Y}} \cup \{(\mathbf{0}, t)\}$. Again, let $z \in C_k$. As none of the sets $(\breve{\mathcal{Y}} \setminus \{\breve{x}, (\mathbf{0}, t)\}) \cap B_\rho(\mathbf{0})$ and $\breve{\mathcal{Y}} \cap C_k \cap (B_r(\mathbf{0}) \setminus B_{4\rho}(\mathbf{0}))$ is empty, the conditions of Proposition 2.1 are fulfilled, so that $\breve{x}$ and $\breve{z}$ are adjacent neither in SIG$(\breve{\mathcal{Y}})$ nor in SIG$(\breve{\mathcal{Y}} \cup \{(\mathbf{0}, t)\})$. Thus, we conclude that $\breve{\mathcal{X}}$ is basically $(r+\rho)$-externally stable at $(\mathbf{0}, t)$ with respect to NSIG.

By Proposition 1.1 and confinement, it remains to show that NSIG stabilizes with respect to $\check{\mathcal{X}}$ and $\check{\mathcal{X}} \cup \{(\mathbf{0}, t)\}$ at all $\check{x} \in (\check{\mathcal{X}} \cup \{(\mathbf{0}, t)\}) \cap B_{r+\rho}(\mathbf{0})$. Clearly, for all such $\check{x}$, the intersection $(\check{\mathcal{X}} \setminus \{\check{x}\}) \cap B_{r+\rho}(\check{x})$ is non-empty. Take $i = 1, 2, \dots, s$ and recall that there exists $\check{y} \in \check{\mathcal{X}} \cap (C_i \setminus B_{4(r+\rho)}(\mathbf{0}))$. However, we then have $y \in (C_i^+ + x) \setminus B_{3(r+\rho)}(x)$. By Proposition 2.1, we then have $\mathrm{NSIG}(\check{x}, \check{\mathcal{Y}}) \cap (C_i^+ + x) = \mathrm{NSIG}(\check{x}, \check{\mathcal{X}}) \cap (C_i^+ + x)$ and $\mathrm{NSIG}(\check{x}, \check{\mathcal{Y}} \cup \{(\mathbf{0}, t)\}) \cap (C_i^+ + x) = \mathrm{NSIG}(\check{x}, \check{\mathcal{X}} \cup \{(\mathbf{0}, t)\}) \cap (C_i^+ + x)$ for all $\check{\mathcal{Y}}$ with $\check{\mathcal{Y}} \cap B_{2\|y-x\|}(x) = \check{\mathcal{X}} \cap B_{2\|y-x\|}(x)$. Since this can be deduced for all $i = 1, \dots, s$, NSIG stabilizes at $\check{x}$ with respect to $\check{\mathcal{X}}$ and $\check{\mathcal{X}} \cup \{(\mathbf{0}, t)\}$. As a result, $\check{\mathcal{X}}$ is externally stable at $(\mathbf{0}, t)$ with respect to ξ. The proof is now completed by Theorem 1.2. □

Theorem 2.3 adds to the existing results on non-degeneracy of the limiting variance (see [27]), central limit theorems (see Chapter 4 of [21] as well as [3]) and, of course, large deviation results. The sphere of influence graphs were not considered in [1].

Example 2.3 (Total number of edges). *Define $\xi(x, \mathcal{X})$ to be half the degree of x in* $\mathrm{SIG}(\mathcal{X})$ (*assume that there are no marks*). *Then $\langle 1, \mu_\lambda \rangle$ is precisely the total number of edges in* $\mathrm{SIG}(\mathcal{X})$.

First, we turn to moment bounds. Take a bounded convex domain Ω with $\mathrm{vol}(\Omega) > 0$ and a probability density function κ with $\inf_\Omega \kappa > 0$, but vanishing outside Ω. From the construction of the radius of stabilization, it follows that for all $\mathcal{X} \subseteq \Omega$, we have:

$$\left|\xi_\lambda(x, \mathcal{X})\right| = \left|\xi(x, \mathcal{X})\right| \le \left|\mathcal{X} \cap B_{R^\Omega(x,\mathcal{X})}(x)\right| = \sum_{y \in \mathcal{X}} \mathbf{1}\left(R^\Omega(x, \mathcal{X}) \ge \|y - x\|\right).$$

Let $k \in \mathbb{N}$. Applying Lemma 3.12 with $a = k$ and $b = 2k$ combined with (2.3) (notice that b in Lemma 3.12 is different from b in (2.3)), we find that for some A_1 and A_2 not depending on k, we have:

$$\begin{aligned}\left(\mathbb{E}\left|\xi_\lambda(x, \mathcal{P}_{\lambda\kappa})\right|^k\right)^{1/k} &\le A_1 k \left\{\left[\lambda \int_{\mathbb{R}^d} \mathrm{e}^{-b\lambda\|y-x\|^d} \kappa(y)\,\mathrm{d}y\right]^{1/(2k)} + \lambda \int_{\mathbb{R}^d} \mathrm{e}^{-b\lambda\|y-x\|^d/(2k)} \kappa(y)\,\mathrm{d}y\right\} \\ &\le A_2 k^2. \end{aligned} \tag{2.4}$$

Combining this estimate with the observation $|\xi_\lambda(x, \mathcal{X} \cup \mathcal{Y})| \le |\xi_\lambda(x, \mathcal{X})| + |\mathcal{Y}|$ and Stirling's formula, we find that the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption $\mathrm{MGP}(2, \kappa)$. *By part* (3) *of Theorem* 2.3, *it then satisfies Assumption* $\mathrm{G}(3, \kappa)$. *Again, the range where moderate deviation results apply is independent of the dimension.*

Although non-degeneracy of the limiting variance is already proved in [27], *we here demonstrate that it also follows from part* (2) *of Theorem* 2.3: *choose any $\rho > 0$ and $r > 4\rho$. Letting $c = \cos\frac{\pi}{12}$, observe that if $\mathcal{X} \cap B_{\rho/(2c)}(\mathbf{0})$ is empty, but each of the sets $\mathcal{X} \cap C_i \cap (B_\rho(\mathbf{0}) \setminus B_{\rho/(2c)}(\mathbf{0}))$ contains at least two points, then $\mathbf{0}$ is the nearest neighbor of no point in $\mathcal{X}$. Therefore, insertion of the origin cannot remove any edges in* SIG, *but it adds at least the edge between $\mathbf{0}$ and its nearest neighbor in $\mathcal{X}$, so that $\Delta(\mathbf{0}, \mathcal{X}) \ne 0$. Clearly, this condition along with non-emptiness of the sets $\mathcal{X} \cap C_i \cap (B_r(\mathbf{0}) \setminus B_{4\rho}(\mathbf{0}))$ and $\mathcal{X} \cap (C_i \setminus B_{4(r+\rho)}(\mathbf{0}))$ is fulfilled for notably many configurations $\mathcal{X}$.*

Next, from (2.4) *and again the fact that $|\xi_\lambda(x, \mathcal{X} \cup \mathcal{Y})| \le |\xi_\lambda(x, \mathcal{X})| + |\mathcal{Y}|$, it follows that the family $(\xi_\lambda)_{\lambda>0}$ satisfies Assumption* $\mathrm{MH1}(p, \tau, \Omega)$ *for all $p, \tau > 0$. By part* (2) *of Theorem* 2.3, *$V(\tau) > 0$ for all $\tau > 0$.*

3. Proofs of the results

3.1. *Moment measures and Palm distributions*

For a random measure μ taking values in the space of Borel measures over $\mathbb{R}^d$, define its *kth moment measure* $M^k(\mu)$ as the one characterized by:

$$\left\langle f_1 \otimes \cdots \otimes f_k, M^k(\mu)\right\rangle = \mathbb{E}\left[\langle f_1, \mu\rangle \cdots \langle f_k, \mu\rangle\right] \tag{3.1}$$

for all $f_1, \dots, f_k \in \mathcal{B}(\mathbb{R}^d)$, where $f_1 \otimes \cdots \otimes f_k : (\mathbb{R}^d)^k \to \mathbb{R}$ is given by $f_1 \otimes \cdots \otimes f_k(v_1, \dots, v_k) = f_1(v_1) \cdots f_k(v_k)$ (formula (5.4.3) on p. 133 of [8]); the kth moment measure exists if the *mixed moments* in the right-hand side of (3.1) exist for all $f_1, \dots, f_k \in \mathcal{B}(\mathbb{R}^d)$.

It will be helpful to consider products of $\mathbb{R}^d$ and $\breve{\mathbb{R}}^d$ indexed by arbitrary finite sets: for a finite index set L consisting of distinct elements $i_1, \dots, i_l$, denote by $(\mathbb{R}^d)^L$ (resp. $(\breve{\mathbb{R}}^d)^L$) the product of l copies of $\mathbb{R}^d$ (resp. $\breve{\mathbb{R}}^d$). Thus, for functions $f_i : \mathbb{R}^d \to \mathbb{R}$, $i \in L$, $\bigotimes_{i\in L} f_i : (\mathbb{R}^d)^L \to \mathbb{R}$ is the counterpart of the function $f_{i_1} \otimes \cdots \otimes f_{i_l} : (\mathbb{R}^d)^l \to \mathbb{R}$. In the special case where all functions are equal, define $f^{\otimes l} := \underbrace{f \otimes \cdots \otimes f}_{l}$ and its counterpart $f^{\otimes L} := \bigotimes_{i\in L} f$.

For a random measure μ on $\mathbb{R}^d$, let $M^L(\mu)$ be the measure on $(\mathbb{R}^d)^L$, which is the counterpart of $M^l(\mu)$, i.e., $\langle \bigotimes_{i\in L} f_i, M^L(\mu)\rangle = \mathbb{E}[\prod_{i\in L}\langle f_i, \mu\rangle]$.

For the random measures which are the subject of the present paper, write $M_\lambda^k := M^k(\mu_\lambda)$ for $k \in \mathbb{N}$ and $M_\lambda^L := M^L(\mu_\lambda)$ for a finite set L. These moment measures can be expressed in terms of singular measures, see (3.5). That formula, also stated in [8], p. 143, is a special case of the Palm disintegration formula for a product of k copies of $\breve{\mathcal{P}}_{\lambda\kappa}$. First, recall the Palm formula for $\breve{\mathcal{P}}_{\lambda\kappa}$: for each functional G, such that the integral and the expectation below exist, we have:

$$\mathbb{E}\int_{\breve{\mathbb{R}}^d} G(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{x}) = \lambda \int_{\breve{\mathbb{R}}^d} \mathbb{E}G\bigl(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa} \cup \{\breve{x}\}\bigr)\kappa(x)\,\mathrm{d}\breve{x} \tag{3.2}$$

(for the unmarked case, see [9], pp. 280–281; the extension to marked Poisson processes can be achived by conditioning on the marks; see Section 6.4 of [8]). To generalize this disintegration formula to k-fold integrals, we need singular measures. First, recall (1.1) and for a measurable function $g : \mathbb{R}^d \to \mathbb{R}$, define the singular differential $\bar{\mathrm{d}}[g]\breve{v}$ of a $(\breve{\mathbb{R}}^d)^k$-valued variable as being characterized by the relation:

$$\int_{(\breve{\mathbb{R}}^d)^k} F(\breve{v}_1, \breve{v}_2, \dots, \breve{v}_k)\,\bar{\mathrm{d}}[g](\breve{v}_1, \dots, \breve{v}_k) = \int_{\breve{\mathbb{R}}^d} F(\breve{x}, \breve{x}, \dots, \breve{x})g(x)\,\mathrm{d}\breve{x}$$

for all measurable $F : (\breve{\mathbb{R}}^d)^k \to \mathbb{R}$. Next, for $\breve{v} = (\breve{v}_1, \dots, \breve{v}_k)$ running over $(\breve{\mathbb{R}}^d)^k$, put:

$$\tilde{\mathrm{d}}[g]\breve{v} := \sum_{L_1,\dots,L_p \preceq \{1,\dots,k\}} \bar{\mathrm{d}}[g]\breve{v}_{L_1} \cdots \bar{\mathrm{d}}[g]\breve{v}_{L_p},$$

where $\breve{v}_L := (\breve{v}_l)_{l\in L}$; by $\sum_{L_1,\dots,L_p\preceq L}$, we shall denote the sum of all *unordered* partitions of a set L. Below we prove the following assertion, which generalizes the disintegration formula (3.2) to the k-fold integral (see also p. 83 of [18]):

Proposition 3.1. *For each functional G, such that the integral and the expectation below exist, we have*:

$$\mathbb{E}\int_{(\breve{\mathbb{R}}^d)^k} G(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa})\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_1)\cdots\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_k) = \int_{(\breve{\mathbb{R}}^d)^k} \mathbb{E}G\bigl(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa} \cup \{\breve{v}_1, \dots, \breve{v}_k\}\bigr)\,\tilde{\mathrm{d}}[\lambda\kappa]\breve{v}, \tag{3.3}$$

where $\breve{v} = (\breve{v}_1, \dots, \breve{v}_k)$.

Proof. As a first step, we prove (3.3) for the case where $G(\breve{v}, \breve{\mathcal{X}})$ vanishes if any two components v_i and v_j are equal. This can be proved by induction. For $k = 1$, this is merely the formula (3.2). For the induction step from k to $k+1$, use (3.2) with $\int_{(\breve{\mathbb{R}}^d)^k} G(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa})\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_1)\cdots\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_k)$ in place of $G(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa})$ and notice that the integration over $\mathcal{P}_{\lambda\kappa} \cup \{v_{k+1}\}$ coincides with the integration over $\mathcal{P}_{\lambda\kappa}$.

Next, observe the following straightforward extension. Let $L_1, \dots, L_p$ be a partition of $\{1, \dots, k\}$. We say that a point $\breve{v} = (\breve{v}_1, \dots, \breve{v}_k)$ *follows* this partition if any two components v_i and v_j are equal if and only if the indices i and j lie in the same set L_r. Now take arbitrary G, and define $G_{L_1,\dots,L_p}(\breve{v}, \breve{\mathcal{X}})$ to be $G(\breve{v}, \breve{\mathcal{X}})$ if $\breve{v}$ follows $L_1, \dots, L_p$ and zero otherwise. Then we have:

$$\begin{aligned}
&\mathbb{E}\int_{(\breve{\mathbb{R}}^d)^k} G_{L_1,\dots,L_p}(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa})\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_1)\cdots\breve{\mathcal{P}}_{\lambda\kappa}(\mathrm{d}\breve{v}_k)\\
&\quad= \int_{(\breve{\mathbb{R}}^d)^k} \mathbb{E}G_{L_1,\dots,L_p}\bigl(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa} \cup \{\breve{v}_1, \dots, \breve{v}_k\}\bigr)\,\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_1}\cdots\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_p}\\
&\quad= \int_{(\breve{\mathbb{R}}^d)^k} \mathbb{E}G\bigl(\breve{v}, \breve{\mathcal{P}}_{\lambda\kappa} \cup \{\breve{v}_1, \dots, \breve{v}_k\}\bigr)\,\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_1}\cdots\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_p}.
\end{aligned}$$

Now write $G = \sum_{L_1,\dots,L_p \preceq \{1,\dots,k\}} G_{L_1,\dots,L_p}$, sum up over all non-trivial partitions of $\{1,\dots,k\}$ and the proof is complete. $\square$

Now take a geometric functional ξ and recall the definition (1.4) of its associated random measure μ_λ. From Proposition 3.1, we deduce that the corresponding moment measures $M_\lambda^k = M^k(\mu_\lambda)$ can be expressed as:

$$\int_{(\mathbb{R}^d)^k} F(v) M_\lambda^k(\mathrm{d}v) = \int_{(\breve{\mathbb{R}}^d)^k} F(v) m_\lambda(\breve{v})\, \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}, \tag{3.4}$$

where $v = (v_1,\dots,v_k)$ and again $\breve{v} = (\breve{v}_1,\dots,\breve{v}_k)$, and where the Radon–Nikodým derivative m_λ is given by:

$$m_\lambda(\breve{v}_1,\dots,\breve{v}_k) := \mathbb{E}\left[\prod_{i=1}^k \xi_\lambda\big(\breve{v}_i, \breve{\mathcal{P}}_{\lambda\kappa} \cup \{\breve{v}_1,\dots,\breve{v}_k\}\big)\right]. \tag{3.5}$$

Analogously, we define m_λ on products indexed by arbitrary index sets, i.e., $\mathbb{R}^L$.

3.2. *The method of cumulants*

We will refine the method of cumulants and cluster measures as developed in [3] in the context of the central limit theorem. We recall the formal definition of cumulants in the context specified for our purposes. For a random variable Y with all moments, expanding the logarithm of the Laplace transform in a formal power series in t gives

$$\log\left[1 + \sum_{k=1}^\infty \frac{\mathbb{E}Y^k}{k!} t^k\right] = \sum_{k=1}^\infty \frac{c^k(Y)}{k!} t^k, \tag{3.6}$$

where $c^k(Y)$ denotes the *kth cumulant* of Y. As the series (3.6) is considered as formal, no additional condition on convergence is required for the cumulants to exist. Defining differentiation, evaluation at zero, and the exponential and the logarithmic function of a formal power series in the obvious way, one may also write:

$$c^k(Y) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_{t=0} \log \mathbb{E}\exp(tY).$$

Similarly as mixed moments, one can also consider *mixed cumulants*. In the spirit of the above, one can define it by means of formal power series of several variables:

$$c(Y_1,\dots,Y_k) = \frac{\partial^k}{\partial t_1\,\partial t_2\cdots\partial t_k}\bigg|_{t_1=t_2=\cdots=t_k=0} \log \mathbb{E}\exp(t_1Y_1 + \cdots + t_kY_k). \tag{3.7}$$

In other words, the mixed cumulant of random variables $Y_1,\dots,Y_k$ is the coefficient in the formal power series expansion of $\log \mathbb{E}\exp(t_1Y_1+\cdots+t_kY_k)$ at $t_1t_2\cdots t_k$. Notice also that $c^k(Y) = c(\underbrace{Y,\dots,Y}_{k})$.

To define the mixed cumulant of random variables $Y_1,\dots,Y_k$, we do not even need all the moments to exist. All we need is the existence of the expectations of the products $\prod_{i\in L} Y_i$, where $L \subseteq \{1,\dots,k\}$. This is because one can replace the exponential function $\exp(t_1Y_1+\cdots+t_kY_k)$ by the polynomial $g(t_1,\dots,t_k) = \sum_{L\subseteq\{1,\dots,k\}} \mathbb{E}\prod_{i\in L} Y_it_i$: the mixed cumulant $c(Y_1,\dots,Y_l)$ is then also the coefficient in the formal power series expansion of $\log g(t_1,\dots,t_k)$ at $t_1t_2\cdots t_k$.

In view of the above, mixed cumulants can be expressed in terms of mixed moments. This can be made explicit by means of the following extension of the celebrated *Faà di Bruno's formula* to functions of several variables:

$$\frac{\partial^k}{\partial t_1\cdots\partial t_k} f\big(g(t_1,\dots,t_k)\big) = \sum_{L_1,\dots,L_p\preceq\{1,\dots,k\}} f^{(p)}\big(g(t_1,\dots,t_k)\big) \frac{\partial^{|L_1|}g}{\prod_{i\in L_1}\partial t_i}\cdots\frac{\partial^{|L_p|}g}{\prod_{i\in L_p}\partial t_i} \tag{3.8}$$

(see [15] and notice that although the result ibidem is stated for real functions, the extension to formal power series is straightforward: once we know the chain and the product rule, Faà di Bruno's formula is a matter of combinatorics, no longer analysis). Combining (3.7) and (3.8), we obtain the formula for mixed cumulants:

$$c(Y_1,\dots,Y_k)=\sum_{L_1,\dots,L_p\preceq\{1,\dots,k\}}(-1)^{p-1}(p-1)!\mathbb{E}\Big[\prod_{i\in L_1}Y_i\Big]\cdots\mathbb{E}\Big[\prod_{i\in L_p}Y_i\Big] \tag{3.9}$$

(see p. 12 of [33]). For a random measure μ, its kth cumulant measure $c^k(\mu)$ is defined analogously as its kth moment measure, i.e., $\langle f_1\otimes\cdots\otimes f_k, c^k(\mu)\rangle = c(\langle f_1,\mu\rangle,\dots,\langle f_k,\mu\rangle)$. In particular, for equal functions, we have:

$$\big\langle f^{\otimes k}, c^k(\mu)\big\rangle = c^k\big(\langle f,\mu\rangle\big). \tag{3.10}$$

In view of (3.9), cumulant measures can be expressed in terms of moment measures in the following way:

$$c^k(\mu)=\sum_{L_1,\dots,L_p\preceq\{1,\dots,k\}}(-1)^{p-1}(p-1)!M^{L_1}(\mu)\cdots M^{L_p}(\mu), \tag{3.11}$$

where the multiplication denotes the usual product of measures: for disjoint finite sets G and H, and for measurable sets $A\subseteq(\mathbb{R}^d)^G$ and $B\subseteq(\mathbb{R}^d)^H$, we have $MN(A\times B)=M(A)N(B)$, identifying $(\mathbb{R}^d)^{G\cup H}\equiv(\mathbb{R}^d)^G\times(\mathbb{R}^d)^H$ (see p. 30 of [19]).

Although we use the same notation for cumulants as well as for cumulant measures, this should not lead to a confusion: for a *real-valued* random variable Y, $c^k(Y)$ denotes a cumulant, while for a random *measure* μ, $c^k(\mu)$ denotes a cumulant measure. Observe also that the first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the covariance measure.

Throughout this subsection, ξ will (as usual) denote a geometric functional and R its radius of stabilization. Recall the random measures μ_λ defined in (1.4) and the corresponding moment measures $M_\lambda^k := M^k(\mu_\lambda)$. Similarly, consider the cumulant measures $c_\lambda^k := c^k(\mu_\lambda)$. Recalling the notation $\bar\mu_\lambda=\mu_\lambda-\mathbb{E}\mu_\lambda$, observe that $c^k(\bar\mu_\lambda)=c_\lambda^k$ for $k\ge 2$. Analogously, define measures M_λ^L and c_λ^L defined on product spaces indexed finite sets L.

Now we can state our result controlling the growth of $\langle f^{\otimes k}, c_\lambda^k\rangle$, which is crucial to prove Theorem 1.3.

Lemma 3.1. *If ξ satisfies Assumption* G(γ,κ), *we have*:

$$\big|\big\langle f^{\otimes k}, c_\lambda^k\big\rangle\big|\le\lambda C^k\|f\|_\infty^k(k!)^{1+\gamma}$$

for all bounded measurable functions $f:\mathbb{R}^d\to\mathbb{R}$, all $k=3,4,\dots$ and all $\lambda\ge\lambda_0$, where the constant C and the lower endpoint λ_0 only depend on κ, ξ and R.

Before proving the preceding lemma, we need a couple of auxiliary results. Following [3], we decompose cumulant measures into *semi-cluster measures*, i.e., cluster measures multiplied by moment measures. For non-empty disjoint finite sets S and T, define the cluster measure by:

$$U_\lambda^{S,T}=M_\lambda^{S\cup T}-M_\lambda^S M_\lambda^T$$

(where multiplication again means product measure). The following result is a refinement of Lemma 5.1 of [3] in the sense that we provide control over the number of summands.

Lemma 3.2. *For each non-trivial partition G, H of a finite set K, the cumulant measure c_λ^K can be decomposed as*:

$$c_\lambda^K=\sum_{L_1,\dots,L_p\preceq K}(-1)^{p-1}(p-1)!W_\lambda^{L_1,\dots,L_p},$$

where $W_\lambda^{L_1,\dots,L_p}$ is a sum of at most p terms of the form $U_\lambda^{S,T}M_\lambda^{K_1}M_\lambda^{K_2}\cdots M_\lambda^{K_r}$, where $S\subseteq G$ and $T\subseteq H$ are non-empty and disjoint, and where $S\cup T, K_1,\dots,K_r$ is a refinement of the partition $L_1,\dots,L_p$.

Proof. Starting from (3.11), we first note that each moment measure $M_\lambda^{L_i}$ with $S := L_i \cap G \neq \varnothing$ and $T := L_i \cap H \neq \varnothing$ can be expressed as $U_\lambda^{S,T} + M_\lambda^S M_\lambda^T$. Repeating the procedure, we may write:

$$M_\lambda^{L_1} \cdots M_\lambda^{L_p} = M_\lambda^{L_1 \cap G} M_\lambda^{L_1 \cap H} \cdots M_\lambda^{L_p \cap G} M_\lambda^{L_p \cap H} + W_\lambda^{L_1,\ldots,L_p}, \tag{3.12}$$

where the measures $W_\lambda^{L_1,\ldots,L_p}$ are as desired and where we set $M_\lambda^\varnothing := 1$. Now consider the measure:

$$c_\lambda^{K;G,H} := \sum_{L_1,\ldots,L_p \preceq K} (-1)^{p-1}(p-1)! M_\lambda^{L_1 \cap G} M_\lambda^{L_1 \cap H} \cdots M_\lambda^{L_p \cap G} M_\lambda^{L_p \cap H} \tag{3.13}$$

and take functions $f_i \in \mathcal{B}(\mathbb{R}^d)$, $i \in K$. By Faà di Bruno's formula (3.8), $\langle \bigotimes_{i \in K} f_i, c_\lambda^{K;G,H} \rangle$ matches the coefficient in the formal power series expansion of $\log g_\lambda^{K;G,H}$ at $\prod_{i \in K} t_i$, where:

$$g_\lambda^{K;G,H} := \sum_{L \subseteq K} \left\langle \bigotimes_{i \in L} f_i, M_\lambda^{L \cap G} M_\lambda^{L \cap H} \right\rangle \prod_{i \in L} t_i.$$

However, $g_\lambda^{K;G,H} = g_\lambda^G g_\lambda^H$, where $g_\lambda^Z := \sum_{L \subseteq Z} \langle \bigotimes_{i \in L} f_i, M_\lambda^L \rangle \prod_{i \in L} t_i$. Since G and H are both non-empty, the coefficient at $\prod_{i \in K} t_i$ in the formal power series expansion of both $\log g_\lambda^G$ and $\log g_\lambda^H$ vanishes; clearly, the same is true for $\log g_\lambda^{K;G,H} = \log g_\lambda^G + \log g_\lambda^H$. Therefore, $c_\lambda^{K;G,H} = 0$. Combining this with (3.12) and (3.13), the result follows. □

Thus, in order to estimate the cumulants, it suffices to estimate semi-cluster measures. Recalling (3.4) and (3.5), it makes sense, as the first step towards the latter estimation, to bound the differences $m_\lambda(v_{S \cup T}) - m_\lambda(v_S) m_\lambda(v_T)$; throughout this subsection, we shall denote:

$$v_L = (v_i)_{i \in L}, \qquad \breve{v}_L = (\breve{v}_i)_{i \in L} \quad \text{and} \quad V_L = \{v_i; i \in L\}, \qquad \breve{V}_L = \{\breve{v}_i; i \in L\}$$

for vectors $v = (v_i)_{i \in K} \in (\mathbb{R}^d)^K$ and $\breve{v} = (\breve{v}_i)_{i \in K} \in (\breve{\mathbb{R}}^d)^K$, where $L \subseteq K$ (the letters v and V are fixed, while the letters K and L can be arbitrary). Next, define the *separation* between two subsets A and B of a $\mathbb{R}^d$ by:

$$\operatorname{sep}(A, B) := \inf\{\|a - b\|; a \in A, b \in B\}.$$

Now recall the definition of ξ_λ along with the conventions on R and R_λ from Section 1.2; in particular, recall that $\lambda^{-1/d} R_\lambda$ a radius of stabilization for ξ_λ inside Ω. In addition, recall that κ vanishes outside Ω, so that $\breve{\mathcal{P}}_{\lambda\kappa} \subseteq \breve{\Omega}$ almost surely. For a finite set L, $\breve{v} = (\breve{v}_l)_{l \in L} \in (\breve{\mathbb{R}}^d)^L$, $i, j \in L$, $\lambda > 0$ and for a function $\psi : [0, \infty) \to [0, \infty)$, define:

$$a_{\lambda,i}(\breve{v}) := \left[\mathbb{E}\left|\xi_\lambda(\breve{v}_i, \breve{\mathcal{P}}_{\lambda\kappa} \cup \breve{V}_L)\right|^{2|L|}\right]^{1/(2|L|)}, \qquad b_{\lambda,j,\psi}(\breve{v}) := \left[\mathbb{E}\left(\psi\left(2R_\lambda(\breve{v}_j, \breve{\mathcal{P}}_{\lambda\kappa} \cup \breve{V}_L)\right)\right)^{-2}\right]^{1/2}. \tag{3.14}$$

Lemma 3.3. *Let S and T be non-empty finite disjoint sets and let $\psi : [0, \infty) \to [0, \infty)$ be a non-increasing function. Then for each $\breve{v} = \breve{v}_{S \cup T} \in (\breve{\mathbb{R}}^d)^{S \cup T}$, we have*:

$$\begin{aligned}\left|m_\lambda(\breve{v}_{S \cup T}) - m_\lambda(\breve{v}_S) m_\lambda(\breve{v}_T)\right| \le{}& \left[\prod_{i \in S \cup T} a_{\lambda,i}(\breve{v}_{S \cup T}) + \left(\prod_{i \in S} a_{\lambda,i}(\breve{v}_S)\right)\left(\prod_{i \in T} a_{\lambda,i}(\breve{v}_T)\right)\right] \\ &\times \left[\sum_{j \in S} b_{\lambda,j,\psi}(\breve{v}_S) + \sum_{j \in T} b_{\lambda,j,\psi}(\breve{v}_T)\right] \psi\left(\lambda^{1/d} \delta\right),\end{aligned} \tag{3.15}$$

where $\delta = \operatorname{sep}(\{v_i; i \in S\}, \{v_j; j \in T\})$ denotes the separation with respect to the Euclidean metric.

Remark 3.1. *This is a refinement of Lemma* 5.2 *of* [3] *in at least two directions*: *first*, *we state a more explicit upper bound*, *and second*, *we allow for arbitrary decay of R* (*described in terms of ψ*), *not just exponential. Moreover*, *a closer look reveals that the argument used for the proof of that result in fact needs stronger assumptions than just* *exponential stabilization (apart from moment bounds), as claimed ibidem. More precisely, in our notation, one has to assume suitable stabilization of the functional $(\check{x}, \check{\mathcal{X}}) \mapsto \xi(\check{x}, \check{\mathcal{X}} \cup \check{V})$, not ξ, for finite sets $\check{V}$ with suitable cardinality. This is due to a confusion between $\check{\mathcal{P}}_{\lambda\kappa}$ and $\check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}$ (see below Eq. (5.4) ibidem). Moreover, in order to derive large deviation results from appropriately corrected Lemma 5.2 of [3], one also needs certain control over the dependence of the stabilization of $(\check{x}, \check{\mathcal{X}}) \mapsto \xi(\check{x}, \check{\mathcal{X}} \cup \check{V})$ on the cardinality of $\check{V}$. These additional conditions can be tedious to verify in actual applications. On the other hand, our argument, though much more extensive, works under more or less standard conditions and leads to a neat result.*

Proof of Lemma 3.3. Take independent Poisson point processes $\check{\mathcal{P}}_{\lambda\kappa}$ and $\check{\mathcal{P}}^{\dagger}_{\lambda\kappa}$ (both with intensity $\lambda\kappa \times \mathbb{P}_{\mathcal{M}}$) and define two new point processes:

$$\check{\mathcal{P}}'_{\lambda\kappa} := \big(\check{\mathcal{P}}_{\lambda\kappa} \cap B_{\delta/2}(V_S)\big) \cup \big(\check{\mathcal{P}}^{\dagger}_{\lambda\kappa} \setminus B_{\delta/2}(V_S)\big),$$
$$\check{\mathcal{P}}''_{\lambda\kappa} := \big(\check{\mathcal{P}}_{\lambda\kappa} \setminus B_{\delta/2}(V_S)\big) \cup \big(\check{\mathcal{P}}^{\dagger}_{\lambda\kappa} \cap B_{\delta/2}(V_S)\big),$$

where, as usual, $B_r(V) := \bigcup_{v \in V} B_r(v)$. Observe that $\check{\mathcal{P}}'_{\lambda\kappa}$ and $\check{\mathcal{P}}''_{\lambda\kappa}$ are independent Poisson point processes with intensity $\lambda\kappa \times \mathbb{P}_{\mathcal{M}}$. Setting:

$$X_i := \xi_\lambda(\check{v}_i, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_{S\cup T}), \qquad X'_i := \xi_\lambda\big(\check{v}_i, \check{\mathcal{P}}'_{\lambda\kappa} \cup \check{V}_S\big), \qquad X''_i := \xi_\lambda\big(\check{v}_i, \check{\mathcal{P}}''_{\lambda\kappa} \cup \check{V}_T\big),$$

we may write:

$$m_\lambda(\check{v}_{S\cup T}) - m_\lambda(\check{v}_S) m_\lambda(\check{v}_T) = \mathbb{E}\Bigg[\bigg(\prod_{i\in S} X_i\bigg)\bigg(\prod_{i\in T} X_i\bigg) - \bigg(\prod_{i\in S} X'_i\bigg)\bigg(\prod_{i\in T} X''_i\bigg)\Bigg].$$

Now observe that for $i \in S$, X'_i agrees with X_i if ξ_λ stabilizes at $\check{v}_i$ within radius less than $\delta/2$ with respect to $\check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_S$. Similarly, for $i \in T$, X''_i agrees with X_i if ξ_λ stabilizes at $\check{v}_i$ within radius less than $\delta/2$ with respect to $\check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_T$. Letting:

$$I'_j := \mathbf{1}\bigg(R_\lambda\big(\check{v}_j, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_S\big) \ge \frac{\lambda^{1/d}\delta}{2}\bigg), \qquad I''_j := \mathbf{1}\bigg(R_\lambda\big(\check{v}_j, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_T\big) \ge \frac{\lambda^{1/d}\delta}{2}\bigg),$$

we can estimate:

$$\big|m_\lambda(\check{v}_{S\cup T}) - m_\lambda(\check{v}_S) m_\lambda(\check{v}_T)\big| \le \mathbb{E}\Bigg\{\Bigg[\prod_{i\in S\cup T} |X_i| + \bigg(\prod_{i\in S} |X'_i|\bigg)\bigg(\prod_{i\in T} |X''_i|\bigg)\Bigg]\Bigg[\sum_{j\in S} I'_j + \sum_{j\in T} I''_j\Bigg]\Bigg\}.$$

Since ψ is non-increasing, we can estimate:

$$I'_j \le \frac{\psi(\lambda^{1/d}\delta)}{\psi(2R_\lambda(\check{v}_j, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_S))} \quad \text{and} \quad I''_j \le \frac{\psi(\lambda^{1/d}\delta)}{\psi(2R_\lambda(\check{v}_j, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_S))}. \tag{3.16}$$

The proof is now completed by application of Hölder's inequality. □

To estimate the semi-cluster measures, we now need to integrate the estimate (3.15). Before tackling this job, we introduce some more notation. First, we extend the convention on the breve accents to the products $(\mathbb{R}^d)^K$ and $(\check{\mathbb{R}}^d)^K$: if v and $\check{v}$ appear in the same context and if $\check{v}$ denotes a marked K-tuple $(\check{v}_i)_{i\in K} \in (\check{\mathbb{R}}^d)^K$, then we shall assume that $v = (v_i)_{i\in K} \in (\mathbb{R}^d)^K$.

Now denote by $\Delta^k_d := \{(x, x, \ldots, x); x \in \mathbb{R}^d\}$ the diagonal in $(\mathbb{R}^d)^k$; similarly, for a finite set K, denote by Δ^K_d the diagonal in $(\mathbb{R}^d)^K$. We also consider the marked diagonal $\check{\Delta}^K_d = \{\check{v} \in \check{\mathbb{R}}^d; v \in \Delta^K_d\}$. Next, for $v \in (\mathbb{R}^d)^K \setminus \Delta^K_d$, denote by $\delta(v)$ the maximal separation between the sets V_G and V_H, where (G, H) runs over all non-trivial partitions of K (i.e., G and H are non-empty with union K). Finally, denote by ω the volume of the unit ball in $\mathbb{R}^d$.

The estimation of suitable integrals in the right-hand side of (3.15) will be based on the following result.

Lemma 3.4. *Let $K_1, K_2, \dots, K_r$ be finite disjoint sets with union K. Put $k_l := |K_l|$ and $k := |K|$. Take a non-increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty}\psi(t) = 0$ and with finite Riemann–Stieltjes integral $\int_0^\infty t^{(k-1)d}\,\mathrm{d}(-\psi)(t)$, $\lambda > 0$, a marked Poisson point process $\check{\mathcal{P}}_{\lambda\kappa}$, a non-negative geometric functional g and $i \in K_1$. Then we have*:

$$\begin{aligned}&\int_{(\check{\mathbb{R}}^d)^K\setminus\check{\Delta}_d^K} \mathbb{E}g(\check{v}_i, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_{K_1})\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_l}\\ &\quad\le \lambda Q(k,\kappa,\psi)\left[\int_{\check{\mathbb{R}}^d}\mathbb{E}\big(g(\check{x},\check{\mathcal{P}}_{\lambda\kappa})\big)^2\kappa(x)\,\mathrm{d}\check{x}\right]^{1/2},\end{aligned}$$

where $Q(k,\kappa,\psi) = 2^{k-1}k!\int_0^\infty (1+\mathrm{e}\|\kappa\|_\infty\omega t^d)^{k-1}\,\mathrm{d}(-\psi)(t)$.

Before proving Lemma 3.4, we need one more auxiliary result.

Lemma 3.5. *For all $k \in \mathbb{N}$ and all $u \in \mathbb{R}$, we have*:

$$\sum_{L_1,\dots,L_p\preceq\{1,\dots,k\}} p!|L_1|!|L_2|!\cdots|L_p|!u^{p-1} = (1+u)^{k-1}k!. \tag{3.17}$$

Proof. Let $f(y) = 1/(u(u+1-uy))$, $g(x) = 1/(1-x)$ and observe that the kth derivative of $f(g(x))$ at $x = 0$ matches the right hand side of (3.17). Then apply Faà di Bruno's formula (3.8). □

Corollary 3.1. *For all $k \in \mathbb{N}$ and $\alpha \ge 0$, we have*:

$$\sum_{L_1,\dots,L_p\preceq\{1,\dots,k\}} p!\big(|L_1|!\big)^\alpha\big(|L_2|!\big)^\alpha\cdots\big(|L_p|!\big)^\alpha \le 2^{k-1}(k!)^{\max\{\alpha,1\}}. \tag{3.18}$$

Remark 3.2. *Clearly, the exponent* $\max\{\alpha, 1\}$ *cannot be reduced.*

Proof of Lemma 3.4. Let $K' := K\setminus\{i\}$, $K_1' := K_1\setminus\{i\}$ and $K_l' := K_l$ for $l = 2,3,\dots,r$. Next, take independent Poisson point processes $\check{\mathcal{P}}_{\lambda\kappa}^{(1)},\dots,\check{\mathcal{P}}_{\lambda\kappa}^{(r)}$ (all with intensity $\lambda\kappa\times\mathbb{P}_{\mathcal{M}}$). Applying (3.3), we may write:

$$\begin{aligned}J &:= \int_{(\check{\mathbb{R}}^d)^K\setminus\check{\Delta}_d^K} \mathbb{E}g(\check{v}_i, \check{\mathcal{P}}_{\lambda\kappa} \cup \check{V}_{K_1})\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_l}\\ &= \mathbb{E}\int_{(\check{\mathbb{R}}^d)^K\setminus\check{\Delta}_d^K} g\big(\check{v}_i,\check{\mathcal{P}}_{\lambda\kappa}^{(1)}\big)\psi\big(\lambda^{1/d}\delta(v)\big)\left(\bigotimes_{l=1}^r\big(\check{\mathcal{P}}_{\lambda\kappa}^{(l)}\big)^{\otimes K_l}\right)(\mathrm{d}\check{v})\end{aligned}$$

(where $\check{v} = (\check{v}_j)_{j\in(\check{\mathbb{R}}^d)^K}$). Next, we may asssume without loss of generality that ψ is left continuous, so that we can write $\psi(x) = \int_{[0,\infty)}\mathbf{1}(x\le t)\mu(\mathrm{d}t)$ for some positive measure μ. Plugging this into the preceding equation, we obtain:

$$J = \int_{[0,\infty)}\mathbb{E}\int_{\check{\mathbb{R}}^d} g\big(\check{v}_i,\check{\mathcal{P}}_{\lambda\kappa}^{(1)}\big)\int_{(\check{\mathbb{R}}^d)^{K'}}\mathbf{1}\big(\delta(v)\le\lambda^{-1/d}t\big)\left(\bigotimes_{l=1}^r\big(\check{\mathcal{P}}_{\lambda\kappa}^{(l)}\big)^{\otimes K_l'}\right)(\mathrm{d}\check{v}_{K'})\check{\mathcal{P}}_{\lambda\kappa}^{(1)}(\mathrm{d}\check{v}_i)\mu(\mathrm{d}t)$$

(identifying $v\in(\mathbb{R}^d)^K$ with $(v_{K'},v_i)\in(\mathbb{R}^d)^{K'}\times\mathbb{R}^d$). Now consider the graph with vertex set K, where vertices j and l are adjacent if $\|v_j - v_l\|\le\lambda^{-1/d}t$. Observe that $\delta(v)\le\lambda^{-1/d}t$ if and only if this graph is connected. Therefore, if $\delta(v)\le\lambda^{-1/d}t$, then $\|v_j - v_i\|\le N(v)\lambda^{-1/d}t$ for all $j\in K$, where $N(v) := |\{v_j; j\in K\}| - 1$. Next, estimating the expression under the second integral sign by the Cauchy–Schwarz inequality, we find that:

$$J \le \sqrt{A}\int_{[0,\infty)}\sqrt{B(t)}\mu(\mathrm{d}t), \tag{3.19}$$

where:

$$
\begin{aligned}
A &= \mathbb{E}\int_{\breve{\mathbb{R}}^d} \big(g\big(\breve{v}_i, \breve{\mathcal{P}}^{(1)}_{\lambda\kappa}\big)\big)^2 \breve{\mathcal{P}}^{(1)}_{\lambda\kappa}(\mathrm{d}\breve{v}_i) = \lambda \int_{\breve{\mathbb{R}}^d} \mathbb{E}\big(g(\breve{x}, \mathcal{P}_{\lambda\kappa})\big)^2 \kappa(x)\,\mathrm{d}\breve{x}, \\
B(t) &= \mathbb{E}\int_{\breve{\mathbb{R}}^d}\left[\int_{(\breve{\mathbb{R}}^d)^{K'}}\left(\prod_{j\in K}\mathbf{1}\big(\|v_j - v_i\| \le N(v)\lambda^{-1/d}t\big)\right)\left(\bigotimes_{l=1}^{r}\big(\breve{\mathcal{P}}^{(l)}_{\lambda\kappa}\big)^{\otimes K'_l}\right)(\mathrm{d}\breve{v}_{K'})\right]^2 \breve{\mathcal{P}}^{(1)}_{\lambda\kappa}(\mathrm{d}\breve{v}_i).
\end{aligned}
\tag{3.20}
$$

To estimate $B(t)$, let $K''_1, K''_2, \dots, K''_r$ be copies of the sets $K'_1, K'_2, \dots, K'_r$, disjoint with K. Put $K^\dagger := \{i\} \cup K''_1 \cup K''_2 \cup \dots \cup K''_r$, $\hat{K} := K \cup K^\dagger$ and $\hat{K}_l := K_l \cup K''_l$. Then we may write:

$$
\begin{aligned}
B(t) = \mathbb{E}\int_{(\breve{\mathbb{R}}^d)^{\hat{K}}} &\left(\prod_{j\in K}\mathbf{1}\big(\|v_j - v_i\| \le N(v_K)\lambda^{-1/d}t\big)\right)\left(\prod_{l\in K^\dagger}\mathbf{1}\big(\|v_l - v_i\| \le N(v_{K^\dagger})\lambda^{-1/d}t\big)\right) \\
&\times \left(\bigotimes_{l=1}^{r}\big(\breve{\mathcal{P}}^{(l)}_{\lambda\kappa}\big)^{\otimes \hat{K}_l}\right)(\mathrm{d}\breve{v}).
\end{aligned}
$$

Noting that $N(v_K), N(v_{K^\dagger}) \le N(v)$ and disintegrating by (3.3), we obtain:

$$
\begin{aligned}
B(t) &\le \int_{(\breve{\mathbb{R}}^d)^{\hat{K}}}\left(\prod_{j\in\hat{K}}\mathbf{1}\big(\|v_j - v_i\| \le N(v)\lambda^{-1/d}t\big)\right)\tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{\hat{K}_1}\cdots\tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{\hat{K}_r} \\
&\le \sum_{L_1,\dots,L_p \preceq \hat{K}} \int_{(\breve{\mathbb{R}}^d)^{\hat{K}}}\left(\prod_{j\in\hat{K}}\mathbf{1}\big(\|v_j - v_i\| \le N(v)\lambda^{-1/d}t\big)\right)\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_1}\cdots\bar{\mathrm{d}}[\lambda\kappa]\breve{v}_{L_p} \\
&\le \lambda \sum_{L_1,\dots,L_p \preceq \hat{K}} \big((p-1)\|\kappa\|_\infty \omega t^d\big)^{p-1}
\end{aligned}
$$

(where $0^0 := 1$). Noting that $(p-1)^{p-1} \le p!\mathrm{e}^{p-1}$ and $|\hat{K}| = 2k-1$, application of Lemma 3.5 yields:

$$
B(t) \le \lambda(2k-1)!\big(1 + \mathrm{e}\|\kappa\|_\infty \omega t^d\big)^{2k-2}.
$$

Noting that $(2k-1)! \le 4^{k-1}(k!)^2$ and combining this with (3.19) and (3.20), the proof is complete. □

Lemma 3.6. *Let $K_1, \dots, K_r$ be a partition of a non-empty finite set K. Put $k_l := |K_l|$ and $k := |K|$. Take $j \in K_1$ and $\alpha, \beta \ge 0$. Suppose that κ satisfies Assumption D, that R satisfies Assumption MGI(β, κ) and that ξ satisfies either Assumption MGP(α, κ) or Assumption MGI(α, κ). Letting*:

$$
J_{\lambda,\psi} := \int_{(\breve{\mathbb{R}}^d)^K \setminus \breve{\Delta}^K_d}\left(\prod_{l=1}^{r}\prod_{i\in K_l} a_{\lambda,i}(\breve{v}_{K_l})\right) b_{\lambda,j,\psi}(\breve{v}_{K_1})\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^{r}\tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l},
$$

there exists a non-increasing function $\psi : [0,\infty) \to [0,\infty)$, such that for all $\lambda \ge \lambda_0$,

$$
J_{\lambda,\psi} \le \lambda C^k (k_1!)^\alpha (k_2!)^\alpha \cdots (k_r!)^\alpha (k!)^{1+\beta d} \quad \textit{under Assumption MGP}(\alpha,\kappa), \tag{3.21}
$$

$$
J_{\lambda,\psi} \le \lambda C^k (k!)^{1+\alpha+\beta d} \quad \textit{under Assumption MGI}(\alpha,\kappa). \tag{3.22}
$$

In both estimates, the constant C and the lower endpoint λ_0 only depend on κ, ξ and R.

Proof. Let $\lambda \ge \lambda_0$, where for λ_0, we take the maximal corresponding lower endpoint from Assumption MGI(β,κ) imposed on R and Assumption MGP(α,κ) or Assumption MGI(α,κ), whichever imposed on ξ. If ξ satisfies Assumption MGP(α,κ), one can estimate, using Jensen's inequality:

$$a_{\lambda,i}(\breve{v}_{K_l}) \le \big[\mathbb{E}\big|\xi_\lambda(\breve{v}_i, \breve{\mathcal{P}}_{\lambda\kappa} \cup \breve{V}_{K_l})\big|^{pk_l}\big]^{1/(pk_l)} \le A\big[(pk_l)!\big]^{\alpha/(pk_l)} \le Ap^\alpha (k_l!)^{\alpha/k_l},$$

where $p := \max\{2, \lceil 1/q \rceil\}$, and where A and q are as in Assumption MGP(α,κ). As a result, we have:

$$J_{\lambda,\psi} \le A^k p^{k\alpha}\left(\prod_{l=1}^k (k_l!)^\alpha\right)\int_{(\breve{\mathbb{R}}^d)^K\setminus\breve{\Delta}_d^K}\big[\mathbb{E}\big(\psi\big(2R_\lambda(\breve{v}_j, \breve{\mathcal{P}}_{\lambda\kappa}\cup \breve{V}_{K_1})\big)\big)^{-2}\big]^{1/2}\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l}.$$

By the Cauchy–Schwarz inequality and Lemma 3.4, we can estimate:

$$\begin{aligned}
J_{\lambda,\psi} &\le A^k p^{k\alpha}\left(\prod_{l=1}^k (k_l!)^\alpha\right)\left[\int_{(\breve{\mathbb{R}}^d)^K\setminus\breve{\Delta}_d^K}\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l}\right]^{1/2}\\
&\quad\times\left[\int_{(\breve{\mathbb{R}}^d)^K\setminus\breve{\Delta}_d^K}\mathbb{E}\big(\psi\big(2R_\lambda(\breve{v}_j, \breve{\mathcal{P}}_{\lambda\kappa}\cup \breve{V}_{K_1})\big)\big)^{-2}\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l}\right]^{1/2}\\
&\le \lambda A^k p^{k\alpha}\left(\prod_{l=1}^k (k_l!)^\alpha\right)Q(k,\kappa,\psi)\left[\int_{\breve{\mathbb{R}}^d}\mathbb{E}\big(\psi\big(2R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\big)\big)^{-4}\kappa(x)\,\mathrm{d}\breve{x}\right]^{1/4}. \qquad (3.23)
\end{aligned}$$

Choosing $\psi(t) := (1 + \mathrm{e}\|\kappa\|_\infty\omega_d t^d)^{-k}$, a straightforward calculation yields $Q(k,\kappa,\psi) = 2^{k-1}k!k$. Writing $(\psi(2r))^{-4} = \sum_{l=0}^{4k}\binom{4k}{l}(2\mathrm{e}\|\kappa\|_\infty\omega_d)^l r^{ld}$ and recalling that R satisfies Assumption MGI(β,κ), we find that:

$$\begin{aligned}
\int_{\breve{\mathbb{R}}^d}\mathbb{E}\big(\psi\big(2R_\lambda(\breve{x}, \breve{\mathcal{P}}_{\lambda\kappa})\big)\big)^{-4}\kappa(x)\,\mathrm{d}\breve{x} &\le \sum_{l=0}^{4k}\binom{4k}{l}\big(2\mathrm{e}\|\kappa\|_\infty\omega_d B^d\big)^l\big[(4ld)!\big]^\beta\\
&\le \big(1+2\mathrm{e}\|\kappa\|_\infty\omega_d B^d\big)^{4k}\big[(4kd)!\big]^\beta\\
&\le (4d)^{4k\beta d}\big(1+2\mathrm{e}\|\kappa\|_\infty\omega_d B^d\big)^{4k}(k!)^{4\beta d},
\end{aligned}$$

where B is the constant A in (1.17). Plugging this into (3.23) and applying $k \le 3^{k/3}$, we obtain (3.21).

Now suppose that ξ satisfies Assumption MGI(α,κ). Then we apply Jensen's and Hölder's inequality to estimate:

$$\begin{aligned}
J_{\lambda,\psi} &\le \left[\prod_{s=1}^r\prod_{i\in K_s}\int_{(\breve{\mathbb{R}}^d)^K\setminus\breve{\Delta}_d^K}\mathbb{E}\big|\xi_\lambda(\breve{v}_i, \breve{\mathcal{P}}_{\lambda\kappa}\cup\breve{V}_{K_s})\big|^{2k}\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l}\right]^{1/(2k)}\\
&\quad\times\left[\int_{(\breve{\mathbb{R}}^d)^K\setminus\breve{\Delta}_d^K}\mathbb{E}\big(\psi\big(2R_\lambda(\breve{v}_j, \breve{\mathcal{P}}_{\lambda\kappa}\cup \breve{V}_{K_1})\big)\big)^{-2}\psi\big(\lambda^{1/d}\delta(v)\big)\prod_{l=1}^r \tilde{\mathrm{d}}[\lambda\kappa]\breve{v}_{K_l}\right]^{1/2}.
\end{aligned}$$

Now choose ψ as before and apply Lemma 3.4. The estimate (3.22) follows in more or less the same way that (3.21) above. □

Now we are ready to state and prove bounds on semi-cluster measures. Let $\mathrm{Sep}(G,H)$ be the set of all $(G\cup H)$-tuples of points where the maximum is attained, i.e.:

$$\mathrm{Sep}(G,H) := \big\{v \in \big(\mathbb{R}^d\big)^{G\cup H};\ \delta(v) = \mathrm{sep}(V_G, V_H)\big\}, \qquad (3.24)$$

recalling that $\delta(v)$ denotes the maximum separation, precisely defined before Lemma 3.4.

Lemma 3.7. *Let G, H and $S, T, K_1, \dots, K_r$ be two non-trivial partitions of a finite set K with $S \subseteq G$ and $T \subseteq H$. Put $s = |S|$, $t = |T|$, $k_l = |K_l|$, $k = |K|$. Take $\alpha, \beta \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Suppose that κ satisfies Assumption D, that R satisfies Assumption MGI(β, κ), and that ξ satisfies either Assumption MGP(α, κ) or Assumption MGI(α, κ). Letting*:

$$J_\lambda := \int_{\mathrm{Sep}(G,H)} f^{\otimes K} \, \mathrm{d}\big(U_\lambda^{S,T} M_\lambda^{K_1} \cdots M_\lambda^{K_r}\big),$$

where the product in the right hand side means the usual product of measures (*like in* (3.11)), *we have*:

$$|J_\lambda| \leq \lambda C^k \|f\|_\infty^k \big((s+t)!\big)^\alpha (k_1!)^\alpha (k_2!)^\alpha \cdots (k_r!)^\alpha (k!)^{1+\beta d} \quad \text{under Assumption MGP}(\alpha, \kappa), \tag{3.25}$$

$$|J_\lambda| \leq \lambda C^k \|f\|_\infty^k (k!)^{1+\alpha+\beta d} \quad \text{under Assumption MGI}(\alpha, \kappa) \tag{3.26}$$

for all $\lambda \geq \lambda_0$, where the constant C and the lower endpoint λ_0 only depend on κ, ξ and R.

Proof. Applying (3.4), write:

$$J_\lambda = \int_{(\check{\mathbb{R}}^d)^K} \mathbf{1}\big(v \in \mathrm{Sep}(G,H)\big) f^{\otimes K}(v) D_\lambda(\check{v}) \, \tilde{\tilde{\mathrm{d}}}\check{v}, \tag{3.27}$$

where:

$$D_\lambda(\check{v}) = \big[m_\lambda(\check{v}_{S\cup T}) - m_\lambda(\check{v}_S) m_\lambda(\check{v}_T)\big] \prod_{l=1}^r m_\lambda(\check{v}_{K_l}),$$

$$\tilde{\tilde{\mathrm{d}}}\check{v} = \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{S\cup T} \, \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_1} \cdots \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_r} = \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_S \, \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_T \, \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_1} \cdots \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{K_r}.$$

Observe that since $\mathrm{sep}(V_S, V_T) \geq \mathrm{sep}(V_G, V_H) = \delta(v) > 0$ for all $v \in \mathrm{Sep}(G,H)$, the product differential $\tilde{\mathrm{d}}[\lambda\kappa]\check{v}_S \, \tilde{\mathrm{d}}[\lambda\kappa]\check{v}_T$ coincides with $\tilde{\mathrm{d}}[\lambda\kappa]\check{v}_{S\cup T}$.

By Lemma 3.3 and the fact that $\mathrm{sep}(V_S, V_T) \geq \delta(v)$ for $v \in \mathrm{Sep}(G,H)$, the quantity $\mathbf{1}(v \in \mathrm{Sep}(G,H))|D_\lambda(\check{v})|$ can be bounded by a sum of $2(s+t)$ terms of the form $(\prod_{i\in K} \tilde{a}_i)\tilde{b}_j \psi(\lambda^{1/d}\delta(v))$, where either $\tilde{a}_i = a_{\lambda,i}(\check{v}_S)$ or $\tilde{a}_i = a_{\lambda,i}(\check{v}_T)$ or $\tilde{a}_i = a_{\lambda,i}(\check{v}_{S\cup T})$ or $\tilde{a}_i = a_{\lambda,i}(\check{v}_{K_l})$ for some $l = 1, \dots, r$, and where either $\tilde{b}_j = b_{\lambda,j,\psi}(\check{v}_S)$ or $\tilde{b}_j = b_{\lambda,j,\psi}(\check{v}_T)$. Bounding those terms by Lemma 3.6 and applying $2(s+t) \leq 2k \leq 2^k$, the result follows. □

Proof of Lemma 3.1. Put $K = \{1, \dots, k\}$ and write:

$$\langle f^{\otimes k}, c_\lambda^k \rangle = \int_{(\mathbb{R}^d)^k} f^{\otimes k} \, \mathrm{d}c_\lambda^k = \int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}c_\lambda^K + \sum_{G,H \preceq K} \int_{\mathrm{Sep}(G,H)} f^{\otimes K} \, \mathrm{d}c_\lambda^K, \tag{3.28}$$

where the sum ranges over all unordered non-trivial partitions of K into two sets. For the first term, we directly apply (3.11):

$$\int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}c_\lambda^K = \sum_{L_1,\dots,L_p \preceq K} (-1)^{p-1}(p-1)! \int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}\big(M_\lambda^{L_1} \cdots M_\lambda^{L_p}\big).$$

However, only the partition into one single set gives a non-zero integral. Therefore,

$$\int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}c_\lambda^K = \int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}M_\lambda^K = \int_{\check{\Delta}_d^K} f^{\otimes K}(v) m_\lambda(\check{v}) \, \tilde{\mathrm{d}}[\lambda\kappa]\check{v}$$

by (3.4). On the diagonal, $\tilde{\mathrm{d}}[\lambda\kappa]\check{v}$ reduces to $\bar{\mathrm{d}}[\lambda\kappa]\check{v}$, so that:

$$\int_{\Delta_d^K} f^{\otimes K} \, \mathrm{d}c_\lambda^K = \lambda \int_{\check{\mathbb{R}}^d} \big(f(x)\big)^k \mathbb{E}\big[\xi_\lambda(\check{x}, \check{\mathcal{P}}_{\lambda\kappa})\big]^k \kappa(x) \, \mathrm{d}\check{x}.$$

Now recall that Assumption $G(\gamma,\kappa)$ include that ξ satisfies either Assumption $MGP(\alpha,\kappa)$ or the weaker Assumption $MGI(\alpha,\kappa)$ for some $\alpha \le \gamma$. The latter one implies:

$$\left|\int_{\Delta_d^K} f^{\otimes K}\,\mathrm{d}c_\lambda^K\right| \le \lambda A^k \|f\|_\infty^k (k!)^\alpha \le \lambda A^k \|f\|_\infty^k (k!)^\gamma. \tag{3.29}$$

Now we turn to the rest of the terms. We shall combine Lemmas 3.2 and 3.7, followed by Corollary 3.1. If ξ satisfies Assumption $MGP(\alpha,\kappa)$, we can estimate:

$$\begin{aligned}\left|\int_{\mathrm{Sep}(G,H)} f^{\otimes K}\,\mathrm{d}c_\lambda^K\right| &\le \lambda C_1^k \|f\|_\infty^k \sum_{L_1,\dots,L_p\preceq K} p!\big(|L_1|!\big)^\alpha\big(|L_2|!\big)^\alpha\cdots\big(|L_p|!\big)^\alpha (k!)^{1+\beta d}\\ &\le \lambda 2^{k-1} C_1^k \|f\|_\infty^k (k!)^{1+\max\{\alpha,1\}+\beta d}\\ &= \lambda 2^{k-1} C_1^k \|f\|_\infty^k (k!)^{1+\gamma},\end{aligned}$$

where C_1 is the constant C from Lemma 3.7. Similarly, if ξ satisfies Assumption $MGI(\alpha,\kappa)$, we estimate:

$$\begin{aligned}\left|\int_{\mathrm{Sep}(G,H)} f^{\otimes K}\,\mathrm{d}c_\lambda^K\right| &\le \lambda C_1^k \|f\|_\infty^k \sum_{L_1,\dots,L_p\preceq K} p!(k!)^{2+\alpha+\beta d}\\ &\le \lambda 2^{k-1} C_1^k \|f\|_\infty^k (k!)^{2+\alpha+\beta d}\\ &= \lambda 2^{k-1} C_1^k \|f\|_\infty^k (k!)^{1+\gamma}.\end{aligned}$$

Plugging the latter bounds along with (3.29) into (3.28), we find that:

$$\left|\left\langle f^{\otimes k}, c_\lambda^k\right\rangle\right| \le \lambda\big(2\max\{A, 2C_1\}\big)^k \|f\|_\infty^k (k!)^{1+\gamma}.$$

This completes the proof. □

3.3. *Proof of bounds on deviation probabilities* (*Theorem* 1.3)

As mentioned in Section 1.5, the proof of the result will be based on the estimation of the cumulants, applying the celebrated lemma of Rudzkis, Saulis and Statulevičius [32]. Consider a general random variable Y with finite absolute moments of all orders and recall that $c^k(Y)$ stands for the kth cumulant of Y. Below we state a simplified form of the version of that lemma which appears as Lemma 2.3 on p. 18 of [33]:

Lemma 3.8. *Let Y be a random variable as above, with $\mathbb{E}Y = 0$ and $\mathrm{Var}(Y) = 1$, and with its cumulants satisfying*:

$$\left|c^k(Y)\right| \le \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \quad k = 3, 4, \dots \tag{3.30}$$

for some $\gamma \ge 0$ and $\Delta > 0$. Then the large deviation relations:

$$\frac{\mathbb{P}(Y \ge y)}{1-\Phi(y)} = \exp\big(L_\gamma(y)\big)\left(1 + \theta_1\psi(y)\frac{y+1}{\Delta_\gamma}\right), \tag{3.31}$$

$$\frac{\mathbb{P}(Y \le -y)}{\Phi(-y)} = \exp\big(L_\gamma(-y)\big)\left(1 + \theta_2\psi(y)\frac{y+1}{\Delta_\gamma}\right) \tag{3.32}$$

hold true in the interval $0 \le y < \Delta_\gamma$. Here:

$$\Delta_\gamma = \frac{1}{6}\left(\frac{\sqrt{2}}{6}\Delta\right)^{1/(1+2\gamma)}, \tag{3.33}$$

$$\psi(y) = \frac{60[1 + 10\Delta_\gamma^2 \exp(-(1-y/\Delta_\gamma)\sqrt{\Delta_\gamma})]}{1 - y/\Delta_\gamma}, \tag{3.34}$$

the quantities θ_1 *and* θ_2 *belong to* $[-1,1]$ *and the function* $L_\gamma(y)$, *which is closely related to the Cramér–Petrov series*, *satisfies*:

$$\left|L_\gamma(y)\right| \le \frac{|y|^3}{3\varDelta_\gamma} \tag{3.35}$$

for all y *with* $|y| \le \varDelta_\gamma$.

The following weaker form of the preceding result will be used to prove the first part of Theorem 1.3:

Corollary 3.2. *Under the conditions of Lemma* 3.8, *there exist constants* C_0, C_1 *and* C_2 *depending only on* γ, *such that for* $\varDelta \ge C_0$ *and* $0 \le y \le C_1 \varDelta^{1/(1+2\gamma)}$, *we can estimate*:

$$\left|\log \frac{\mathbb{P}(Y \ge y)}{1-\Phi(y)}\right| \le C_2 \frac{1+y^3}{\varDelta^{1/(1+2\gamma)}}, \tag{3.36}$$

$$\left|\log \frac{\mathbb{P}(Y \le -y)}{\Phi(-y)}\right| \le C_2 \frac{1+y^3}{\varDelta^{1/(1+2\gamma)}}. \tag{3.37}$$

Proof. The key observation is that $\psi(y)$ from (3.34) is uniformly bounded in $0 \le y \le q\varDelta_\gamma$, where $q \in [0,1)$ is fixed. Indeed, for such y, one can estimate $\psi(y) \le c_1 + c_2 \varDelta_\gamma^2 \exp(-c_3\sqrt{\varDelta_\gamma})$, where c_1, c_2 and c_3 depend only on q. But the right-hand side of the last estimate can be bounded uniformly in $\varDelta_\gamma$.

Boundedness of ψ along with (3.31), (3.33) and (3.35) implies that there exist universal constants D_1, D_2 and D_3, such that:

$$\exp\left(-\frac{D_2 y^3}{\varDelta^{1/(1+2\gamma)}}\right)\left(1-\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}\right) \le \frac{\mathbb{P}(Y \ge y)}{1-\Phi(y)} \le \exp\left(\frac{D_2 y^3}{\varDelta^{1/(1+2\gamma)}}\right)\left(1+\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}\right) \tag{3.38}$$

for all $0 \le y \le D_1 \varDelta^{1/(1+2\gamma)}$.

Now take $\varDelta \ge (3D_3)^{1+2\gamma}$ and $0 \le y \le \varDelta^{1/(1+2\gamma)}/(3D_3)$, so that $D_3(1+y)\varDelta^{-1/(1+2\gamma)} \le 2/3$. By convexity of the logarithmic function, we have:

$$-\log\left(1-\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}\right) \le \frac{3\log 3}{2}\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}. \tag{3.39}$$

An easy exercise shows that $y \le (2+y^3)/3$, so that:

$$\log\left(1+\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}\right) \le -\log\left(1-\frac{D_3(1+y)}{\varDelta^{1/(1+2\gamma)}}\right) \le \frac{\log 3}{2}\frac{D_3(5+y^3)}{\varDelta^{1/(1+2\gamma)}}. \tag{3.40}$$

The estimate (3.36) now follows from (3.38) and (3.40). Similarly, we obtain (3.37) and the proof is complete. □

For the second part of Theorem 1.3, we shall need another result, which is due to Bentkus and Rudzkis [4] and appears as Lemma 2.4 on p. 19 of [33]. Like Lemma 3.8, we state it in a simplified form, which appears as a corollary of the afore-mentioned result.

Lemma 3.9. *Let* Y *be a random variable with* $\mathbb{E}Y = 0$ *and with its cumulants satisfying*:

$$\left|c^k(Y)\right| \le \left(\frac{k!}{2}\right)^{1+\gamma} \frac{H}{\varDelta^{k-2}} \tag{3.41}$$

for some $\gamma \ge 0$, $H > 0$ *and* $\varDelta > 0$. *Then for all* $y \ge 0$, *we have*:

$$\mathbb{P}(Y \ge y) \le \exp\left(-\frac{1}{4}\min\left\{\frac{y^2}{H}, (\varDelta y)^{1/(1+\gamma)}\right\}\right).$$

Proof of Theorem 1.3. (1) Applying Lemma 3.1 along with (1.7) and (3.10) and recalling that $\sigma_-[f] > 0$, we find that for λ large enough and $k \geq 3$, the cumulants of $\langle f, \bar{\mu}_\lambda \rangle$, i.e., the cumulants of $\langle f, \mu_\lambda \rangle$, i.e., $\langle f^{\otimes k}, c_\lambda^k \rangle$, satisfy:

$$\frac{|c^k(\langle f, \bar{\mu}_\lambda \rangle)|}{\sigma_\lambda^k[f]} \leq \frac{C_1^k}{\lambda^{(k-2)/2}} \frac{\|f\|_\infty^k}{\sigma_-^k[f]} (k!)^{1+\gamma} \tag{3.42}$$

for some constant $C_1 \geq 0$ depending only on κ, ξ and R, where we recall from Section 1.3 that $\sigma_\lambda^2[f]$ denotes the variance of $\langle f, \bar{\mu}_\lambda \rangle$ (and $\sigma_\lambda^k[f]$ its $(k/2)$th power; similarly, $\sigma_-^k[f] = (\sigma_-[f])^k$). To apply Corollary 3.2, rewrite the right-hand side of (3.42) as:

$$C_1^2 \frac{\|f\|_\infty^2}{\sigma_-^2[f]} \left(\frac{C_1}{\sqrt{\lambda}} \frac{\|f\|_\infty}{\sigma_-[f]} \right)^{k-2} (k!)^{1+\gamma} \leq \left(\max\left\{1, C_1^2 \frac{\|f\|_\infty^2}{\sigma_-^2[f]}\right\} \frac{C_1}{\sqrt{\lambda}} \frac{\|f\|_\infty}{\sigma_-[f]} \right)^{k-2} (k!)^{1+\gamma}.$$

Thus, recalling that the first cumulant of the centered measure $\bar{\mu}_\lambda$ equals zero whereas its higher order cumulants coincide with those of μ_λ, we can apply Corollary 3.2 to $Y := \langle f, \bar{\mu}_\lambda \rangle / \sigma_\lambda[f]$ with $y = x/\sigma_\lambda[f]$ and with Δ taken to be $\sqrt{\lambda}$ multiplied by some constant depending only on κ, ξ, R and the ratio $\|f\|_\infty / \sigma_-[f]$. It follows that there exist constants $\lambda_1, D_1, D_2 \geq 0$, such that for all $\lambda \geq \lambda_1$ and all $0 \leq x \leq D_1 \sigma_\lambda[f] \lambda^{1/(2+4\gamma)}$,

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_\lambda \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])} \right| \leq \frac{D_2}{\lambda^{1/(2+4\gamma)}} \left[1 + \left(\frac{x}{\sigma_\lambda[f]} \right)^3 \right].$$

Applying (1.7) once again, we obtain that there exist constants $\lambda_2, D_3, D_4 \geq 0$ such that for all $\lambda \geq \lambda_2$ and all $0 \leq x \leq D_3 \sigma_-[f] \lambda^{(1+\gamma)/(1+2\gamma)}$,

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_\lambda \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])} \right| \leq D_4 \left[\frac{1}{\lambda^{1/(2+4\gamma)}} + \frac{x^3}{\lambda^{(2+3\gamma)/(1+2\gamma)} \sigma_-^3[f]} \right].$$

An analogous bound holds for the lower tail probabilities and part (1) follows.

(2) Taking C from Lemma 3.1, and letting $D_5 := 2^{\gamma+1} C^2 \|f\|_\infty^2$ and $D_6 := 1/(C\|f\|_\infty)$, we deduce that the random variable $Y := \langle f, \bar{\mu}_\lambda \rangle$ satisfies (3.41) with:

$$H = D_5 \lambda \quad \text{and} \quad \Delta = D_6. \tag{3.43}$$

Next, if $\sigma_\lambda^2[f] \leq D_5 \lambda$, then Y also satisfies (3.41) with:

$$H = \sigma_\lambda^2[f] \quad \text{and} \quad \Delta = D_6 \frac{\sigma_\lambda^2[f]}{D_5 \lambda}. \tag{3.44}$$

Now take arbitrary $0 \leq t \leq 1$. Combining (3.43) and (3.44) with the inequality $\min\{a, b\} \leq a^{1-t} b^t$, we find that (3.41) is also satisfied with:

$$H = \sigma_\lambda^{2t}[f] (D_5 \lambda)^{1-t} \quad \text{and} \quad \Delta = D_6 \left(\frac{\sigma_\lambda^2[f]}{D_5 \lambda} \right)^t.$$

Plugging this into Lemma 3.9, we obtain:

$$\mathbb{P}\big(\pm \langle f, \bar{\mu}_\lambda \rangle \geq x\big) \leq \min_{0 \leq t \leq 1} \exp\left[-\frac{1}{4} \min\left\{ \frac{x^2}{\sigma_\lambda^{2t}[f] (D_5 \lambda)^{1-t}}, (D_6 x)^{1/(1+\gamma)} \left(\frac{\sigma_\lambda^2[f]}{D_5 \lambda} \right)^{t/(1+\gamma)} \right\} \right]. \tag{3.45}$$

An easy exercise in optimization shows that for all $a_1, a_2 > 0$, $b \geq 1$ and $c_1, c_2 \geq 0$, we have $\max_{0 \leq t \leq 1} \min\{a_1 b^{c_1 t}, a_2 b^{-c_2 t}\} = \min\{a_1 b^{c_1}, a_2, a_1^{c_2/(c_1+c_2)} a_2^{c_1/(c_1+c_2)}\}$. Plugging this into (3.45), we obtain (1.21). $\square$

3.4. *Proof of moderate deviations*

The results of Section 1.6 will follow from the following consequence of Theorem 1.3.

Lemma 3.10. *Let $\bar{\mu}_\lambda$ be defined as in Section 1 with ξ satisfying Assumptions $\mathrm{G}(\gamma,\kappa)$ and $\mathrm{CV}(\kappa)$, let a_λ satisfy (1.23) and take $f \in \mathcal{B}(\mathbb{R}^d)$. Then, recalling (1.7), for $\sigma[f] > 0$ and $t \geq 0$, we have*:

$$\lim_{\lambda\to\infty} \frac{1}{a_\lambda^2} \log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \geq t\big) = \lim_{\lambda\to\infty} \frac{1}{a_\lambda^2} \log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle > t\big) = -\frac{t^2}{2\sigma^2[f]}, \tag{3.46}$$

while for $\sigma[f] = 0$ and $t > 0$, we have:

$$\lim_{\lambda\to\infty} \frac{1}{a_\lambda^2} \log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \geq t\big) = \lim_{\lambda\to\infty} \frac{1}{a_\lambda^2} \log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle > t\big) = -\infty. \tag{3.47}$$

Proof. Suppose first that $\sigma[f] > 0$. In this case, we plug $x = ta_\lambda\lambda^{1/2}$ into (1.19) and make use of the fact that for all $y \geq 0$,

$$\frac{1}{2+\sqrt{2\pi}y} \leq \mathrm{e}^{y^2/2}\big(1-\Phi(y)\big) \leq \frac{1}{2}.$$

Combining both and noting that $\sigma_-[f] = \sigma[f]$ by (1.7), we obtain the bound:

$$\left|\log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \geq t\big) + \frac{a_\lambda^2\lambda t^2}{2\sigma_\lambda^2[f]}\right| \leq \log\left(2 + \frac{\sqrt{2\pi}ta_\lambda\lambda^{1/2}}{\sigma_\lambda[f]}\right) + C_2\frac{1+a_\lambda^3t^3/\sigma^3[f]}{\lambda^{1/(2+4\gamma)}}.$$

Dividing by a_λ^2, making use of (1.7) once again and applying condition (1.23), this implies (3.46) for 'greater or equal'. The corresponding result for the strict inequality follows by continuity.

In the case where $\sigma[f] = 0$, plug $x = ta_\lambda\lambda^{1/2}$ into (1.21) to obtain:

$$\frac{\log \mathbb{P}(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \geq t)}{a_\lambda^2} \leq -t^2\min\left\{C_4\frac{\lambda}{\sigma_\lambda^2[f]}, C_5\left(\frac{\lambda}{(ta_\lambda)^{2+4\gamma}}\right)^{1/(2+2\gamma)}, C_6\left(\frac{\lambda}{(ta_\lambda)^{2+4\gamma}}\right)^{1/(4+2\gamma)}\right\}$$

and the desired limiting behavior follows again from (1.7) and (1.23). This completes the proof. □

Proof of Theorem 1.4. We apply the preceding lemma along with Theorem 4.1.11 of [10], which allows us to derive a LDP from the limiting behavior of probabilities for a basis of topology. For the latter, we choose all open intervals (u_1, u_2), where at least one of the endpoints is finite and where none of the endpoints lies at the origin. Denote the family of all such intervals by $\mathcal{U}$. From Lemma 3.10, it follows that for each $U = (u_1, u_2) \in \mathcal{U}$,

$$\mathcal{L}_U := -\lim_{\lambda\to\infty} \frac{1}{a_\lambda^2} \log \mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \in U\big) = \begin{cases} u_2^2/(2\sigma^2[f]); & u_1 < u_2 < 0,\\ 0; & u_1 < 0 < u_2,\\ u_1^2/(2\sigma^2[f]); & 0 < u_1 < u_2,\end{cases}$$

for all $U \in \mathcal{U}$, recalling our convention on division by zero from the end of Section 1.2. By Theorem 4.1.11 of [10], the random variables $a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle$ satisfy a *weak* LDP (MDP) as $\lambda \to \infty$ with speed a_λ^2 and rate function:

$$t \mapsto \sup_{\substack{U\in\mathcal{U}\\ t\in U}} \mathcal{L}_U = \frac{t^2}{2\sigma^2[f]},$$

which matches the function I_f from (1.24). Here, weak LDP means that the lower bound in (1.22) holds for all measurable sets Γ, while the upper bound holds for all relatively compact measurable Γ. However, from Lemma 3.10, it follows that the family $a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle$ is exponentially tight for speed a_λ^2, i.e., for each $M < \infty$, there exists a measurable relatively compact set K, such that $\limsup_{\lambda\to\infty} a_\lambda^{-2}\log \mathbb{P}(a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle \notin K) \le -M$. By Lemma 1.2.18 of [10], the family $a_\lambda^{-1}\lambda^{-1/2}\langle f, \bar{\mu}_\lambda\rangle$ must then satisfy a *full* LDP with the same speed and the same *good* rate function. This completes the proof. □

Now we turn to the proof of Theorem 1.5. We begin with the same argument based on Theorem 4.1.11 of [10], which requires a certain limiting behavior of probabilities of sets from a basis of topology. The derivation of that behavior requires a specific shape of the sets. Thus, we have to show that certain sets of that shape form a basis of topology.

Lemma 3.11. *Let F and V be pairwise dual finite-dimensional vector spaces, equipped with the usual topology, and let Q be a positively semi-definite quadratic form on F. Let $\mathcal{U}_0$ be the family of all open subsets of the half-spaces $\{\nu \in V; \langle f, \nu\rangle > b\}$, where $b > 0$ and $Q(f) = 0$, and denote by $\mathcal{U}_1$ the family of all sets of the form*

$$\bigl\{\nu \in V; \langle f_0, \nu\rangle > b_0, \langle f_1, \nu\rangle < b_1, \langle f_2, \nu\rangle < b_2, \dots, \langle f_n, \nu\rangle < b_n\bigr\}, \tag{3.48}$$

where either $0 < b_0/\sqrt{Q(f_0)} < b_i/\sqrt{Q(f_i)}$ for all $i = 1, 2, \dots, n$, or $b_0 < 0 < b_i$ for all $i = 1, 2, \dots, n$. Then the family $\mathcal{U}_0 \cup \mathcal{U}_1$ is a basis of the topology on V.

Proof. Define $F_0 := \{f \in F; Q(f) = 0\}$ and $F_0^\perp := \{\nu \in V; \langle f, \nu\rangle = 0 \text{ for all } f \in F_0\}$. Now take $\mu \in V$ and its open neighborhood W. We have to show that there exists $U \in \mathcal{U}_0 \cup \mathcal{U}_1$, such that $x \in U \subseteq W$. We distinguish three cases.

Case 1: $\mu \notin F_0^\perp$. Then there exist $f \in F_0$ and $a > 0$, such that $\langle f, \mu\rangle > a$, so that we can take $U := W \cap \{\nu \in V; \langle f, \mu\rangle > a\} \in \mathcal{U}_0$.

Case 2: $\mu = 0$. Then there exists $\varepsilon > 0$ and elements $f_0, f_1, \dots, f_n$, such that $U := \bigcap_{i=0}^n \{\nu; \langle f_i, \nu\rangle < \varepsilon\} \subseteq W$. Clearly, $0 \in U$. Since we can write $U = \{\nu; \langle -f_0, \nu\rangle > -\varepsilon\} \cap \bigcap_{i=1}^n \{\nu; \langle f_i, \nu\rangle < \varepsilon\}$, we also have $U \in \mathcal{U}_1$.

Case 3: $\mu \in F_0^\perp \setminus \{0\}$. Recalling our convention on division by zero from the end of Section 1.2, it follows from standard linear algebra and topology that the map $f \mapsto \langle f, \mu\rangle/\sqrt{Q(f)}$, defined on $F\setminus F_0$, is continuous, bounded and attains its maximum, say, at f_0. Since $\mu \ne 0$, we have $\langle f_0, \mu\rangle > 0$ (and remember that $Q(f_0) > 0$).

There exist functions $f_1, f_2, \dots, f_n$ and $\delta \in (0, \langle f_0, \mu\rangle)$, such that $U_0 := \{\nu \in V; \langle f_0, \nu - \mu\rangle > -\delta\} \cap \bigcap_{i=0}^n \{\nu \in V; \langle f_i, \nu - \mu\rangle < \delta\} \subseteq W$. Now consider the sets:

$$U_{\varepsilon,t} := \bigl\{\nu \in V; \langle f_0, \nu - \mu\rangle > -\varepsilon\bigr\} \cap \bigcap_{i=1}^n \bigl\{\langle f_0 + t f_i, \nu - \mu\rangle < \varepsilon\bigr\}.$$

Clearly, $\mu \in U_{\varepsilon,t}$ for all $\varepsilon, t > 0$. Next, for each $\nu \in U_{\varepsilon,t}$, we can estimate:

$$\langle f_i, \nu - \mu\rangle = \frac{1}{t}\bigl[\langle f_0 + t f_i, \nu - \mu\rangle - \langle f_0, \nu - \mu\rangle\bigr] < \frac{2\varepsilon}{t}.$$

Therefore, if $\varepsilon < \delta$ and $2\varepsilon/t < \delta$, then $U_{\varepsilon,t} \subseteq U_0$.

Now we turn our attention to the question when $U_{\varepsilon,t} \in \mathcal{U}$. By (3.48), this will be surely true if:

$$\frac{\langle f_0 + t f_i, \mu\rangle + \varepsilon}{\sqrt{Q(f_0 + t f_i)}} > \frac{\langle f_0, \mu\rangle - \varepsilon}{\sqrt{Q(f_0)}}. \tag{3.49}$$

Since $\langle f, \mu\rangle/\sqrt{Q(f)}$ is maximal at f_0, it follows from smoothness that there exist $a, t_0 > 0$, such that for all $i = 1, 2, \dots, n$,

$$\frac{\langle f_0 + t f_i, \mu\rangle}{\sqrt{Q(f_0 + t f_i)}} \ge \frac{\langle f_0, \mu\rangle}{\sqrt{Q(f_0)}} - a t^2$$

for all $0 \le t \le t_0$. Therefore, the condition (3.49) is satisfied if $t \le t_0$ and $at^2 < \varepsilon/\sqrt{Q(f_0)}$. Collecting everything together, we conclude after some calculation that if $0 < t < \min\{t_0, \frac{\delta}{2a\sqrt{Q(f)}}, \sqrt{\frac{\delta}{a\sqrt{Q(f)}}}\}$, then there exists $\varepsilon > 0$, such that $U_{\varepsilon,t} \subseteq U_0 \subseteq W$ and $U_{\varepsilon,t} \in \mathcal{U}$, so that the desired set U exists in this case, too. This completes the proof. □

Proof of Theorem 1.5. We split the argument into several steps.

Step 1: Derive a MDP for finite-dimensional restrictions of the measures. Denote by $\mathcal{F}$ the set of all finite-dimensional subspaces of $\mathcal{B}(\mathbb{R}^d)$, fix $F \in \mathcal{F}$ and consider the random measures μ_λ as linear functionals on F. Let $\mathcal{U}_0 \cup \mathcal{U}_1$ be the basis of the usual topology on F', the dual of F, where $\mathcal{U}_0$ and $\mathcal{U}_1$ are defined as in Lemma 3.11, taking the quadratic form $Q := \sigma^2$. For $U = \{\nu \in F'; \langle f_0, \nu\rangle > b_0, \langle f_1, \nu\rangle < b_1, \ldots, \langle f_n, \nu\rangle < b_n\} \in \mathcal{U}_1$, where the numbers $b_0, b_1, \ldots, b_n$ satisfy the conditions below (3.48), we have, by Lemma 3.10:

$$\mathcal{L}_U := -\lim_{\lambda\to\infty} a_\lambda^{-2}\mathbb{P}\big(a_\lambda^{-1}\lambda^{-1/2}\bar{\mu}_\lambda \in U\big) = \frac{(\max\{b_0, 0\})^2}{2\sigma^2[f_0]};$$

for $U \in \mathcal{U}_0$, we have $\mathcal{L}_U = \infty$. By Theorem 4.1.11 of [10], the random functionals $a_\lambda^{-1}\lambda^{-1/2}\bar{\mu}_\lambda \in F'$ satisfy a weak LDP (MDP) as $\lambda \to \infty$ with speed a_λ^2 and rate function:

$$\nu \mapsto \sup_{\substack{U\in\mathcal{U}\\ \nu\in U}} \mathcal{L}_U = \sup_{f\in F} \frac{\langle f, \nu\rangle^2}{2\sigma^2[f]}.$$

Finally, by Lemma 1.2.18 of [10], these random functionals satisfy a full LDP (MDP) with the same good rate function because they are exponentially tight for speed a_λ^2. To see this, take a basis $f_1, \ldots, f_n$ of F with $\sigma[f_i] \le 1$ for all i and consider the compact sets $K_M := \bigcap_{i=1}^n \{\nu \in F'; |\langle f_i, \nu\rangle| \le M\}$. By Lemma 3.10, $\limsup_{\lambda\to\infty} a_\lambda^{-2}\mathbb{P}(a_\lambda^{-1}\lambda^{-1/2}\bar{\mu}_\lambda \notin K_M) \le -M^2/2$ and exponential tightness follows. This completes Step 1.

Step 2: Combine the MDP's for finite-dimensional restrictions into a MDP for entire random measures. We apply a version of the Dawson–Gärtner theorem for projective limits, namely Theorem 4.6.9 of [10], naturally embedding $\mathrm{Meas}(\mathbb{R}^d)$ into $(\mathcal{B}(\mathbb{R}^d))'$, the algebraic dual of $\mathcal{B}(\mathbb{R}^d)$, and identifying the projections of the functionals to finite-dimensional spaces with their restrictions to finite-dimensional subspaces of $\mathcal{B}(\mathbb{R}^d)$. Thus we find that, as $\lambda \to \infty$, the random measures $a_\lambda^{-1}\lambda^{-1/2}\bar{\mu}_\lambda$ satisfy the MDP in $(\mathcal{B}(\mathbb{R}^d))'$ with speed a_λ^2 and the good rate function:

$$J(\nu) := \sup_{F\in\mathcal{F}} \sup_{f\in\operatorname{Lin} F} \frac{\langle f, \nu\rangle^2}{2\sigma^2[f]} = \sup_{f\in\mathcal{B}(\mathbb{R}^d)} \frac{\langle f, \nu\rangle^2}{2\sigma^2[f]}.$$

Step 3: Compute the rate function. Take $\nu \in (\mathcal{B}(\mathbb{R}^d))'$ and distinguish five separate cases.

Case 1: ν is unbounded with respect to the supremum norm on $\mathcal{B}(\mathbb{R}^d)$. Since the latter is stronger than the seminorm σ, we have $J(\nu) = +\infty$ in this case.

Case 2: ν is bounded with respect to the supremum norm, but is not a measure. This means that there exists a sequence of bounded functions f_n with $f_n \downarrow 0$ pointwise, such that the $\langle f_n, \nu\rangle$ does not converge to 0. We may assume that $|\langle f_n, \nu\rangle| \ge 1$ for all n. Denoting $L^2 := \{f : \mathbb{R}^d \to \mathbb{R}; \sigma[f] < \infty\}$ and noting that $\mathcal{B}(\mathbb{R}^d) \subseteq L^2$, we find that $\sigma[f] \to 0$ by the dominated convergence theorem. Therefore $J(\nu) = +\infty$.

Case 3: ν is a measure, but is not absolutely continuous with respect to $V(\kappa(x))\kappa(x)\,\mathrm{d}x$, where V is as in (1.8). In this case, there exists a measurable set A with $\int_A V(\kappa(x))\kappa(x)\,\mathrm{d}x = 0$, but $\nu(A) \ne 0$. In other words, $\sigma[1_A] = 0$, but $\langle \mathbf{1}_A, \nu\rangle \ne 0$, so that again $J(\nu) = +\infty$.

Case 4: $\nu \ll V(\kappa(x))\kappa(x)\,\mathrm{d}x$, but $\sigma[\rho] = \infty$, where $\rho(x) := \nu(\mathrm{d}x)/(V(\kappa(x))\kappa(x)\,\mathrm{d}x)$. In this case, there exists a sequence $\rho_1, \rho_2, \ldots \in L^2$ which converges pointwise to ρ and satisfies $\rho\rho_n \ge 0$ and $|\rho_1| \le |\rho_2| \le \cdots$. By the monotone convergence theorem, we have $\sigma[\rho_n] \uparrow \infty$; we may assume that $\sigma[\rho_1] > 0$. Then the functions $g_n := \rho_n/\sigma^2[\rho_n]$ satisfy $\sigma[g_n] = 1/\sigma[\rho_n] \to 0$ but $\langle g_n, \nu\rangle \ge 1$, so that again $J(\nu) = +\infty$.

Case 5: $\nu \ll V(\kappa(x))\kappa(x)\,\mathrm{d}x$ and $\rho(x) := \nu(\mathrm{d}x)/(V(\kappa(x))\kappa(x)\,\mathrm{d}x)$ satisfies $\sigma[\rho] < \infty$. In this case we may write:

$$\begin{aligned} J(\nu) &= \sup_{f\in\mathcal{B}(\mathbb{R}^d)} \frac{(\int_{\mathbb{R}^d} f(x)\rho(x)V(\kappa(x))\kappa(x)\,\mathrm{d}x)^2}{2\int_{\mathbb{R}^d} f^2(x)V(\kappa(x))\kappa(x)\,\mathrm{d}x} = \sup_{f\in L^2} \frac{(\int_{\mathbb{R}^d} f(x)\rho(x)V(\kappa(x))\kappa(x)\,\mathrm{d}x)^2}{2\int_{\mathbb{R}^d} f^2(x)V(\kappa(x))\kappa(x)\,\mathrm{d}x} \\ &= \frac{1}{2}\sigma^2[\rho] = I(\nu), \end{aligned}$$

where the latter is defined in (1.25). The second equality holds because $\mathcal{B}(\mathbb{R}^d)$ is dense in L^2; the third one is due to the Cauchy–Schwarz inequality. This completes Step 3.

Step 4: Restrict the MDP. To see that we may replace $(\mathcal{B}(\mathbb{R}^d))'$ and J with $\mathrm{Meas}(\mathbb{R}^d)$ and I, we apply Lemma 4.1.5 of [10], noting that J agrees with I on $\mathrm{Meas}(\mathbb{R}^d)$ and is infinite outside $\mathrm{Meas}(\mathbb{R}^d)$. This completes the proof. □

3.5. *Proof of non-degeneracy of the limiting variance*

Throughout this subsection, we stick to the conventions on ξ, H, Δ, R, ξ_λ, Δ_λ, R_λ and Ω specified in Section 1.2. Before proving Theorem 1.2, we need the following auxiliary result.

Lemma 3.12. *Let f be a non-negative locally integrable function. Take $1 < a < b$. Then there exists a universal constant C, such that for every non-negative geometric functional g,*

$$\biggl[\mathbb{E}\biggl(\sum_{\breve{x}\in\breve{\mathcal{P}}_f} g(\breve{x},\breve{\mathcal{P}}_f)\biggr)^a\biggr]^{1/a} \le C\biggl(\frac{(a-1)b}{b-a}\biggr)^{(a-1)/a}$$
$$\times\biggl\{\frac{b-1}{b-a}\biggl[\int_{\breve{\mathbb{R}}^d}\mathbb{E}\bigl(g(\breve{x},\breve{\mathcal{P}}_f)\bigr)^b f(x)\,\mathrm{d}\breve{x}\biggr]^{1/b} + \int_{\breve{\mathbb{R}}^d}\bigl[\mathbb{E}\bigl(g(\breve{x},\breve{\mathcal{P}}_f)\bigr)^b\bigr]^{1/b} f(x)\,\mathrm{d}\breve{x}\biggr\}.$$

Proof. Let m be a non-negative measurable function on $\breve{\mathbb{R}}^d$, such that for each $\breve{x}$ with $m(\breve{x}) = 0$, $g(\breve{x},\breve{\mathcal{P}}_f)$ almost surely vanishes. For each $t \ge 0$, let $\breve{M}(t) := \{\breve{x}; m(\breve{x}) > t\}$. Write:

$$J := \biggl[\mathbb{E}\biggl(\sum_{\breve{x}\in\breve{\mathcal{P}}_f} g(\breve{x},\breve{\mathcal{P}}_f)\biggr)^a\biggr]^{1/a} = \biggl[\mathbb{E}\biggl(\sum_{\breve{x}\in\breve{\mathcal{P}}_f\cap\breve{M}(0)} \frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\int_0^{m(\breve{x})}\mathrm{d}t\biggr)^a\biggr]^{1/a}$$
$$= \biggl[\mathbb{E}\biggl(\int_0^\infty \sum_{\breve{x}\in\breve{\mathcal{P}}_f\cap\breve{M}(t)} \frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\,\mathrm{d}t\biggr)^a\biggr]^{1/a}.$$

By Minkowski's inequality for integrals and then by Jensen's inequality, we can estimate:

$$J \le \int_0^\infty \biggl[\mathbb{E}\biggl(\sum_{\breve{x}\in\breve{\mathcal{P}}_f\cap\breve{M}(t)} \frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\biggr)^a\biggr]^{1/a}\mathrm{d}t$$
$$\le \int_0^\infty \biggl[\mathbb{E}\biggl(\sum_{\breve{x}\in\breve{\mathcal{P}}_f\cap\breve{M}(t)} \bigl|\breve{\mathcal{P}}_f\cap\breve{M}(t)\bigr|^{a-1}\biggl(\frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\biggr)^a\biggr)\biggr]^{1/a}\mathrm{d}t.$$

By the Palm formula (3.2), we can rewrite this estimate as:

$$J \le \int_0^\infty \biggl[\int_{\breve{M}(t)} \mathbb{E}\biggl\{\bigl(1+\bigl|\breve{\mathcal{P}}_f\cap\breve{M}(t)\bigr|\bigr)^{a-1}\biggl(\frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\biggr)^a\biggr\} f(x)\,\mathrm{d}\breve{x}\biggr]^{1/a}\mathrm{d}t.$$

Applying Hölder's inequality, we obtain:

$$J \le \int_0^\infty \biggl[\int_{\breve{M}(t)} \bigl\{\mathbb{E}\bigl(1+\bigl|\breve{\mathcal{P}}_f\cap\breve{M}(t)\bigr|\bigr)^{(a-1)b/(b-a)}\bigr\}^{(b-a)/b}\biggl\{\mathbb{E}\biggl(\frac{g(\breve{x},\breve{\mathcal{P}}_f)}{m(\breve{x})}\biggr)^b\biggr\}^{a/b} f(x)\,\mathrm{d}\breve{x}\biggr]^{1/a}\mathrm{d}t.$$

Now set $m(\breve{x}) := [\mathbb{E}(g(\breve{x},\breve{\mathcal{P}}_f))^b]^{1/b}$. In addition, observe that $|\breve{\mathcal{P}}_f\cap\breve{M}(t)|$ is Poisson with expectation $I(t) := \int_{\breve{M}(t)} f(x)\,\mathrm{d}\breve{x}$. If $X \sim \mathrm{Pois}(\lambda)$, then for any $n\in\mathbb{N}$, we may express the nth moment of $1+X$ in terms of a contour integral:

$$\mathbb{E}(1+X)^n = \frac{n!}{2\pi\mathrm{i}}\oint_K \frac{\mathrm{e}^{\lambda(\mathrm{e}^z-1)+z}}{z^{n+1}}\,\mathrm{d}z.$$

Choosing K to be the circle with radius $1/(1+\lambda)$ centered at the origin, we can estimate $\mathbb{E}(1+X)^n \le n!(1+\lambda)^n f(\lambda)$, where $f(\lambda) = \mathrm{e}^{\lambda(\mathrm{e}^{1/(\lambda+1)}-1)+1/(\lambda+1)}$. Noting that f is bounded in $\lambda > 0$ and applying Stirling's formula, we obtain that there exists a universal constant A, such that $\mathbb{E}(1+X)^\gamma \le (A\gamma(1+\lambda))^\gamma$ for all $\gamma > 0$. Therefore,

$$J \le \left(\frac{A(a-1)b}{b-a}\right)^{(a-1)/a} \int_0^\infty \left[\left(1+I(t)\right)^{a-1} I(t)\right]^{1/a} \mathrm{d}t \le A\left(\frac{(a-1)b}{b-a}\right)^{(a-1)/a} \int_0^\infty \left[\left(I(t)\right)^{1/a} + I(t)\right] \mathrm{d}t.$$

Observe that:

$$\int_0^\infty I(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} m(\check{x}) f(x)\,\mathrm{d}x.$$

For the rest, apply Young's inequality:

$$\int_0^\infty \left(I(t)\right)^{1/a} \mathrm{d}t \le \frac{a-1}{a} \int_0^\infty \phi(t)\,\mathrm{d}t + \frac{1}{a} \int_0^\infty \left(\phi(t)\right)^{1-a} I(t)\,\mathrm{d}t, \tag{3.50}$$

where we choose $\phi(t) = \min\{1, (c/t)^{(b-1)/(a-1)}\}$; $c > 0$ will be chosen later. It is easy to see that:

$$\int_0^\infty \phi(t)\,\mathrm{d}t = \frac{b-1}{b-a} c. \tag{3.51}$$

For the second term, we have:

$$\int_0^\infty \left(\phi(t)\right)^{1-a} I(t)\,\mathrm{d}t = \int_{\check{\mathbb{R}}^d} \int_0^{m(\check{x})} \max\left\{1, \left(\frac{t}{c}\right)^{b-1}\right\} \mathrm{d}t\, f(x)\,\mathrm{d}x \le \int_{\check{\mathbb{R}}^d} \left(m(\check{x}) + \frac{(m(\check{x}))^b}{bc^{b-1}}\right) f(x)\,\mathrm{d}x.$$

Choosing $c := [\int_{\check{\mathbb{R}}^d} (m(\check{x}))^b f(x)\,\mathrm{d}\check{x}]^{1/b}$, combining with (3.51) and plugging into (3.50), we obtain:

$$\int_0^\infty \left(I(t)\right)^{1/a} \mathrm{d}t \le B\left[\int_{\check{\mathbb{R}}^d} \left(m(\check{x})\right)^b f(x)\,\mathrm{d}\check{x}\right]^{1/b} + \frac{1}{a} \int_{\check{\mathbb{R}}^d} m(\check{x}) f(x)\,\mathrm{d}\check{x},$$

where $B = \frac{a-1}{a}\frac{b-1}{b-a} + \frac{1}{ab} \le \frac{b-1}{b-a}$. Collecting all together, the result follows. □

Corollary 3.3. *Let $(g_\lambda)_{\lambda>\lambda_0}$ and $(R^*_\lambda)_{\lambda>\lambda_0}$ be two families of non-negative geometric functionals. Define*:

$$h_\lambda(\check{x}, \check{\mathcal{X}}) := \sum_{\check{y}\in\check{\mathcal{X}}} g_\lambda(\check{y}, \check{\mathcal{X}}) \mathbf{1}\left(R^*_\lambda(\check{y}, \check{\mathcal{X}}) \ge \lambda^{1/d}\|y-x\|\right),$$

$$h^-_\lambda(\check{x}, \check{\mathcal{X}}) := \sum_{\check{y}\in\check{\mathcal{X}}} g_\lambda\left(\check{y}, \check{\mathcal{X}} \setminus \{\check{x}\}\right) \mathbf{1}\left(R^*_\lambda(\check{y}, \check{\mathcal{X}}) \ge \lambda^{1/d}\|y-x\|\right)$$

(assuming $\check{x} \in \check{\mathcal{X}}$). Next, let $0 < \tau < \infty$ and let $p, s, q > 0$ with $s/p + d/q < 1$. Suppose that the family $(R^*_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(q, τ, Ω). Then, if the family $(g_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(p, τ, Ω), the family $(h^-_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(s, τ, Ω); if the family $(g_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH1(p, τ, Ω), the family $(h_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH1(s, τ, Ω).

Proof. For arbitrary non-negative geometric functional η and any $\tau, u > 0$, set:

$$m_{u,\tau}(\eta) := \operatorname*{ess\,sup}_{\mathbf{1}(x\in\Omega)\,\mathrm{d}\check{x}} \left[\mathbb{E}\left(\eta(\check{x}, \check{\mathcal{P}}_\tau \cap \Omega)\right)^u\right]^{1/u},$$

$$m^+_{u,\tau}(\eta) := \operatorname*{ess\,sup}_{\mathbf{1}(x,y\in\Omega)\,\mathrm{d}\check{x}\otimes\mathrm{d}\check{y}} \left[\mathbb{E}\left(\eta\left(\check{x}, \left(\check{\mathcal{P}}_\tau \cup \{\check{y}\}\right) \cap \Omega\right)\right)^u\right]^{1/u}.$$

Thus, we have to bound $m_{s,\lambda\tau}(h_\lambda)$ and $m_{s,\lambda\tau}(h_\lambda^-)$. We shall apply Lemma 3.12. Observe that one can choose $0 < r < q$ and $t > s$ with $t/p + d/r = 1$. To bound $m_{s,\lambda\tau}(h_\lambda^-)$, we first use Hölder's and Markov's inequality to estimate:

$$\left[\mathbb{E}\big(g_\lambda(\check{y}, \check{\mathcal{P}}_{\lambda\tau} \cap \Omega)\mathbf{1}\big(R_\lambda^*(\check{y}, \check{\mathcal{P}}_{\lambda\tau} \cap \check{\Omega}) \geq \lambda^{1/d}\|y - x\|\big)\big)^t\right]^{1/t}$$
$$\leq m_{p,\lambda\tau}(g_\lambda)\left[\mathbb{P}\big(R_\lambda^*(\check{y}, \check{\mathcal{P}}_{\lambda\tau} \cap \check{\Omega}) \geq \lambda^{1/d}\|y - x\|\big)\right]^{d/r} \leq \frac{m_{p,\lambda\tau}(g_\lambda)(m_q(1 + R_\lambda^*))^{qd/r}}{(1 + \lambda^{1/d}\|y - x\|)^{qd/r}}$$

for almost all $\check{x}, \check{y} \in \check{\Omega}$. By Lemma 3.12, we have:

$$m_{s,\lambda\tau}\big(\check{x}, h_\lambda^-\big) \leq K_1\left[\int_\Omega \frac{(m_{p,\lambda\tau}(g_\lambda))^t(m_q(1 + R_\lambda^*))^{qtd/r}}{(1 + \lambda^{1/d}\|y - x\|)^{qtd/r}}\lambda\tau\,\mathrm{d}y\right]^{1/t}$$
$$+ K_2\int_\Omega \frac{m_{p,\lambda\tau}(g_\lambda)(m_q(1 + R_\lambda^*))^{qd/r}}{(1 + \lambda^{1/d}\|y - x\|)^{qd/r}}\lambda\tau\,\mathrm{d}y.$$

Substituting $z = \lambda^{1/d}(y - x)$, observe that the both integrals converge uniformly in x and λ, leading to the desired bound on $m_{s,\lambda\tau}(h_\lambda^-)$. Similarly, we can bound $m_{s,\lambda\tau}(h_\lambda)$, using $m_{p,\lambda\tau}^+(g_\lambda)$ instead of $m_{p,\lambda\tau}(g_\lambda)$. This completes the proof. $\square$

Proof of Theorem 1.2. First, observe that there exists $\rho > 0$, such that for notably many pairs $(t, \check{\mathcal{X}})$, $\Delta((\mathbf{0}, t), \check{\mathcal{X}}) \neq 0$, $\check{\mathcal{X}}$ is ρ-externally stable at $(\mathbf{0}, t)$ and $\check{\mathcal{X}} \subset B_\rho(\mathbf{0})$. This also means that with non-zero probability, $\Delta((\mathbf{0}, T)$, $\check{\mathcal{P}}_1 \cap B_\rho(\mathbf{0})) \neq 0$ and $\check{\mathcal{P}}_1 \cap B_\rho(\mathbf{0})$ is ρ-externally stable at $(\mathbf{0}, T)$, where T is a generic random mark with distribution $\mathbb{P}_{\mathcal{M}}$, independent of $\check{\mathcal{P}}_1$ (see also Lemma 4.2 of [26]). However, from the latter, we can deduce that $\Delta((\mathbf{0}, T), \check{\mathcal{P}}_1) \neq 0$ and $\check{\mathcal{P}}_1$ is ρ-externally stable at $(\mathbf{0}, T)$: see Remarks 1.4 and 1.6.

Now let $v := \mathrm{vol}(\Omega)$ and $\Omega^* := (\tau v)^{-1/d}\Omega$. Define κ to be the uniform density on Ω^* (that is, $\kappa(x^*) := \tau\mathbf{1}(x^* \in \Omega^*)$) and let f to be the indicator function of Ω^*. Then condition (1.13) need not be verified, while, by Remark 1.8, the equivalent conditions (1.12) and (1.14) hold provided that the family $(\Delta_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(s, τ, Ω). To verify the latter, first estimate (assuming that $\check{x} \in \check{\mathcal{X}}$):

$$\big|\Delta_\lambda(\check{x}, \check{\mathcal{X}})\big| \leq \big|\xi_\lambda(\check{x}, \check{\mathcal{X}})\big| + \sum_{\check{y}\in\check{\mathcal{X}}}\big(\big|\xi_\lambda(\check{y}, \check{\mathcal{X}})\big| + \big|\xi_\lambda\big(\check{y}, \check{\mathcal{X}} \setminus \{\check{x}\}\big)\big|\big)\mathbf{1}\big(R(\check{y}, \check{\mathcal{X}}) \geq \lambda^{1/d}\|y - x\|\big).$$

Clearly, there exists $s > 2$ with $s/p + d/q < 1$. Since $p > s$, the family $(\xi_\lambda)_{\lambda>\lambda_0}$ satisfies Assumption MH(s, τ, Ω). To verify the latter for the remaining sum, apply Corollary 3.3.

We have now verified all the conditions required for Theorem 1.1. To complete the proof, observe that ξ satisfies Assumption CV(κ) by Remark 1.8. Formula (1.7) applied to our choice of f and κ yields $V(\tau) = \lim_{\lambda\to\infty}\sigma_\lambda^2[f]/\lambda > 0$ and the proof is complete. $\square$

Acknowledgements

The authors are grateful to J. E. Yukich for many inspiring discussions related to the subject of this paper. They would also wish to thank the referees for pointing out several deficiencies of the previous version. Their thoughtful remarks led to significant improvement of the results.

Peter Eichelsbacher and Martin Raič dedicate their contribution to this paper to the memory of Tomasz Schreiber.

References

[1] Y. Baryshnikov, P. Eichelsbacher, T. Schreiber and J. E. Yukich. Moderate deviations for some point measures in geometric probability. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (2008) 422–446. MR2451052

[2] Y. Baryshnikov and J. E. Yukich. Gaussian fields and random packing. *J. Statist. Phys.* **111** (2003) 443–463. MR1964280

[3] Y. Baryshnikov and J. E. Yukich. Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.* **15** (2005) 213–253. MR2115042

[4] V. Bentkus and R. Rudzkis. On exponential estimates of the distribution of random variables. *Litovsk. Mat. Sb.* **20** (1980) 15–30 (in Russian). MR0575427

[5] T. Bodineau, D. Ioffe and Y. Velenik. Rigorous probabilistic analysis of equilibrium crystal shapes. *J. Math. Phys.* **41** (2000) 1033–1098. MR1757951

[6] S. N. Chiu and M. P. Quine. Central limit theory for the number of seeds in a growth model in $\mathbb{R}^d$ with inhomogeneous Poisson arrivals. *Ann. Appl. Probab.* **7** (1997) 802–814. MR1459271

[7] E. G. Coffman Jr., L. Flatto, P. Jelenković and B. Poonen. Packing random intervals on-line. Average-case analysis of algorithms. *Algorithmica* **22** (1998) 448–476. MR1701622

[8] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*, 2nd edition. *Elementary Theory and Methods* **I**. Springer, New York, 2003. MR1950431

[9] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*, 2nd edition. *General Theory and Structure* **II**. Springer, New York, 2008. MR2371524

[10] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, 2nd edition. Springer, New York, 1998. MR1619036

[11] R. Dobrushin, R. Kotecký and S. Shlosman. *Wulff Construction. A Global Shape from Local Interaction*. American Mathematical Society, Providence, RI, 1992. MR1181197

[12] A. Dvoretzky and H. Robbins. On the "parking" problem. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964) 209–225. MR0173275

[13] P. Eichelsbacher and T. Schreiber. Process level moderate deviations for stabilizing functionals. *ESAIM Probab. Stat.* **14** (2010) 1–15. MR2640365

[14] A. B. Gorchakov. Upper bounds for cumulants of the sum of multi-indexed random variables. *Discrete Math. Appl.* **5** (1995) 317–331.

[15] M. Hardy. Combinatorics of partial derivatives. *Electron. J. Combin.* **13** (2006) Research Paper 1 (electronic). MR2200529

[16] L. Heinrich. Large deviations of the empirical volume fraction for stationary Poisson grain models. *Ann. Appl. Probab.* **15** (2005) 392–420. MR2115047

[17] L. Heinrich and M. Spiess. Berry–Esséen bounds and Cramér-type large deviations for the volume distribution of Poisson cylinder processes. *Lith. Math. J.* **49** (2009) 381–398. MR2591874

[18] K. Krickeberg. Moments of point processes. In *Probability and Information Theory, II* 70–101. Springer, Berlin, 1973. MR0378095

[19] V. A. Malyshev and R. A. Minlos. *Gibbs Random Fields*. Kluwer Academic, Dordrecht, 1991. MR1191166

[20] M. D. Penrose. Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.* **218** (2001) 153–176. MR1824203

[21] M. D. Penrose. *Random Geometric Graphs*. Oxford Univ. Press, Oxford, 2003. MR1986198

[22] M. D. Penrose. Multivariate spatial central limit theorems with applications to percolation and spatial graphs. *Ann. Probab.* **33** (2005) 1945–1991. MR2165584

[23] M. D. Penrose. Convergence of random measures in geometric probability. Preprint, 2005. Available at arXiv:math/0508464.

[24] M. D. Penrose. Laws of large numbers in stochastic geometry with statistical applications. *Bernoulli* **13** (2007) 1124–1150. MR2364229

[25] M. D. Penrose. Gaussian limits for random geometric measures. *Electron. J. Probab.* **12** (2007) 989–1035 (electronic). MR2336596

[26] M. D. Penrose and A. R. Wade. Multivariate normal approximation in geometric probability. *J. Stat. Theory Pract.* **2** (2008) 293–326. MR2524467

[27] M. D. Penrose and J. E. Yukich. Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11** (2001) 1005–1041. MR1878288

[28] M. D. Penrose and J. E. Yukich. Limit theory for random sequential packing and deposition. *Ann. Appl. Probab.* **12** (2002) 272–301. MR1890065

[29] M. D. Penrose and J. E. Yukich. Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.* **13** (2003) 277–303. MR1952000

[30] M. D. Penrose and J. E. Yukich. Normal approximation in geometric probability. In *Stein's Method and Applications* 37–58. Singapore Univ. Press, Singapore, 2005. MR2201885

[31] A. Rényi. Théorie des éléments saillants d'une suite d'observations. *Ann. Fac. Sci. Univ. Clermont-Ferrand* **8** (1962) 7–13. MR0286162

[32] R. Rudzkis L. Saulis and V. Statulevičius. A general lemma on probabilities of large deviations. *Lith. Math. J.* **18** (1978) 226–238. MR0501287

[33] L. Saulis and V. Statulevičius. *Limit Theorems on Large Deviations*. Kluwer Academic, Dordrecht, 1991. MR1171883

[34] T. Schreiber and J. E. Yukich. Large deviations for functionals of spatial point processes with applications to random packing and spatial graphs. *Stochastic Process. Appl.* **115** (2005) 1332–1356. MR2152378