SOLVED PROBLEMS IN COUNTING PROCESSES

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This is an English version of the e-book:

http://valjhun.fmf.uni-lj.si/~raicm/Poucevanje/SPI/SPI_vaje_2015.pdf

with minor supplements and corrections.

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Preface

This collection of problems is based on the tutorial which I was delivering for several years to the undergraduate students of financial mathematics at the University of Ljubljana. Several problems in the book are due to my predecessor Aleš Toman; I am deeply grateful to him.

The book covers counting processes in time. The mainpart of the matter is included in [4]; however, other references listed here can also be used to advantage. Usually, problems are grouped into clusters introduced by frames, which contain the summary of the necessary theory as well as notation. All problems are solved, some of them in several ways. However, there may always be yet another method, so the reader is always encouraged to find an alternative solution.

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1 Selected Topics in Probability Theory

Conditional distributions. Stopping times. Computation of expectation by means of survival function. Wald's equation. Ranks, order statistics.

1. Let N be an $\mathbb{N}_0$-valued random variable.

   a) Derive the formula:
   $$\mathbb{E}(N) = \sum_{n=0}^{\infty} \mathbb{P}(N > n)\,.$$

   b) Compute $\mathbb{E}(N)$ in the special case where $\mathbb{P}(N > n) = \dfrac{2}{(n+2)(n+3)}$ for all $n = 0, 1, 2, \ldots$

2. Let T be arbitrary non-negative random variable.

   a) Derive the formula:
   $$\mathbb{E}(T) = \int_0^{\infty} \mathbb{P}(T > t)\,\mathrm{d}t\,.$$

   b) Compute $\mathbb{E}(T)$ in the special case where $\mathbb{P}(T > t) = \dfrac{1}{(1+t)^3}$ for all $t \ge 0$.

3. Andrew is tossing a symmetric coin, while Bridget is rolling a standard die. Each time Andrew tosses his coin and Bridget rolls her die at one time. The tosses and the rollings are all independent. They toss and roll until either Andrew's coin comes up heads or they finish the third round. Find:
   - the probability that Bridget never rolls a six;
   - the expected number of the total number of the sixes rolled.

4. Let $\lambda > 0$. Take a random variable $T \sim \mathrm{Exp}(\lambda)$. Given T, let a random variable X follow the Poisson distribution $\mathrm{Pois}(T^2)$. Compute $\mathbb{E}(X)$ and $\mathrm{var}(X)$.

Stopping time

A $\{p, p+1, p+2, \ldots\}$-valued random variable T is a **stopping time** *with respect to a sequence of random variables $Z_p, Z_{p+1}, Z_{p+2}, \ldots$ if the event $T = n$ can be deterministically expressed in terms of $Z_p, Z_{p+1}, Z_{p+2}, \ldots, Z_n$ for all $n \in \{p, p+1, p+2, \ldots\}$. The phrase 'can be deterministically expressed' means that there exists a set A_n, such that $\{T = n\} = \{(Z_p, Z_{p+1}, Z_{p+2}, \ldots) \in A_n\}$.*

It is equivalent to say that the event $T \le n$ can be deterministically expressed in terms of $Z_p, Z_{p+1}, Z_{p+2}, \ldots, Z_n$ for all $n \in \{p, p+1, p+2, \ldots\}$.

Furthermore, it is equivalent to say that the event $T > n$ can be deterministically expressed in terms of $Z_p, Z_{p+1}, Z_{p+2}, \ldots, Z_n$ for all $n \in \{p, p+1, p+2, \ldots\}$.

The domain of T or equivalently the number p is a constitutional part of the definition.

5. Again, Andrew is tossing a fair coin and Bridget is rolling a standard die. Each time, they do it at one time, and the tossings and the rollings are all independent. The times when they toss toss and roll are numbered by $1, 2, 3, \ldots$ Determine which random variables are stopping times (with respect to the sequence $Z_1, Z_2, \ldots$, where Z_n denotes the pair of both outcomes at time t – the coin and the die – 12 possibilities):

   a) the time when Bridget rolls her first six;
   b) the time when Bridget rolls her second six;
   c) the time when Bridget rolls her first six, provided that this happens up to time 100; otherwise 100;
   d) the time when Bridget rolls her first six, provided that this happens up to time 100; otherwise 1;
   e) the first time after time 100 when Bridget rolls a six;
   f) the time when Bridget rolls her second six increased by 1;
   g) the time when Bridget rolls her second six decreased by 1;
   h) the time when Bridget rolls her first six after Andrew's coin has come up heads ten times;
   i) the time when the Bridget's total score first exceeds 42;
   j) the first time when Bridget rolls her highest result at times $1, 2, \ldots, 10$.

6. Take:

   - identically distributed random variables $X_1, X_2, \ldots$;
   - random variables $Z_0, Z_1, Z_2, \ldots$, such that X_n is independent of $Z_0, \ldots, Z_{n-1}$ for all $n \in \mathbb{N}$;
   - an $\mathbb{N}_0$-valued random variable T, which is a stopping time with respect to the sequence $Z_0, Z_1, Z_2, \ldots$

   An important special case is when $T, X_1, X_2, \ldots$ are independent: in this case, we can just set $Z_0 = T, Z_1 = X_1, Z_2 = X_2, \ldots$ In general, however, $Z_0, Z_1, \ldots, Z_n$ can be interpreted as 'everything that has occurred up to step n'. Define

$$S = X_1 + X_2 + \cdots + X_T$$

   (for $T = 0$, set $S = 0$). Assume that there exists $\mu = \mathbb{E}(X_i)$ as well as $\mathbb{E}(T)$. Compute $\mathbb{E}(S)$ (the result is called *Wald's equation*).

7. Again, a fair coin is tossed until tails come up. Compute the expected number of tosses along with the expected number of heads that have appeared.

8. A fair coin is tossed repeatedly and all tosses are independent. For each $n \in \mathbb{N}$, denote by S_n the number of tails in the first n tosses. Next, let T be the number of the tosses before the second tails (not including the toss when the coin comes up tails second time). Determine $\mathbb{E}(S_T)$. Does Wald's equation hold in this case? Make a comment!

9. A fair coin is tossed until two *consequent* tails. Compute the expected number of all tosses along with the expected number of all tails.

10. Just like in Exercise 6, take:

    - identically distributed random variables $X_1, X_2, \ldots$;
    - random variables $Z_0, Z_1, Z_2, \ldots$, such that X_n is independent of $Z_0, \ldots, Z_{n-1}$ for all $n \in \mathbb{N}$;
    - an $\mathbb{N}_0$-valued random variable T, which is a stopping time with respect to the sequence $Z_0, Z_1, Z_2, \ldots$

    Now assume that there exist $\mathbb{E}(T)$ and $\sigma^2 = \operatorname{var}(X_i)$, that $\mathbb{E}(X_i) = 0$ and that X_n is a function of random variables $Z_0, \ldots, Z_n$ for all $n \in \mathbb{N}$ (implying that the random variables $X_1, X_2, \ldots$ are independent). Compute $\operatorname{var}(S)$.

    Does the result remain true even without the assumption that $\mathbb{E}(X_i) = 0$?

11. Let $X_1, X_2, \ldots$ be independent and identically distributed, and let T be another $\mathbb{N}_0$-valued random variable independent of the sequence $X_1, X_2, \ldots$. Define:

    $$S = X_1 + X_2 + \cdots + X_T .$$

    (for $T = 0$, set $S = 0$). Compute:

    a) the variance $\operatorname{var}(S)$, provided that $\sigma^2 = \operatorname{var}(X_i)$ and $\operatorname{var}(T)$ exist (hint: letting $\mu = \mathbb{E}(X_i)$, $\mathbb{E}\big[(S - \mu T)^2\big]$, i. e., the variance unexplained by T), can be computed from the preceding exercise – the result is called *Blackwell–Girshick equation*;

    b) the generating function $G_S(z) = \mathbb{E}(z^S)$, provided that $z > 0$, that $\mathbb{E}(z^S) < \infty$ and that $G_T(w) = \mathbb{E}(w^T) < \infty$ for all $w > 0$.

    Do these two results still hold if the assumption that T is independent of the sequence $X_1, X_2, \ldots$ is replaced by a weaker assumption that T is a stopping time with respect to $X_0, X_1, X_2, \ldots$, where X_0 is an additional random variable independent of $X_1, X_2, \ldots$?

12. Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed *continuous* random variables.

    a) Show that the random variables $X_1, X_2, \ldots$ are all distinct with probability 1.

    b) We shall say that a *record value* occurs at time t if $X_t > \max\{X_1, X_2, \ldots, X_{t-1}\}$. As we assume the maximum of the empty set to be $-\infty$, a record value always occurs at time 1. This gives rise to a counting process with $\mathbb{N}$-valued arrival times. Compute the probability that a record value occurs at a given time t.

    c) Show that the events that a record value occurs at a given time are independent.

d) As usual, denote by N_t the number of record values up to and including time t. Compute $\mathbb{E}(N_t)$ and $\operatorname{var}(N_t)$ and find the asymptotic behavior of these two characteristics.

e) Let S be the time of the first record value from time 2 on (i. e., the second arrival time of our counting process). Determine the distribution of S. In particular, show that $\mathbb{P}(S < \infty) = 1$, while $\mathbb{E}(S) = \infty$.

f) Denote by $S^{(y)}$ the time of the first record value with value higher than y, that is:

$$S^{(y)} := \min\{t \,;\, X_t > y\}\,.$$

Show that the random variable $S^{(y)}$ is independent of $X_{S^{(y)}}$. In other words, the first record value greater than y is independent of the time of its occurrence.

2 Counting Processes

Basic concepts, fundamental equivalence. Bernoulli sequence as a counting process. Binomial process. Memorylessness of geometric distribution.

> *A* **counting process** *describes things which are randomly distributed over time, more precisely, over* $[0, \infty)$. *They will be called* **arrivals**. *It is only important when an arrival occurs. Moreover, only finitely many arrivals can occur in a finite time interval. However, infinitely many arrivals typically occur in the whole time-line* $[0, \infty)$. *Such a process can be formally described in various ways:*
>
> - *by a random locally finite subset of* $[0, \infty)$*;*
> - *by random variables* N_t, $t \in [0, \infty)$*, denoting the number of arrivals up to and including time* t*;*
> - *by ordered arrival times:* $S_1 < S_2 < S_3 \cdots$. *The random variable* S_n *is called the* n**-th arrival time**. *Observe the* **fundamental equivalence***:*
>
> $$N_t \geq n \iff S_n \leq t$$

1. A simple case of a counting process is the **Bernoulli sequence of trials**, i. e., the sequence of independent random trials, such that each of them *succeeds* with the same probability. In the corresponding counting process, arrivals can only occur at times from $\mathbb{N}$, and at a given time t, an arrival occurs if the t-th trial succeeds.

   a) Find the distribution of N_t, the number of arrivals up to time t ($t \in \mathbb{N}$) and the number of arrivals between times s and t, i. e., $N_t - N_s$ ($s < t$). Reformulate the conclusions in terms of sums of independent random variables.

   b) Let $t_1 \leq t_2 \leq t_3 \leq \cdots \in \mathbb{N}$. What is the relationship between random variables $N_{t_1}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots$?

   c) Find the distribution of the first arrival time S_1.

   d) Find the distribution of further arrival times S_n.

2. Consider a counting process with $N_t + 1 \sim \mathrm{Geom}(e^{-t})$. Compute the expected first and second arrrival time.

3. n guests are invited to a party, which starts at a given time. However, each guest arrives with some delay. The delays are independent and distributed uniformly over $[0, 1]$. Regarding the arrivals as a continuous-time counting process, determine the distribution of the random variables N_t, $0 \leq t \leq 1$, and S_k, $k = 1, 2, \ldots, n$.

4. Consider a Bernoulli sequence of trials and denote by $\mathcal{P}$ the set of times of successes (with the time running over $\mathbb{N}$). Moreover, for $t \in \mathbb{N}$, denote by $\mathcal{P}^{\to\!\!\!\!\not\,t}$ the set of

arrivals up to and including time t, and by $\mathcal{P}^{t\to}$ the set of arrivals from time t on (excluding t), with the time reset to zero:

$$\mathcal{P}^{\nrightarrow t} := \mathcal{P} \cap (0, t]\,, \qquad \mathcal{P}^{t\to} := \mathcal{P} \cap (t, \infty) - t\,.$$

a) Show that $\mathcal{P}^{t\to}$ is independent of $\mathcal{P}^{\nrightarrow t}$ and follows the same distribution as $\mathcal{P}$. This is called the *time-homogeneous Markov property.*

b) Take a stopping time T with respect to the sequence $0, Z_1, Z_2, \ldots$, where, for $t \in \mathbb{N}$, we set $Z_t = 1$ in the case of success and $Z_t = 0$ in the case of failure at time t. Prove that the process $\mathcal{P}^{T\to}$ is independent of $\big(T, \mathcal{P}^{\nrightarrow T}\big)$ and follows the same distribution as $\mathcal{P}$. This is called the *strong time-homogeneous Markov property.*

5. Show that the geometric distribution is *memoryless*, i. e., if $T \sim \mathrm{Geom}(p)$, then:

$$\mathbb{P}(T = t + s \mid T > t) = \mathbb{P}(T = s) \qquad (*)$$

for all $t \in \mathbb{N}_0$ and all $s \in \mathbb{N}$. In addition, show that the geometric distribution is the only memoryless distribution taking values in $\mathbb{N}$. We assume that the conditional distribution in $(*)$ makes sense, i. e., that $\mathbb{P}(T > t) > 0$ for all $t \in \mathbb{N}_0$.

Inter-arrival times

Another way of defining a counting process is in terms of inter-arrival times, i. e., the times between two consequent arrivals:

$$T_1 = S_1\,,\; T_2 = S_2 - S_1\,,\; T_3 = S_3 - S_2\,, \ldots$$

6. Consider a Bernoulli sequence of trials.

a) Show that S_n are stopping times. Combining with Exercise 4, what follows?

b) Find the distribution of all inter-arrival times T_n and show that they are mutually independent. Reformulate the conclusion as a statement on sums of independent random variables.

c) Find the distribution of the differences $S_n - S_m$, where $m < n$.

3 Homogeneous Poisson Process

Motivation and definition of homogenous Poisson process. Corresponding distributions.

Homogeneous Poisson process

A **homogeneous Poisson process with intensity** $\lambda > 0$ *can be defined as a certain limit of processes arising from Bernoulli trials. It is characterized by the following two properties:*

- $N_0 = 0$ *and* $N_t - N_s \sim \mathrm{Pois}\big(\lambda(t-s)\big)$ *for any* $0 \le s \le t$ *(implying* $N_t \sim \mathrm{Pois}(\lambda t)$*).*
- *For any sequence* $0 \le t_1 \le t_2 \le t_3 \le \cdots$*, the random variables* $N_{t_1}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots$ *are independent.*

Some more properties:

- *The inter-arrival times* $T_1, T_2, T_3, \ldots$ *are independent and folow the exponential distribution* $\mathrm{Exp}(\lambda)$*.*
- $S_n \sim \mathrm{Gama}(n, \lambda)$ *for all* $n \in \mathbb{N}$*. More generally,* $S_n - S_m \sim \mathrm{Gama}(n-m, \lambda)$ *for any* $m \le n$*.*
- *For any sequence* $n_1 < n_2 < n_3 < \ldots$*, the random variables* $S_{n_1}, S_{n_2} - S_{n_1}, S_{n_3} - S_{n_2}, \ldots$ *are independent.*
- *Like the Bernoulli sequence, this process also enjoys the* **strong time-homogeneous Markov property***: representing the process by a random set* $\mathcal{P}$*, for any stopping time* T*, the process* $\mathcal{P}^{T\rightarrow}$ *is independent of* $\big(T, \mathcal{P}^{\rightarrow\!\!|T}\big)$ *and follows the same distribution as* $\mathcal{P}$*.*

1. Patients arrive in a surgery according to a homogeneous Poisson process with intensity 6 patients an hour. The doctor starts to examine the patients only when the third patient arrives.

   a) Compute the expected time from the opening of the surgery until the first patient starts to be examined.

   b) Compute the probability that in the first opening hour, the doctor does not start examining at all.

2. Consider a homogeneous Poisson process with intensity λ.

   a) Compute the auto-covariance function of the family N_t, i. e., all covariances $\mathrm{cov}(N_t, N_s)$.

   b) Compute the auto-correlation function of the family N_t, i. e., all correlations $\mathrm{cov}(N_t, N_s)$.

   c) For $t_1 \le t_2 \le \cdots \le t_n$, find the covariance matrix of the random vector $(N_{t_1}, N_{t_2}, \ldots, N_{t_n})$.

3. *Casinò Poisson.* In a casino, a bell rings every now and then. Each time the bell rings, the player can press a button. The player wins the game if they press the

button before time 1 and, in addition, after they have pressed the button, the bell no longer rings until time 1. Assume that the bell rings according to a Poisson process with intensity $\lambda > 0$.

The strategy of the player is first to wait until time s and then to press the button immediately after the bell rings.

a) Find the probability that the player wins the game (depending on s)?

b) Find the optimal value of s and the corresponding winning probability.

4. Let $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ be independent random variables. Find the conditional distribution of the random variable X given $Z := X + Y$.

5. Let N be a random variable denoting the number of arrivals, ditributed by Poisson $\mathrm{Pois}(\lambda)$. Each arrival is *successful* with probability p, independently of other arrivals, as well as of the number of arrivals. Denote by S the number of successful and by T the number of unsuccessful arrivals, that is, $T = N - S$.

   a) Find the distribution of S and T.

   b) Show that the random variables S and T are independent.

   c) Show that under some other choice of the distribution of N, S and T are no longer necessarily independent.

   Remark. The transformation converting N to S is called *thinning*.

6. Show that the exponential distribution is also memoryless: for $T \sim \mathrm{Exp}(\lambda)$ we have for sufficiently large $t, s \in \mathbb{N}$:

$$\mathbb{P}(T \le t + s \mid T > t) = \mathbb{P}(T \le s)\,. \qquad (*)$$

   Moreover, show that the exponential distribution is the only memoryless distribution which is continuous and with density being continuous on $(0, \infty)$ and zero elsewhere. We assume that the conditional distribution in $(*)$ makes sense, i. e., that $\mathbb{P}(T > t) > 0$ for all $t \ge 0$.

7. For independent random variables $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$, find the distribution of $U := \min\{X, Y\}$ and $V := \max\{X, Y\}$.

8. A fire station receives emergency calls according to a homogeneous Poisson process with intensity half a call per hour. Each time, the fire brigade needs certain time to process the call and prepare for further calls (this total period will be called *intervention*). During an intervention, the calls are redirected to other fire stations. Suppose that the intervention times are distributed uniformly over the interval from half an hour to one hour and independent of each other as well as of the emergency calls.

   Suppose that the fire brigade is able to respond to the calls at the moment (i. e., there is no intervention). Find the distribution of the number of the calls to which the fire brigade responds before any call is redirected.

9. Consider a homogeneous Poisson process with intensity λ.

   a) Suppose that up to time t, exactly one arrival occured. Given this information, find the conditional distribution of the arrival time.

   b) Now suppose that exactly two arrivals occured up to time t. Compute the conditional expectations of both arrival times.

Order statistic property

In a homogeneous Poisson process, given the event that there are exactly n arrivals in a certain time interval, the conditional distribution of the restriction of the process to this interval (considered as a random set) matches the distribution of the set $\{U_1, U_2, \ldots, U_n\}$, where $U_1, \ldots, U_n$ are independent and uniformly distributed over the interval.
Considering the time interval $[0, t]$, given the event that there are exactly n arrivals, the random vector of the arrival times $(S_1, S_2, \ldots, S_n)$ follows the same distribution as the random vector $(U_{(1)}, U_{(2)}, \ldots, U_{(n)})$ of suitable order statistics.

10. Passengers arrive at a railway station according to a homogeneous Poisson process with intensity λ. At the beginning (time 0), there are no passengers at the station. The train departs at time t. Denote by W be the total waiting time of all passengers arrived up to the departure of the train. Compute $\mathbb{E}(W)$.

11. *Poisson shocks.* Each arrival in a homogeneous Poisson process with intensity λ causes a *shock*. Its effect s time units later equals $e^{-\theta s}$. Denote by $X(t)$ the total effect of all the shocks from the interval $[0, t]$ at time t. Compute the expectation $\mathbb{E}\big[X(t)\big]$.

12. A hen wants to cross a one-way road, where cars drive according to a homogeneous Poisson process with intensity λ cars a time unit, all with the same speed. It takes c time units for the hen to cross the road. Assume that the hen starts to cross the road immediately when there is a chance to do it without being run over by a car. Compute the expected total waiting and crossing time for the hen.

13. In a certain place at a fair, prizes are shared every now and then. Everyone being in that place when the prizes are being shared gets a prize. The sharing times form a homogeneous Poisson process with intensity λ. At time zero, Tony observes that a sharing is in progress. He rushes towards the sharing place, but is too late. Then he waits for the next sharing, but at most for time δ: after that time, he gets bored and moves eslewhere. As soon as another sharing starts, he rushes towards the place, but is again too late and starts waiting for the next sharing, again at most for time δ. So he repeats until ge gets the prize. Denote by T the time when Tony eventually gets his prize. Compute $\mathbb{E}(T)$, assuming that the fair is open infinitely long.

14. For a counting process on $(0, \infty)$, denote:

- by A_t the *age* of the process at time t, i. e., the time that has passed from the last arrival until time t if an arrival has indeed occured until t; otherwise, define the age to be t (alternatively, setting $S_0 := 0$, we have $A_t = t - S_{N_t}$);
- by E_t the *exceedance* at time t, i. e., the time that passes from t to the first arrival after t (i. e., $E_t = S_{N_t+1} - t$).

For a homogeneous Poisson process with intensity λ:

a) Find the distributions of the random variables A_t in E_t.

b) Show that the random variables A_t and E_t are independent.

c) Find the distribution of the sum $A_t + E_t$ (i. e., the gap between two consequent arrivals surrounding t).

4 Marking, Thinning, Superposition

Homogeneous Poisson process with discrete marks

Consider a homogeneous Poisson process with intensity λ, where each arrival is given a random **mark**, *where the marks are independent of each other as well as of the original process, following the distribution:*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ p_1 & p_2 & \cdots & p_r \end{pmatrix},$$

This marked process follows the same distribution as a union of r independent homogeneous Poisson processes with intensities $p_1\lambda, p_2\lambda, \ldots, p_r\lambda$, where the arrivals in the i-th process are assinged mark a_i.

1. Suppose that the night traffic on the Jadranska street can be modelled by a homogeneous Poisson process with intensity 40 vehicles per hour, 10% thereof being lorries and 90% being cars. Suppose that the types of particular vehicles are independent.
   a) Find the probability that in one our, at least one lorry passes the Faculty of Mathematics and Physics.
   b) Suppose that during the first hour of observation, exactly 10 lorries have passed by. Find the expected number of the cars passing the FMF during the same period.
   c) Suppose that in the first hour of observation, exactly 50 vehicles have passed by. Find the probability that among these vehicles, there were exactly 5 lorries and 45 cars.
   d) Compute the expected number of cars until the first lorry has passed by.
2. The life time of a light bulb follows the exponential distribution with expectation 200 days. When the bulb blows out, it is replaced immediately by a maintainer. Meanwhile, another maintainer replaces the bulbs regardless of their condition, according to a homogeneous Poisson process with intensity 0˙01 replacement per day. Of course, we assume that the bulbs, actually their life times, are independent of each other.
   a) How frequently is a bulb replaced?
   b) For a longer period, compute the percentage of the bulbs replaced because of blowing out and the bulbs replaced because of 'precaution.'
3. A director is searching for three actors, one man and two women. Men apply according to a homogeneous Poisson process with intensity 2 per day, while women apply according to a homogeneous Poisson process with intensity 1 per day, independently of the men. Compute the expected time needed for the director to get the man as well as the two women. We assume that all candidates are acceptable.

4. A married couple is searching for a used car. Each one of them is looking up offers for their favourite brand. The offers for wife's brand are coming according to a homogeneous Poisson process with intensity λ, while the offers for husband's brand are coming according to a homogeneous Poisson process with intensity μ. The wife is ready to buy a car when she encounters the third offer for her brand, while the husband is ready when he encounters the second offer for his brand. The couple buys a car when either of them is ready to buy. We assume that the offer processes for both brands are independent.

   a) Compute the probability that the couple buys a car according to wife's choice.

   b) Compute the expected time needed to buy a car.

5. Consider two parallel Poisson processes with intensities λ and μ. Find:

   - the probability that in the second process, exactly one arrival in occurs before the first arrival in the first process;
   - the expected number of the arrivals in the second process before the first arrival in the first process.

6. From a given time on, students of financial and general mathematics have an opportunity to lookup their corrected test papers. The students of finantial mathematics arrive according to a homogeneous Poisson process with intensity 4 students an hour, while the students of general mathematics arrive according to a homogeneous Poisson process with intensity 2 students an hour. Assume that both groups of students arrive independently of each other.

   Suppose that during the first half an hour, exactly one student came to lookup his/her test paper. Compute the conditional arrival time of the first student of *financial* mathematics. The time is measured from the beginning, assuming that the students are arriving infinitely long in the future.

7. Take two independent Poisson process with intensities λ and μ. Let $n \in \mathbb{N}$. Determine the distribution of the number of arrivals in the first process before the n-th arrival in the second process.

8. Consider two homogeneous Poisson processes with intensities λ and μ. Denote by $N_t^{(1)}$ the number of arrivals up to time t in the first, and with $N_t^{(2)}$ the number of arrivals up to time t in the second process. Compute the probability that the two-dimensional walk $\left(N_t^{(1)}, N_t^{(2)}\right)$ reaches the point (i, j).

Homogeneous Poisson process with general marks

Suppose that each arrival in a homogeneous Poisson process with intensity λ is given a mark from a set M, chosen according to a distribution μ, where the marks are independent of each other as well as of the original process. Then the number of marked arrivals belonging to a set $A \subseteq [0, \infty) \times M$ follows the Poisson dostribution with parameter $\theta = (\lambda m \otimes \mu)(A)$, where m denotes the Lebesgue measure.

If $\mu = \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$, then:

$$\theta = \lambda \sum_i p_i \, m(\{t \, ; \, (t, a_i) \in A\}) \, .$$

If μ is a continuous distribution with density f, then:

$$\theta = \lambda \iint_A f(s) \, \mathrm{d}t \, \mathrm{d}s \, .$$

9. The capital of a bank grows proportionally with time: at time t, the bank has at units of capital. The bank has to undergo stress tests, which occur according to a homogeneous Poisson process with intensity λ. The bank passes a stress test provided that it has at least a certain amount of capital at the moment of the test; the desired amounts of capitals are random, independent and following a continuous distribution with density:

$$f(s) = \frac{4}{\pi(1 + s^2)^2} \, .$$

Compute the probability that the bank will pass all the stress test.

5 General Poisson Process

Intensity function. Conditioning on the number of arrivals. Behavior with respect to stopping times.

Let $\rho\colon (0,\infty) \to [0,\infty)$ be a function. A **Poisson process with intensity function** *ρ is a counting process characterized by the following two properties:*

- *For $a \le b$, $N_b - N_a \sim \mathrm{Pois}\left(\int_a^b \rho(t)\,\mathrm{d}t\right)$. Consequently, $N_t \sim \mathrm{Pois}\big(R(t)\big)$, where $R(t) = \int_0^t \rho(s)\,\mathrm{d}s$.*
- *Any restrictions of the process (regarded as a random subset od $(0,\infty)$) to disjoint intervals are independent.*

1. A shop is open from 10am to 6pm. Customers arrive in the shop according to a Poisson process with an intensity function, which equals zero at opening, 4 customers per hour at noon, 6 customers per hour at 2pm, 2 customers per hour at 4pm and zero at closing; between any two consequent afore-mentioned times, it is linear.

   a) Find the distribution of the number of customers on a given day.

   b) Find the probability that no customer arrives until noon.

   c) Assume that during the first two opening hours, exactly two customers have arrived. Find the their expected arrival times.

Conditioning on the number of arrivals

In a Poisson process, given the event that there are exactly n arrivals in a certain time interval, the conditional distribution of the restriction of the process to this interval (considered as a random set) matches the distribution of the set $\{U_1, U_2, \ldots, U_n\}$, where $U_1, \ldots, U_n$ are independent, following a continuous distribution with density:

$$f(t) = \begin{cases} \dfrac{\rho(t)}{\int_a^b \rho(s)\,\mathrm{d}s} & ;\ a < t < b \\ 0 & ;\ \textit{otherwise.} \end{cases}$$

Considering the time interval $[0,t]$, given the event that there are exactly n arrivals, the random vector of the arrival times $(S_1, S_2, \ldots, S_n)$ follows the same distribution as the random vector $(U_{(1)}, U_{(2)}, \ldots, U_{(n)})$ of suitable order statistics.

2. Consider a Poisson process with intensity function $\rho(t) = a/(1+t)$. Determine the distribution of the first arrival time along with its expectation, provided that it exists.

3. Latecomers arrive according to a Poisson process with intensity function $\rho(t) = e^{-t}$, where t denotes the delay in months.

   a) Find the probability that exactly one latecomer appears and that this latecomer arrives more than two months.

   b) Suppose that in fact, exactly one latecomer appears and that this latecomer arrives more than two months. Find the conditional expected time of their delay.

4. Let $a, \lambda, \delta > 0$. Compute the expected number of arrivals in a Poisson process with intensity function $t \mapsto a\, e^{-\lambda t}$, which are not followed by another arrival within time interval of length δ.

General Poisson process and stopping time

Consider a Poisson process with intensity function ρ and let T be a stopping time.

Given $\big(T, \mathcal{P}^{\to\!| T}\big)$, the process $\mathcal{P}^{T\to}$ is a Poisson process with intensity function $t \mapsto \rho(t + T)$.

Equivalently, given $\big(T, \mathcal{P}^{\to\!| T}\big)$, the process $\mathcal{P} \cap (T, \infty)$ is a Poisson process with intensity function $t \mapsto \rho(t)\, \mathbb{1}(t > T)$.

5. Consider a Poisson process with intensity function:

$$\rho(t) = \frac{1}{1+t}\,.$$

   Find the distribution of the first two (inter)-arrival times T_1 and T_2.

6 Renewal Processes

Basic definitions, asymptotic behavior. Renewal–reward processes. Renewal equation. Renewal processes with delay.

A **renewal process** *is a generalized counting process with independent and identically distributed inter-arrival times $T_1, T_2, T_3, \ldots$. Their distribution is called the* **inter-arrival distribution**. *We also allow for the case $T_i = 0$: in this case, there are multiple arrivals at a suitable time.*
Renewal processes with finite expectation of their inter-arrival times satisfy the **strong law of large numbers***:*

$$\frac{N_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{\mathbb{E}(T_1)} .$$

1. A shuttle bus operates between the Virgin Beach and a hotel. Due to unpredictable traffic situation, the return times are distributed uniformly over the interval from 20 minutes to one hour. Assume that they are independent of each other. The time the bus stays at the Virgin Beach (including disembarkation and embarkation) is negligible.

   a) Compute the long-term number of arrivals of the bus per hour.

   b) The Simpsons wanted to catch the bus, but it has just ran away. Therefore, they continue swimming and return to the bus stop 40 minutes later. Find the probability that they will wait for less than 20 minutes.

2. Consider again the fire station from Exercise 8 in Section 3: the station receives emergency calls according to a homogeneous Poisson process with intensity half a call per hour. Each time, the fire brigade needs certain time to process the call and prepare for further calls. Meanwhile, the calls are redirected to other fire stations. The durations of these periods (interventions) are distributed uniformly over the interval from half an hour to one hour and independent of each other as well as of the emergency calls. Compute the long-term proportion of redirected calls.

A **renewal–reward process** *is a renewal process, where each arrival is attached a reward: the reward attached to the i-th arrival will usually be denoted by R_i and may also be negative. The reward attached to the i-th arrival is received during the i-th inter-arrival interval, i. e., between times S_{i-1} and S_i. Not all amount needs to be received once at a time: in can be received gradually and not necessarily monotonously. More formally, denoting by W_t the total amount of the rewards received up to time t, we have:*

$$W_{S_i} - W_{S_{i-1}} = R_t .$$

Assume that the dynamics of the receivement of the rewards along with the corresponding inter-arrival times are independent and identically distributed. Denote by R_i^+ the maximal absolute value of a partially received reward attached to the i-th arrival, that is, $R_i = \sup_{S_{i-1} \le t \le S_i} |W_t - W_{S_{i-1}}|$; observe that $|R_i| \le R_i^+$. Then this random process obeys the following strong law of large numbers:

$$\frac{W_t}{t} \xrightarrow[t\to\infty]{\text{s.g.}} \frac{\mathbb{E}(R_1)}{\mathbb{E}(T_1)} ,$$

provided that $\mathbb{E}(T_1) < \infty$ in $\mathbb{E}(|R_1|) < \infty$.

3. Ben has got a gas oven, which is inspected every now and then, according to a renewal process with inter-arrival time distributed uniformly over the interval from one year to two years and a half. If the inspectors find out that maintainance of the oven has not been performed for more than one year, Ben has to pay a fine of 105 euros. Ben's strategy is to perform maintainance each time exactly one year after the visit of the inspectors. Compute the long-term amount of fine Ben has to pay per year.

4. An *alternating renewal process* spends each moment of its time in one of two possible states. Every now and then, it jumps from one state to another. The durations of all stays (in the first or the second state) are independent of each other. The durations of all stays in State 1 are identically distributed with expectation μ_1, and the durations of all stays in State 2 are identically distributed with expectation μ_2. Find the long-term proportion of the time the process spends in State 1.

5. Monica is selling a certain article by telephone. Up to time $t \le 1/2$, she manages to persuade a customer to buy the article with probability $3(t - t^2)$. For $t > 1/2$, that probability remains at $3/4$ (i. e., after time $1/2$, everything is in vain). Once she manages to persuade the customer, she hangs up and start dialing the next customer immediately. However, she does the same if she does not manage to persuade the customer up to time τ.

   Find the value of τ which maximizes the long-term number of sold articles.

6. Every now and then at a certain point at a fair, all the visitors being there are given a prize. The prize sharings form a renewal process with inter-arrival distribution

being uniform over the interval from 20 to 40 minutes. At time zero, Tony notices that prizes are being shared, rushes to the spot, but he is too late. Then he waits until the next sharing, but at most for 30 minutes: after that time, he gets tired of waiting and moves to another place. When the next sharing takes place, he rushes to the spot again, but he is too late. Then he waits again for a prize, but at most for 30 minutes. The process is repeated until Tony eventually obtains a prize. Denote by T the time when Tony obtains a prize. Compute $\mathbb{E}(T)$, assuming that the fair is open infinitely long.

The **Lebesgue–Stieltjes integral** *of a measurable function* $h\colon \mathbb{R} \to \mathbb{R}$ *over function* $F\colon \mathbb{R} \to \mathbb{R}$*, which is assumed to be either (not necessarily strictly) increasing or with finite total variation, is the integral of* h *over the Lebesgue–Stieltjes measure, which corresponds to the function* h*:*

$$\int_A h \, \mathrm{d}F := \int_A h(x) \, \mathrm{d}F(x) := \int_A h \, \mathrm{d}\mu \,,$$

The underlying (positive or signed) Riemann–Stieltjes measure μ *is determined by:*

$$\mu\big((a,b)\big) = \lim_{x \uparrow b} F(x) - \lim_{x \downarrow a} F(x)$$

for all $a \le b$*. In particular,*

$$\mu(\{a\}) = \lim_{x \downarrow a} F(x) - \lim_{x \uparrow a} F(x) =: \delta F(a)$$

and $\mu\big([a,b]\big) = \lim_{x \downarrow b} F(x) - \lim_{x \uparrow a} F(x)$.
If h *is continuous and* F *continuously differentiable on the interval* (a,b) *(which can be infinite), the Lebesgue–Stieltjes integral reduces to the generalized Riemann integral:*

$$\int_{(a,b)} h \, \mathrm{d}F = \int_{(a,b)} h(x) \, \mathrm{d}F(x) = \int_a^b h(x) \, F'(x) \, \mathrm{d}x \,.$$

Moreover,

$$\int_{\{a\}} h \, \mathrm{d}F = h(a) \, \delta F(a) \,.$$

Furthermore, for each random variable with cumulative distribution function F*,*

$$\mathbb{E}\big[h(X)\big] = \int_{\mathbb{R}} h \, \mathrm{d}F \,.$$

The **renewal function** $M(t) := \mathbb{E}(N_t)$ *of a renewal process satisfies the* **renewal equation**:

$$M(t) = F(t) + \int_{[0,t]} M(t-s)\,\mathrm{d}F(s)\,,$$

where F is the cumulative distribution function of the inter-arrival distribution. If the latter is continuous with density f, we may write:

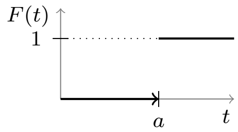
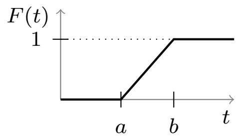
$$M(t) = F(t) + \int_0^t M(t-s)\,f(s)\,\mathrm{d}s\,.$$

The **Laplace–Stieltjes transform** *of a function* $F\colon\ [0,\infty) \to \mathbb{R}$ *is the function given by:*

$$\hat{F}(z) = \int_{[0,\infty)} e^{-zt}\,\mathrm{d}F(t)\,.$$

where we set $F(t) = 0$ for $t < 0$. Basic Laplace–Stieltjes transforms:

$F(t)$	$\hat{F}(z)$
1	1
t^r	$\dfrac{r!}{z^r}$
$t^r\,e^{\alpha t}$	$\dfrac{r!\,z}{(z-\alpha)^{r+1}}$
$\displaystyle\int_0^t s^r\,e^{\alpha s}\,\mathrm{d}s$	$\dfrac{r!}{(z-\alpha)^{r+1}}$
$\displaystyle\int_{[0,t]} e^{\alpha s}\,\mathrm{d}G(s)$	$\hat{G}(z-\alpha)$

<table>
<tr><th>$F(t)$</th><th>$\hat{F}(z)$</th></tr>
<tr><td></td><td>e^{-az}</td></tr>
<tr><td></td><td>$\dfrac{e^{-az} - e^{-bz}}{(b-a)z}$</td></tr>
<tr><td>$G(t-a)\,\mathbb{1}(t \ge a)\,;\ \ a \ge 0$</td><td>$e^{-az}\,\hat{G}(z)$</td></tr>
</table>

The Laplace transform of (the distribution of) a $[0,\infty)$-valued random variable X is defined to be the Laplace–Stieltjes transform of its cumulative distribution function (according to the convention $F_X(x) = \mathbb{P}(X \le x)$). Or, alternatively, this is the function mapping z into $\mathbb{E}\big[e^{-zX}\big]$.

The **Stieltjes-type convolution** *of functions F and $G\colon [0,\infty) \to \mathbb{R}$ is defined by:*

$$(F \star G)(t) := \int_{[0,t]} F(t-u)\,\mathrm{d}G(u) = \int_{[0,t]} G(t-u)\,\mathrm{d}F(u) = (G \star F)(t)\,,$$

where again we set $F(t) = G(t) = 0$ for $t < 0$. This allows us to rewrite the renewal equation in the following neat form:

$$M = F + M \star F\,.$$

The Stieltjes-type convolution is commutative, associative and bilinear.

If F and G are the cumulative distribution functions of independent $[0,\infty)$-valued random variables, $F \star G$ is the cumulative distribution function of their sum.

The Laplace–Stieltjes transform of the Stieltjes-type convolution is the product of the Laplace–Stieltjes transforms of the factors:

$$\widehat{F \star G} = \hat{F}\,\hat{G}\,.$$

Consequently, the Laplace transform of a sum of independent random variables is the product of the transforms.

As a result, the Laplace–Stieltjes transform $\hat{M}$ of a renewal function M satisfies the equation $\hat{M} = \hat{F} + \hat{M}\hat{F}$ and therefore equals:

$$\hat{M}(z) = \frac{\hat{F}(z)}{1 - \hat{F}(z)}\,.$$

7. Determine the renewal function of the renewal process consisting of all *even* arrivals of a homogeneous Poisson process with intensity λ.

8. Determine the renewal function of the renewal process with inter-arrival time being zero with probability p and (conditionally) following the exponentional distribution $\mathrm{Exp}(\lambda)$ with probability $1 - p$.

Delayed renewal processes

These processes are an extension of the renewal processes obtained by omitting the assumption that T_1 follows the same distribution as the other inter-arrival times. The strong law of large numbers:

$$\frac{N_t}{t} \xrightarrow[t\to\infty]{\text{s.g.}} \frac{1}{\mathbb{E}(T_2)}\,.$$

remains true provided that T_1 is almost surely finite (and, of course, that the other inter-arrival times have finite expectation).

The renewal function of a delayed renewal process satisfies the following renewal equation:

$$M(t) = G(t) + \int_{[0,t]} M(t-s)\,\mathrm{d}F(s)\,,$$

where G denotes the cumulative distribution function of the first arrival time, while F denotes the cumulative distribution function of subsequent inter-arrival times. Consequently, the Laplace–Stieltjes transform of the renewal function equals:

$$\hat{M}(z) = \frac{\hat{G}(z)}{1-\hat{F}(z)}\,.$$

9. A novice policeman is looking for offenders, which arrive according to a homogeneous Poisson process with intensity λ. He overlooks the first one and catches all subsequent ones. Denote by N_t the number of offenders caught by the policeman up to time t. Compute $\mathbb{E}(N_t)$.

10. In a small town, there is a bank with only one counter. Customers arrive *to* the bank according to a homogeneous Poisson process with intensity μ. However, a potential customer arriving to the bank *enters* the bank only if there is no other customer at the counter; otherwise, they leave the bank and never return. Assume that the service time at the counter follows the exponential distribution with parameter λ, and that the service times are independent of each other as well as of the arrivals of the customers.

    a) When the bank is opened, there is no customer yet. Compute the renewal function of the corresponding delayed renewal process of the customers that enter the bank.
    b) Find the long-term intensity of the customers entering the bank.
    c) Find the long-term proportion of the customers which enter the bank among all customers that arrive to the bank.

11. Policeman Roy starts to serve in place A. When he is there, a supervisor meets him every now and then according to a homogeneous Poisson process with intensity one arrival per month. When the supervisor meets Roy, he redeploys him to place

B with probability $1/2$. When Roy is there, a supervisor meets him according to a homogeneous Poisson process with intensity one arrival per two months. Again, when the supervisor meets Roy, he redeploys him to place A with probability $1/2$. Assume that the decisions on redeployment are independent of each other as well as of the arrival times.

Show that the arrivals of supervisors that Roy meets form a delayed renewal process, and compute the renewal measure.

12. Determine the renewal measure of the process where the first arrival time follows the uniform distribution over the interval from 0 to a, while the subsequent inter-arrival times follow the exponential distribution with parameter λ.

13. A counting process represented by a set of arrivals $\mathcal{P}$ is *stationary* if the process $\mathcal{P}^{t\to}$ follows the same distribution as $\mathcal{P}$ for all $t \ge 0$. Suppose that a delayed renewal process is stationary. Given the distribution of its inter-arrival times $T_2, T_3, \ldots$, find the distribution of T_1.

SOLUTIONS

-(-)

1 Selected Topics in Probability Theory

1. a) For N taking values in $\mathbb{N}_0$, we have:

$$N = \sum_{n=0}^{\infty} \mathbb{1}(N > n)\,,$$

implying the desired result.

b) By the previous formula, we have:

$$\mathbb{E}(N) = \sum_{n=0}^{\infty} \frac{2}{(n+2)(n+3)} = 2\sum_{n=0}^{\infty}\left[\frac{1}{n+2} - \frac{1}{n+3}\right] = 1\,.$$

Remark. The expectation can also be computed by means of point probabilities, but this is much more complicated: first, for $n \in \mathbb{N}$, we have to compute $\mathbb{P}(N = n) = \mathbb{P}(N > n-1) - \mathbb{P}(N > n) = \frac{4}{(n+1)(n+2)(n+3)}$, then we have to compute $4\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$.

2. a) Observe that:

$$T = \int_0^T \mathrm{d}t = \int_0^\infty \mathbb{1}(t < T)\,\mathrm{d}t = \int_0^\infty \mathbb{1}(T > t)\,\mathrm{d}t\,.$$

The desired result now follows from Fubini's theorem.

b) By the previous formula.

$$\mathbb{E}(T) = \int_0^\infty \frac{\mathrm{d}t}{(1+t)^3} = -\left.\frac{1}{2(1+t)^2}\right|_0^\infty = \frac{1}{2}\,.$$

Remark. Clearly, the expectation can be computed by means of density:

$$f_T(t) = \frac{3}{(1+t)^4}\,,$$

leading to:

$$\begin{aligned}\mathbb{E}(T) &= 3\int_0^\infty \frac{t}{(1+t)^4}\,\mathrm{d}t = 3\int_0^\infty \left(\frac{1}{(1+t)^3} - \frac{1}{(1+t)^4}\right)\mathrm{d}t = \\ &= -\left.\left(\frac{3}{2(1+t)^2} - \frac{1}{(1+t)^3}\right)\right|_0^\infty = \frac{1}{2}\,.\end{aligned}$$

However, the computation is again more complicated.

3. Denote by S the number of sixes and by N the number of tosses. Then the conditional distribution of S given N is binomial $\mathrm{Bin}(N, 1/6)$ and therefore:

$$\mathbb{P}(S = 0 \mid N) = \left(\frac{5}{6}\right)^N \quad \text{and} \quad \mathbb{E}(S \mid N) = \frac{N}{6}\,.$$

Since:

$$N \sim \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1/4 & 1/4 \end{pmatrix},$$

we finally find that:

$$\begin{aligned} \mathbb{P}(S=0) &= \frac{1}{2}\cdot\frac{5}{6}+\frac{1}{4}\cdot\frac{25}{36}+\frac{1}{4}\cdot\frac{125}{216}=\frac{635}{864}\doteq 0{\cdot}735\,, \\ \mathbb{E}(S) &= \frac{1}{2}\cdot\frac{1}{6}+\frac{1}{4}\cdot\frac{2}{6}+\frac{1}{4}\cdot\frac{3}{6}=\frac{7}{24}\doteq 0{\cdot}292\,. \end{aligned}$$

4. Recall that $N \sim \mathrm{Pois}(\lambda)$ satisfies $\mathbb{E}(N)=\mathrm{var}(N)=\lambda$. Thus, $\mathbb{E}(X \mid T)$ $= \mathrm{var}(X \mid T) = T^2$. As a result,

$$\mathbb{E}(X)=\mathbb{E}\big[\mathbb{E}(X \mid T)\big]=\mathbb{E}(T^2)\,.$$

For $m \in \mathbb{N}_0$, compute

$$\mathbb{E}(T^m)=\lambda\int_0^\infty t^m e^{-\lambda t}\,\mathrm{d}t=\frac{1}{\lambda^m}\int_0^\infty x^m e^{-x}\,\mathrm{d}x=\frac{m!}{\lambda^m}\,,$$

leading to

$$\mathbb{E}(X)=\frac{2}{\lambda^2}\,.$$

The variance can be computed by at least two methods. One can go directly:

$$\mathrm{var}(X)=\mathbb{E}(X^2)-\big(\mathbb{E}(X)\big)^2=\mathbb{E}\big[\mathbb{E}(X^2 \mid T)\big]-\frac{4}{\lambda^4}$$

Observing that

$$\mathbb{E}(X^2 \mid T)=\mathrm{var}(X \mid T)+\big(\mathbb{E}(X \mid T)\big)^2=T^2+T^4\,,$$

we eventually find that

$$\mathrm{var}(X)=\mathbb{E}(T^2)+\mathbb{E}(T^4)-\frac{4}{\lambda^4}=\frac{2}{\lambda^2}+\frac{20}{\lambda^4}\,.$$

Alternatively, one can apply the decomposition of variance:

$$\begin{aligned} \mathrm{var}(X) &= \mathrm{var}\big[\mathbb{E}(X \mid T)\big]+\mathbb{E}\big[\mathrm{var}(X \mid T)\big]= \\ &= \mathrm{var}(T^2)+\mathbb{E}(T^2)= \\ &= \mathbb{E}(T^4)-\big(\mathbb{E}(T^2)\big)^2+\mathbb{E}(T^2)= \\ &= \frac{20}{\lambda^4}+\frac{2}{\lambda^2}\,. \end{aligned}$$

As expected, the result is the same as before.

5. All random variables except the ones in d), g) and i) are stopping times.

6. Write

$$S = \sum_{n=1}^{\infty} X_n \, \mathbb{1}(T \ge n) \,.$$

Since T is a stopping time with respect to $Z_0, Z_1, \ldots$, the event $\{T \ge n\} = \{T > n-1\}$ can be deterministically expressed in terms of $Z_0, \ldots, Z_{n-1}$ and is therefore independent of X_n. So we conclude that:

$$\mathbb{E}(S) = \sum_{n=1}^{\infty} \mathbb{E}(X_n) \, \mathbb{P}(T \ge n) = \mu \sum_{n=1}^{\infty} \mathbb{P}(T \ge n) = \mu \, \mathbb{E}(T) \,.$$

(see Exercise 1).

7. Let T be the number of tosses. This random variable is distributed geometrically $\mathrm{Geom}\big(\frac{1}{2}\big)$, so that the expected number of tosses equals 2.

For $n = 1, 2, 3, \ldots$, define $X_n := 1$ if the coin comes up heads in the n-th toss and $X_n := 0$ otherwise. Next, let $Z_0 := 0$ and $Z_n := X_n$ for $n = 1, 2, 3, \ldots$ One can easily verify the validity of the conditions stated in the preceding exercise. Since $\mathbb{E}(X_i) = \frac{1}{2}$, the expected number of heads equals $\frac{1}{2} \cdot 2 = 1$.

Alternatively, one can make use of the fact that the number of heads equals the total number of tosses minus the number of tails. Since the coin comes up tails exactly once almost surely, the expected number of heads should be $2 - 1 = 1$, which is, of course, the same as before.

8. We have $S_T = 1$ (deterministically), so that $\mathbb{E}(S_T) = 1$. However, since $T + 1 \sim \mathrm{Geom}\big(\frac{1}{2}\big)$, we have $\mathbb{E}(T) = \dfrac{2}{\frac{1}{2}} - 1 = 3$. Next, we have $S_n = X_1 + X_2 + \cdots + X_n$, where X_i is the indicator of the event that in the i-th toss, the coin comes up heads. Since $\mathbb{E}(X_i) = \frac{1}{2}$, Wald's equation does not hold. This does not contradict the statement because the random variable T is not a stopping time.

9. *First method.* Denote by T the number of all tosses and consider the following three hypotheses:

$$\begin{aligned} H_1 &= \{\text{the coin comes up heads in the first toss}\} \,. \\ H_2 &= \{\text{first toss tails, second toss heads}\} \,. \\ H_3 &= \{\text{tails in the first two tosses}\} \,. \end{aligned}$$

Under H_3, we have $T = 2$, so that $\mathbb{E}(T \mid H_3) = 2$. Under H_1 or H_2, the further development is again a sequence of independent coin tosses. Therefore, $\mathbb{E}(T \mid H_1) = 1 + \mathbb{E}(T)$ and $\mathbb{E}(T \mid H_2) = 2 + \mathbb{E}(T)$. Putting everything together, we find that:

$$\mathbb{E}(T) = \frac{1}{2}\big(1 + \mathbb{E}(T)\big) + \frac{1}{4}\big(2 + \mathbb{E}(T)\big) + \frac{1}{4} \cdot 2 \,,$$

leading to $\mathbb{E}(T) = 6$.

The expected number of tails can be deduced from Wald's equation similarly as in

Exercise 7, for T is a stopping time. It equals $6 \cdot \frac{1}{2} = 3$.

Second method.[1] We shall say that the coin tosses *basic tails* if this occurs either in the first toss or if it is preceded by heads. Then the number of tails that are tossed up to and including first two consequent tails equals the number of the basic tails up to (and including) the first basic tails followed by tails, plus one. From the strong time-homogeneous Markov property for the Bernoulli sequence (see Exercise 4 in Section 2), it follows that for each $n \in \mathbb{N}$, the n-th basic tails is followed by heads with probability $1/2$ and by tails with probability $1/2$ (the number of tosses up to the n-th basic tails is a stopping time). Moreover, this also holds conditionally given the entire sequence up to the n-th basic tails. Thus, considering each basic tails as a trial and each basic tails followed by another tails as a successful trial, we obtain a Bernoulli sequence with success probability $1/2$. Therefore, the number of basic tails up to the first basic tails followed by tails follows the geometric distribution $\mathrm{Geom}(1/2)$, having expectation 2. As a result, the expected number of tails up to two consequent tails equals $2 + 1 = 3$.

To compute the expected number of *all* tosses until two consequent tails, we can apply Wald's equation in the *opposite* direction as in the first method, leading to $\mathbb{E}(T) = 3/(1/2) = 6$.

10. Since $\mathbb{E}(X_i) = 0$, Wald's equation yields $\mathbb{E}(S) = 0$, so that $\mathrm{var}(S) = \mathbb{E}(S^2)$. Write:

$$S^2 = \sum_{n=1}^{\infty} (S_n^2 - S_{n-1}^2)\, \mathbb{1}(T \geq n) = \sum_{n=1}^{\infty} (2X_n S_{n-1} + X_n^2)\, \mathbb{1}(T \geq n)\,,$$

where $S_n = X_1 + X_2 + \cdots + X_n$. Thanks to independence, we have:

$$\begin{aligned} \mathrm{var}(S) &= \sum_{n=1}^{\infty} \Big(2\, \mathbb{E}(X_n)\, \mathbb{E}\big[S_{n-1}\, \mathbb{1}(T \geq n)\big] + \mathbb{E}(X_n^2)\, \mathbb{P}(T \geq n) \Big) = \\ &= \sigma^2 \sum_{n=1}^{\infty} \mathbb{P}(T \geq n) = \\ &= \sigma^2\, \mathbb{E}(T)\,. \end{aligned}$$

Without the assumption $\mathbb{E}(X_i) = 0$, the result does not necessarily hold: suppose that a fair coin is tossed until tails come up (the tosses are independent). Denote by T the number of all tosses. Let $X_n = 1$ if tails come up in the n-th toss, and $X_n = 0$ otherwise. Then we have $S = 1$ and furthermore $\mathrm{var}(S) = 0$, $\mathbb{E}(T) = 2$ and $\sigma^2 = 1/4$, contradicting the result.

11. a) According to the hint, we apply the preceding exercise 6 to random variables $Y_i := X_i - \mu$. Letting $U := Y_1 + \cdots + Y_T = S - \mu T$, the random variables $Y_1, Y_2, \ldots$ are independent with $\mathbb{E}(Y_i) = 0$. Putting

$$Z_0 := T\,, \quad Z_1 := Y_1\,, \quad Z_2 := Y_2\,, \ldots$$

[1] The idea is due to Timotej Akrapovič.

and considering T as an $\mathbb{N}_0$-valued random variable, T is (trivially) a stopping time with respect to the sequence $Z_0, Z_1, \ldots$ In addition, X_n is independent of the sequence $Z_0, Z_1, \ldots, Z_{n-1}$ for all $n \in \mathbb{N}$. From Wald's equation, we have $\mathbb{E}(U) = 0$; from the preceding exercise, it follows that $\operatorname{var}(U) = \mathbb{E}(U^2) = \sigma^2 \, \mathbb{E}(T)$.

Now we turn to S. We use $\operatorname{var}(S) = \mathbb{E}(S^2) - \big(\mathbb{E}(S)\big)^2$. Again by Wald's equation, we have $\mathbb{E}(S) = \mu \, \mathbb{E}(T)$. Therefore,

$$\begin{aligned}\operatorname{var}(S) &= \mathbb{E}(U^2) + 2\mu \, \mathbb{E}(UT) + \mu^2 \, \mathbb{E}(T^2) - \big(\mu \, \mathbb{E}(T)\big)^2 = \\ &= \sigma^2 \, \mathbb{E}(T) + 2\mu \, \mathbb{E}(UT) - \mu^2 \operatorname{var}(T) \,.\end{aligned}$$

Next, $\mathbb{E}(UT) = \mathbb{E}\big[\mathbb{E}(UT \mid T)\big] = \mathbb{E}\big[\mathbb{E}(U \mid T)T\big]$. Since the random variables $Y_1, Y_2, \ldots$ are independent of T, we have $\mathbb{E}(Y_n \mid T) = \mathbb{E}(Y_n) = 0$ for all n and consequently $\mathbb{E}(U \mid T) = 0$, leading to $\mathbb{E}(UT) = 0$. As a result, we conclude that

$$\operatorname{var}(S) = \sigma^2 \, \mathbb{E}(T) + \mu^2 \operatorname{var}(T) \,.$$

Remark: the first term represents the variance unexplained by T, i. e., $\mathbb{E}\big[\operatorname{var}(S \mid T)\big]$, and the second term is the explained variance, i. e., $\operatorname{var}\big[\mathbb{E}(S \mid T)\big]$.

b) From

$$\mathbb{E}\big[z^S \mid T = n\big] = \mathbb{E}(z^{S_n}) = \big[G_X(z)\big]^n \,,$$

it follows that $\mathbb{E}(z^S \mid T) = \big[G_X(z)\big]^T$, leading to:

$$\mathbb{E}(z^S) = G_T(G_X(z)) \,.$$

If the assumption that T is independent of the sequence $X_1, X_2, \ldots$ is replaced by the weaker assumption that T is a stopping time with respect to $X_0, X_1, X_2, \ldots$, none of the results necessarily holds. As a counter-example, we can, similarly as in the preceding exercise, take a fair coin, which is tossed until it comes up heads (assuming as usual that all tosses are independent). Again, let T be the number of all tosses. Next, let $X_n = 1$ if the coin comes up heads in the n-th toss, and $X_n = 0$ otherwise (this makes $S = 1$). As a $\mathbb{N}_0$-valued random variable, T is a stopping time with respect to the sequence $0, X_1, X_2, \ldots$ However, it is not difficult to check that none of the afore-mentioned results holds in this case.

12. a) From the transformation formula and marginal densities, it follows that the differences $X_t - X_s$, $t \neq s$, are continuously distributed with density

$$g(w) = \int_{-\infty}^{\infty} f(x) f(x - w) \, \mathrm{d}x \,.$$

As a result, $\mathbb{P}(X_t - X_s = 0) = \mathbb{P}(X_t = X_s) = 0$ for any $t \neq s$. However, the event that there exist $t \neq s$ with $X_t = X_s$ is the union of the underlying events over the collection of all possible pairs $t \neq s$, which is countable (in fact, it suffices only to take $t < s$). Therefore, the latter union has probability zero. The event that all random varables X_t are distinct is the complement of that union and therefore occurs with probability 1.

b) Denoting by R_t the event that a record value occurs at time t, observe that this event is uniquely determined by the *ordering* of the random variables $X_1, X_2, \ldots, X_t$. From the preceding part, it follows that it suffices only to consider the $t!$ orderings where $X_1, \ldots, X_n$ are all distinct.

The random vector $(X_1, X_2, \ldots, X_t)$ is continuously distributed with density

$$f_t(x_1, x_2, \ldots, x_t) = f(x_1) f(x_2) \cdots f(x_n) .$$

If π is a permutation of t elements, then, by the transformation formula, the random vector $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(t)})$ has density

$$\begin{aligned}(y_1, y_2, \ldots, y_t) \mapsto f_t(y_{\pi^{-1}(1)}, y_{\pi^{-1}(2)}, \ldots, y_{\pi^{-1}(t)}) &= \\ &= f(y_{\pi^{-1}(1)}) f(y_{\pi^{-1}(2)}) \cdots f(y_{\pi^{-1}(t)}) = \\ &= f(y_1) f(y_2) \cdots f(y_t) .\end{aligned}$$

This means that the random vector $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(t)})$ follows the same distribution as $(X_1, X_2, \ldots, X_t)$. Such random variables are called *exchangeable*. Now it follows that all $t!$ orderings of random variables $X_1, X_2, \ldots, X_t$, where they are all distinct, are equally likely. The event R_t corresponds to $(t-1)!$ out of these orderings. Therefore, $\mathbb{P}(R_t) = 1/t$.

c) What we need to prove is that

$$\mathbb{P}\big(R_{t_1} \cap R_{t_2} \cap \cdots \cap R_{t_k}\big) = \mathbb{P}\big(R_{t_1}\big)\, \mathbb{P}\big(R_{t_2}\big) \cdots \mathbb{P}\big(R_{t_k}\big) = \frac{1}{t_1} \cdot \frac{1}{t_2} \cdots \frac{1}{t_k}$$

for all $1 \le t_1 < t_2 < \cdots < t_k$. Similarly as in the preceding part, observe that the event $R_{t_1} \cap R_{t_2} \cap \ldots \cap R_{t_k}$ is uniquely determined by the ordering of the random variables $X_1, X_2, \ldots, X_{t_k}$. Again, it suffices only to consider the $t_k!$ orderings where their values are all distinct; recall also that they are equally likely. Now there are at least two methods to proceed.

First method: simply count all the orderings which are due to the event $R_{t_1} \cap R_{t_2} \cap \ldots \cap R_{t_k}$. The orderings of the random variables $X_1, X_2, \ldots, X_{t_k}$ can be represented by arrangements of balls with labels $1, 2, \ldots, t_k$ in a row: the greater the value of a random variable, the more to the right the underlying ball is positioned. The orderings which are due to the event $R_{t_1} \cap R_{t_2} \cap \ldots \cap R_{t_k}$ correspond to the arrangements with ball t_i positioned to the right of all balls $1, 2, \ldots, t_i - 1$ for each $i = 1, 2, \ldots, k$. These arrangements can be generated as follows: first, arrange balls $1, 2, \ldots, t_1 - 1$; there are $1 \cdot 2 \cdots (t_1 - 1)$ ways to do it. Put ball t_1 to the right end. Now insert balls $t_1 + 1, t_1 + 2, \ldots, t_2 - 1$ among the t_1 balls which have already been arranged; this can be done in $(t_1 + 1)(t_1 + 2) \cdots (t_2 - 1)$ ways, of course regardless of the arrangement of the first t_1 balls. To the right end, put ball t_2. Continue this way: after we have arranged t_{i-1} balls, insert balls $t_{i-1} + 1, t_{i-1} + 2, \ldots, t_i - 1$ among them, which can be done in $(t_{i-1} + 1)(t_{i-1} + 2) \cdots (t_i - 1)$ ways; then put ball t_i to the right end. Thus, we find that the desired number of orderings equals

$$\begin{aligned}1 \cdot 2 \cdots (t_1 - 1) \times (t_1 + 1)(t_1 + 2) \cdots (t_2 - 1) \times (t_2 + 1)(t_2 + 2) \cdots (t_3 - 1) \times \cdots \\ \cdots \times (t_{k-1} + 1)(t_{k-1} + 2) \cdots (t_k - 1) ,\end{aligned}$$

leading to

$$\mathbb{P}\big(R_{t_1} \cap R_{t_2} \cap \ldots \cap R_{t_k}\big) = \frac{1}{t_1 t_2 \cdots t_k}\,,$$

completing the proof.

Second method: consider the conditional probability of the event R_{t_k} given a certain ordering of the random variables $X_1, X_2, \ldots, X_{t_k-1}$. There are t_k possible ways to extend such an ordering to the ordering of the random variables $X_1, X_2, \ldots, X_{t_k}$, all of them being equally likely and exactly one of them being due to the event R_{t_k}. This means that the conditional probability of R_{t_k} given any ordering of the random variables $X_1, X_2, \ldots, X_{t_k-1}$ equals $1/t_k$.

Since the event $R_{t_1} \cap R_{t_2} \cap \cdots \cap R_{t_{k-1}}$ is a disjoint union of the events associated to the suitable orderings of the random variables $X_1, X_2, \ldots, X_{t_k-1}$, the conditional probability of the event R_{t_k} given $R_{t_1} \cap R_{t_2} \cap \cdots \cap R_{t_{k-1}}$ also equals $1/t_k$. By conditioning, we obtain

$$\begin{aligned}
&\mathbb{P}\big(R_{t_1} \cap R_{t_2} \cap \cdots \cap R_{t_k}\big) = \\
&\quad = \mathbb{P}\big(R_{t_1}\big)\,\mathbb{P}\big(R_{t_2} \bigm| R_{t_1}\big)\,\mathbb{P}\big(R_{t_3} \bigm| R_{t_1} \cap R_{t_2}\big) \times \cdots \\
&\qquad\qquad \cdots \times \mathbb{P}\big(R_{t_k} \bigm| R_{t_1} \cap R_{t_2} \cap \cdots \cap R_{t_{k-1}}\big) = \\
&\quad = \frac{1}{t_1} \cdot \frac{1}{t_2} \cdots \frac{1}{t_k}\,,
\end{aligned}$$

which is the same as before.

Remark. The argument given above shows that the event R_t is independent of the *ordering* of the random variables $X_1, X_2, \ldots, X_{t-1}$. However, this does not imply the independence of the random vector $(X_1, X_2, \ldots, X_{t-1})$, which is *not* true.

d) Denoting by I_t the *indicator* of the event R_t, i. e.:

$$I_t = \mathbb{1}_{R_t} = \begin{cases} 1 & ; \text{a record value occurs at time } t \\ 0 & ; \text{no record value occurs at time } t \end{cases}\,,$$

we have $N_t = I_1 + I_2 + \cdots + I_t$ and therefore $\mathbb{E}(I_t) = 1/t$ and $\operatorname{var}(I_t) = (t-1)/t^2$. Hence,

$$\mathbb{E}(N_t) = \mathbb{E}(I_1) + \mathbb{E}(I_2) + \cdots + \mathbb{E}(I_t) = 1 + \frac{1}{2} + \cdots + \frac{1}{t} \sim \ln t\,.$$

If certain events are independent, so are their indicators. Therefore,

$$\operatorname{var}(N_t) = \operatorname{var}(I_1) + \operatorname{var}(I_2) + \cdots + \operatorname{var}(I_t) = \frac{1}{2^2} + \frac{2}{3^2} + \cdots + \frac{t-1}{t^2} \sim \ln t\,.$$

e) The event $\{S > t\}$ is equivalent to the event that X_1 is highest among $X_1, X_2, \ldots$ $\ldots, X_t$ and has probability $1/t$. Therefore, $\mathbb{P}(S = \infty) = \lim_{t\to\infty} \mathbb{P}(S > t) = 0$. For $t = 2, 3, 4, \ldots$, we compute:

$$\mathbb{P}(S = t) = \mathbb{P}(S > t-1) - \mathbb{P}(S > t) = \frac{1}{t(t-1)}\,.$$

Hence $\mathbb{E}(S) = \sum_{t=2}^{\infty} \frac{1}{t-1} = \infty$.

Remark. Although the expectation is infinite, the *conditional* expectation given the first measurement X_1 is finite – we have:

$$\mathbb{E}(S \mid X_1) = 1 + \frac{1}{1 - F(X_1)}\,,$$

where F denotes the cumulative distribution function of the random variables $X_1, X_2, \ldots$ (given X_1, the number of further attempts needed to overcome the value X_1 follows the geometric distribution $\mathrm{Geom}(1 - F(X_1))$). However, integration into unconditional expectation yields an infinite result: denoting by $f = F'$ the density, we obtain:

$$\mathbb{E}(S) = \mathbb{E}\big[\mathbb{E}(S \mid X_1)\big] = \int_{-\infty}^{\infty} \frac{1}{1 - F(x)}\, f(x)\,\mathrm{d}x\,.$$

and substitution $t = F(x)$ yields:

$$\mathbb{E}(S) = \int_0^1 \frac{1}{1-t}\,\mathrm{d}t = \infty\,.$$

f) The conditional distribution of the random variable $X_{S^{(y)}}$ given $\{S^{(y)} = t\}$ is the same as the conditional distribution of the random variable X_t given $\{X_1 \leq y, X_2 \leq y, \ldots, X_{t-1} \leq y, X_t > y\}$. Because of independence, the latter event can be replaced by $\{X_t > y\}$. However, the conditional distribution of X_t given $X_t > y$ is independent of t.

2 Counting Processes

1. a) Denoting by p the probability of an arrival to occur at a given time, we have $N_t \sim \mathrm{Bin}(n,p)$ and $N_t - N_s \sim \mathrm{Bin}(t-s,p)$. As a result, we have:

- For independent Bernoulli random variables $I_1, I_2, \ldots, I_t$ with $\mathbb{P}(Y_k = 1) = p$, we have $I_1 + I_2 + \cdots + I_t \sim \mathrm{Bin}(t,p)$.
- For independent random variables $X \sim \mathrm{Bin}(k,p)$ and $Y \sim \mathrm{Bin}(l,p)$, we have $X + Y \sim \mathrm{Bin}(k+l,p)$.

b) These random variables are independent.

c) By the fundamental equivalence,

$$\mathbb{P}(S_1 \le t) = \mathbb{P}(N_t \ge 1) = 1 - \mathbb{P}(N_t = 0) = 1 - (1-p)^t .$$

The formula holds for $t = 0, 1, 2, \ldots$, so for all $t \in \mathbb{N}$, we have:

$$\mathbb{P}(S_1 = t) = \mathbb{P}(S_1 \le t) - \mathbb{P}(S_1 \le t-1) = p(1-p)^{t-1} .$$

Therefore $S_1 \sim \mathrm{Geom}(p)$.

d) More generally,

$$\begin{aligned}\mathbb{P}(S_n = t) &= \mathbb{P}(S_n \le t, S_n > t-1) = \mathbb{P}(N_t \ge n, N_{t-1} < n) = \\ &= \mathbb{P}(N_{t-1} = n-1, \text{an arrival occurs at time } t) = \\ &= \mathbb{P}(N_{t-1} = n-1)\, \mathbb{P}(\text{an arrival occurs at time } t) = \\ &= \binom{t-1}{n-1} p^n (1-p)^{t-n} .\end{aligned}$$

Hence $S_n \sim \mathrm{NegBin}(n,p)$.

2. By the fundamental equivalence, the n-th arrival time satisfies:

$$F_{S_n}(t) = \mathbb{P}(S_n \le t) = \mathbb{P}(N_t \ge n) = \mathbb{P}(N_t + 1 > n) = \left(1 - e^{-t}\right)^n .$$

Therefore,

$$\begin{aligned}\mathbb{E}(S_1) &= \int_0^\infty e^{-t}\,\mathrm{d}t = 1 , \\ \mathbb{E}(S_2) &= \int_0^\infty \left[1 - \left(1 - e^{-t}\right)^2\right] \mathrm{d}t = \int_0^\infty \left(2\,e^{-t} - e^{-2t}\right) \mathrm{d}t = \frac{3}{2} .\end{aligned}$$

3. *First method.* Clearly, $N_t \sim \mathrm{Bin}(n,t)$. The fundamental equivalence $\{S_k \le t\} = \{N_t \ge k\}$ yields the cumulative distribution function of the random variables S_k:

$$F_{S_k}(t) = \sum_{l=k}^{n} \binom{n}{l} t^l (1-t)^{n-l} ; \quad 0 \le u \le 1 .$$

Differentiating, we obtain the probability density function (minding the last term):

$$\begin{aligned}
f_{S_k}(t) &= \sum_{l=k}^{n} \binom{n}{l} \Big[l\, t^{l-1}(1-t)^{n-l} - (n-l)t^l(1-t)^{n-l-1} \Big] = \\
&= \sum_{l=k}^{n} \binom{n}{l} l\, t^{l-1}(1-t)^{n-l} - \sum_{l=k}^{n-1} \binom{n}{l} (n-l)t^l(1-t)^{n-l-1} = \\
&= \sum_{l=k}^{n} \frac{n!}{(l-1)!\,(n-l)!}\, t^{l-1}(1-t)^{n-l} - \sum_{l=k}^{n-1} \frac{n!}{l!\,(n-l-1)!}\, t^l(1-t)^{n-l-1} = \\
&= \sum_{l=k}^{n} \frac{n!}{(l-1)!\,(n-l)!}\, t^{l-1}(1-t)^{n-l} - \sum_{j=k+1}^{n} \frac{n!}{(j-1)!\,(n-j)!}\, t^{j-1}(1-t)^{n-j} = \\
&= \frac{n!}{(k-1)!\,(n-k)!}\, t^{k-1}(1-t)^{n-k}\,.
\end{aligned}$$

This is the *beta distribution*: $S_k \sim \mathrm{Beta}(k, n-k+1)$.

Second method. The density can be easily obtained by the following "physical" thought, which is, of cource, not entirely mathematically correct: the event $\{S_k \in [t, t+\mathrm{d}t]\}$ means that exactly $k-1$ guests arrived up to time t, exactly one guest arrived within the infinitesimal time interval from t to $t+\mathrm{d}t$, white the other $n-k$ guests arrived from time $t+\mathrm{d}t$ on. Therefore, we have:

$$\mathbb{P}(S_k \in [t, t+\mathrm{d}t]) = \frac{n!}{(k-1)!\,1!\,(n-k)!} t^{k-1}\,\mathrm{d}t\,(1-t)^{n-k}\,,$$

leading to the density:

$$f_{S_k}(t) = \frac{n!}{(k-1)!\,(n-k)!}\, t^{k-1}(1-t)^{n-k}\,,$$

which is the same as before. However, to make it mathematically correct, take $0 \le t < t+h \le 1$. The fundamental equivalence implies:

$$F_{S_k}(t+h) - F_{S_k}(t) = \mathbb{P}(N_{t+h} \ge k) - \mathbb{P}(N_t \ge k)\,.$$

Since $\{N_t \ge k\} \subseteq \{N_{t+h} \ge k\}$, we also have:

$$\begin{aligned}
F_{S_k}(t+h) - F_{S_k}(t) &= \mathbb{P}(N_t < k, N_{t+h} \ge k) = \\
&= \mathbb{P}(N_t < k, N_1 - N_{t+h} \le n-k) = \\
&= \sum_{i=0}^{k-1} \sum_{j=k}^{n} \mathbb{P}(N_t = i, N_{t+h} - N_t = j-i, N_1 - N_{t+h} = n-j) = \\
&= \sum_{i=0}^{k-1} \sum_{j=k}^{n} \frac{n!}{i!\,(j-i)!\,(n-j)!}\, t^i h^{j-i}(1-t-h)^{n-j}\,.
\end{aligned}$$

Observe that the exponent at h is always at least 1. Dividing by h and taking the limit as $h \to 0$, the left hand side is exactly the *right* derivative of the function

F_{S_k} at t. In the right hand side, however, only the terms with h^1 survive. As the only such term occurs at $i = k-1$ and $j = k$, the above-mentioned right derivative equals:

$$\frac{n!}{(k-1)!\,(n-k)!}\,t^{k-1}(1-t)^{n-k}\,,$$

which is the same as before. It remains to prove that the same holds for the left derivative. Thus, take $0 \le t-h < t \le 1$. Similarly as before, one can compute:

$$F_{T_k}(t) - F_{T_k}(t-h) = \sum_{i=0}^{k-1}\sum_{j=k}^{n} \frac{n!}{i!\,(j-i)!\,(n-j)!}\,(t-h)^i h^{j-i}(1-t)^{n-j}\,.$$

and the limit as h tends to zero is the same as before. This means that the function F_{S_k} is differentiable on the whole interval $(0,1)$ and one can easily check that the derivative is continuous. Clearly, F_{S_k} itself is also continuous. As a result, we conclude that:

$$f_{S_k}(t) = \frac{n!}{(k-1)!\,(n-k)!}\,t^{k-1}(1-t)^{n-k}\,.$$

4. a) The independence of $\mathcal{P}^{t\rightarrow}$ and $\mathcal{P}^{\rightarrow\!\!\!\!\!/ t}$ follows from the independence of the events that an arrival occurs at a given time Finally, the fact that $\mathcal{P}^{t\rightarrow}$ follows the same distribution as $\mathcal{P}$ follows from the fact that it shares all the properties which define $\mathcal{P}$ (the distribution is uniquely determined by these properties).

b) The process $\mathcal{P}^{T\rightarrow}$ takes its values in the set $\wp(\mathbb{N})$, the power set of the set of $\mathbb{N}$. The process $\mathcal{P}^{\rightarrow\!\!\!\!\!/ T}$ takes its values in the set $\wp_{\text{fin}}(\mathbb{N})$, the set of all finite subsets of $\mathbb{N}$. Take arbitrary measurable sets $A \subseteq \wp_{\text{fin}}(\mathbb{N})$ and $B \subseteq \wp(\mathbb{N})$ and arbitrary $t \in \mathbb{N}$. Consider the following conditional probability:

$$\mathbb{P}\big(\mathcal{P}^{T\rightarrow} \in B \mid T = t, \mathcal{P}^{\rightarrow\!\!\!\!\!/ T} \in A\big) = \mathbb{P}\big(\mathcal{P}^{t\rightarrow} \in B \mid T = t, \mathcal{P}^{\rightarrow\!\!\!\!\!/ t} \in A\big)$$

(which makes sense if $\mathbb{P}\big(T = t, \mathcal{P}^{\rightarrow\!\!\!\!\!/ t} \in A\big) > 0$). However, the event $\{T = t\}$ is uniquely determined by $\mathcal{P}^{\rightarrow\!\!\!\!\!/ t}$. This means that it can be expressed as $\{\mathcal{P}^{\rightarrow\!\!\!\!\!/ t} \in C_t\}$ for some set $C_t \subseteq \wp(\{1,\dots,t\})$. From the ordinary time-homogeneous Markov property, it follows that:

$$\begin{aligned}\mathbb{P}\big(\mathcal{P}^{T\rightarrow} \in B \mid T = t, \mathcal{P}^{\rightarrow\!\!\!\!\!/ T} \in A\big) &= \mathbb{P}\big(\mathcal{P}^{t\rightarrow} = B \mid \mathcal{P}^{\rightarrow\!\!\!\!\!/ t} = A \cap C_t\big) = \\ &= \mathbb{P}\big(\mathcal{P}^{t\rightarrow} = B\big) = \\ &= \mathbb{P}\big(\mathcal{P} = B\big)\,,\end{aligned}$$

which yields the desired result.

5. The first part (memorylessness) can be proved in at least two ways.

First method. By direct calculation, we verify that:

$$\mathbb{P}(T = t+s \mid T > t) = \frac{\mathbb{P}(T = t+s)}{\mathbb{P}(T > t)} = \frac{p(1-p)^{t+s-1}}{(1-p)^t} = p(1-p)^{s-1} = \mathbb{P}(T = s)\,.$$

Second method. Consider the counting process from the previous exercise and let $T = S_1$ be the first arrival time. The event $\{T > t\}$ matches the event that no arrival occurs up to and including time t. Denoting again by $\mathcal{P}$ the set of arroval times and by T' the first arrival time in the process $\mathcal{P}^{t\rightarrow}$, we find that in the event $\{T > t\}$, we have $T' = T - t$; therefore, in this event, the events $\{T = t + s\}$ and $\{T' = s\}$ match, i. e., their intersections with $\{T > t\}$ are equal. Since the event $\{T > t\}$ can be deterministically expressed in terms of $\mathcal{P}^{\not\rightarrow t}$, T' must be independent of $\{T > t\}$ and follows the same distribution as T; that is the conditional distribution of T' given $\{T > t\}$ is the same as the unconditional distribution of T. This completes the proof.

Let T be a memoryless random variable. This can be also written in the following form:

$$\mathbb{P}(T > t + s \mid T > t) = \mathbb{P}(T > s)$$

for all $s \in \mathbb{N}_0$ and also for all $t \in \mathbb{N}_0$. In other words,

$$\mathbb{P}(T > t + s) = \mathbb{P}(T > t)\,\mathbb{P}(T > s)\,.$$

As a special case, we have:

$$\mathbb{P}(T > t + 1) = \mathbb{P}(T > t)\,\mathbb{P}(T > 1)$$

and denoting $q := \mathbb{P}(T > 1)$, it follows that:

$$\mathbb{P}(T > t) = q^t\,,$$

where $0 < q < 1$: the case $q = 0$ is ruled out because the conditional probability in $(*)$ must make sense; in the case $q = 1$, however, the probability of a decreasing sequence of events would yield $\mathbb{P}(T \in \mathbb{N}) = 0$. Finally, we obtain:

$$\mathbb{P}(T = t) = \mathbb{P}(T > t - 1) - \mathbb{P}(T > t) = (1 - q)q^{t-1}\,,$$

which means that T indeed follows a geometric distribution.

6. a) This follows from the fundamental equivalence $\{S_n \le t\} = \{N_t \ge n\}$. As a result, the process $\mathcal{P}^{S_n\rightarrow}$ is also independent of $\left(S_n, \mathcal{P}^{\not\rightarrow S_n}\right)$ and follows the same distribution as $\mathcal{P}$.

b) The n-th inter-arrival time of the process $\mathcal{P}$ corresponds to the first inter-arrival time of the process $\mathcal{P}^{S_{n-1}\rightarrow}$. From d), it follows that T_n follows the same distribution as T_1 and is independent of $(S_{n-1}, \mathcal{P}^{S_{n-1}\rightarrow})$ and therefore of $(T_1, \ldots, T_{n-1})$. Thus, the inter-arrival times T_n are independent with the geometric distribution $\mathrm{Geom}(p)$. As a result, we have:

- For $T_1, T_2, \ldots, T_n$ independent with the geometric distribution $\mathrm{Geom}(p)$, we have $T_1 + T_2 + \cdots + T_n \sim \mathrm{NegBin}(n, p)$.
- For independent random variables $X \sim \mathrm{NegBin}(k, p)$ and $Y \sim \mathrm{NegBin}(l, p)$, we have $X + Y \sim \mathrm{NegBin}(k + l, p)$.

c) The difference $S_n - S_m$ corresponds to the $(n - m)$-th inter-arrival time of the process $\mathcal{P}^{S_m\rightarrow}$; thus, it follows the same distribution as S_{n-m}, that is, negative binomial $\mathrm{NegBin}(n - m, p)$.

3 Homogeneous Poisson Process

1. a) The expected time from the opening until the start of examination is exactly the third arrival time S_3, which follows the gamma distribution $\mathrm{Gama}(3, 6)$; its expected value equals $3 \cdot 1/6 = 1/2$, i. e., half an hour.

b) $\mathbb{P}(N_1 < 3) = \mathrm{Pois}(6)\{0, 1, 2\} = \sum_{k=0}^{2} \frac{6^k e^{-6}}{k!} = 25\, e^{-6} \doteq 0{\text{'}}0620.$

2. a) Suppose first that $s > t$. Then $N_s - N_t$ is independent of N_t. Therefore,

$$\mathrm{cov}(N_t, N_s) = \mathrm{cov}(N_t, N_t) + \mathrm{cov}(N_t, N_s - N_t) = \mathrm{var}(N_t) = \lambda t\,.$$

In the general case, we have $\mathrm{cov}(N_t, N_s) = \lambda \min\{t, s\}$.

b) $\mathrm{corr}(N_t, N_s) = \frac{\min\{t, s\}}{\sqrt{ts}} = \sqrt{\frac{\min\{t, s\}}{\max\{t, s\}}}\,.$

c) $\lambda \begin{bmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_n \end{bmatrix}.$

3. The player wins the game if the bell rings exactly once between times s and 1. Since the number of the rings is Poisson $\mathrm{Pois}\big(\lambda(1-s)\big)$, the winning probability is:

$$\lambda(1-s)\, e^{-\lambda(1-s)}\,.$$

b) The function $t \mapsto t\, e^{-t}$ is increasing on $[0, 1]$, attains its maximal value e^{-1} at $t = 1$ and is decreasing on $[1, \infty)$. Now substitute t with $\lambda(1-s)$, which can range the interval from 0 to λ. If $\lambda < 1$, this expression cannot reach 1. In this case, the maximumal winning probability is attained at $s = 0$ and equals $\lambda\, e^{-\lambda}$. For $\lambda \geq 1$, the expression $\lambda(1-s)$ reaches 1 at $s = (\lambda - 1)/\lambda$ and the maximum winning probability equals e^{-1}.

4. Observing that:

$$\begin{aligned}
\mathbb{P}(X = k \mid Z = n) &= \frac{\mathbb{P}(X = k, Z = n)}{\mathbb{P}(Z = n)} = \\
&= \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(Z = n)} = \\
&= \frac{\mathbb{P}(X = k)\,\mathbb{P}(Y = n - k)}{\mathbb{P}(Z = n)} = \\
&= \frac{n!}{k!(n-k)!}\,\frac{\lambda^k \mu^{n-k}}{(\lambda + \mu)^n} = \\
&= \binom{n}{k} \left(\frac{\lambda}{\lambda + \mu}\right)^k \left(\frac{\mu}{\lambda + \mu}\right)^{n-k},
\end{aligned}$$

we find that the desired conditional distribution is binomial $\mathrm{Bin}\left(Z, \dfrac{\lambda}{\lambda+\mu}\right)$.

5. a) Conditionally given N, we have $S \sim \mathrm{Bin}(N, p)$, i. e.:

$$\mathbb{P}(S = k \mid N = n) = \binom{n}{k} p^k (1-p)^{n-k}\,; \quad k = 0, 1, \ldots, n\,.$$

By the total probability theorem, we compute:

$$\begin{aligned}
\mathbb{P}(S = k) &= \sum_{n=k}^{\infty} \mathbb{P}(N = n)\, \mathbb{P}(S = k \mid N = n) = \\
&= \sum_{n=k}^{\infty} \frac{\lambda^n\, e^{-\lambda}}{n!} \binom{n}{k} p^k (1-p)^{n-k} = \\
&= \frac{\lambda^n p^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} = \\
&= \frac{(p\lambda)^k\, e^{-p\lambda}}{k!}\,.
\end{aligned}$$

Thus, $S \sim \mathrm{Pois}(p\lambda)$. Similarly, $T \sim \mathrm{Pois}((1-p)\lambda)$.

b) Observing that:

$$\begin{aligned}
\mathbb{P}(S = k, T = l) = \mathbb{P}(S = k, N = k + l) &= \mathbb{P}(N = k + l)\, \mathbb{P}(S = k \mid N = k + l) = \\
&= \frac{\lambda^{k+l}\, e^{-\lambda}}{(k+l)!} \binom{k+l}{k} p^k (1-p)^l = \frac{\lambda^{k+l}\, p^k (1-p)^l\, e^{-\lambda}}{k!\, l!} = \\
&= \mathbb{P}(S = k)\, \mathbb{P}(T = l)\,,
\end{aligned}$$

we find that S and T are indeed independent.

c) If $N = n$ is a constant S are T dependent, provided that $n \geq 1$ and $0 < p < 1$: in this case, S in T take at least two values (with positive probability), but $S = k$ implies $T = n - k$.

6. As the *survival function* seems to be the simplest way of describing the exponential distribution:

$$\mathbb{P}(T > t) = e^{-\lambda t}\,; \quad t \geq 0\,,$$

the memorylessness can be characterized with:

$$\mathbb{P}(T > t + s \mid T > t) = \mathbb{P}(T > s)$$

This can be checked out in at least two ways.

First method. By direct calculation, we find that:

$$\mathbb{P}(T > t + s \mid T > t) = \frac{\mathbb{P}(T > t + s)}{\mathbb{P}(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(T > s)\,.$$

Second method. Consider a homogeneous Poisson process with intensity λ and proceed as in Exercise 5 from Section 2.

Now let T be a memoryless random variable. Denoting by $G(t) := \mathbb{P}(T > t)$ its survival function, we have:

$$G(t + s) = G(t)\, G(s)\,.$$

for all $t, s \geq 0$. Since the density is continuous on $(0, \infty)$, the survival function is there continuously differentiable and has a right derivative at 0. Differentiation with respect to s yields a differential equation:

$$G'(t) = G(t)\, G'(0)\,.$$

Moreover, the fact that the density vanishes outside $(0, \infty)$ leads to an initial condition $G(0) = 1$. Thus the solution is unique and given by:

$$G(t) = e^{G'(0)\, t}\,; \quad t \geq 0\,,$$

which means that the distribution of T is indeed exponential.

7. The easiest way to find the distribution of U is by means of the survival function:

$$\mathbb{P}(U > u) = \mathbb{P}(X > u, Y > u) = \mathbb{P}(X > u)\,\mathbb{P}(Y > u) = e^{-(\lambda+\mu)u}\,,$$

which yields $U \sim \mathrm{Exp}(\lambda + \mu)$.

To find the distribution of V, we refer to the cumulative distribution function:

$$\mathbb{P}(V \leq v) = \mathbb{P}(X \leq v, Y \leq v) = \mathbb{P}(X \leq v)\,\mathbb{P}(Y \leq v) = \big(1 - e^{-\lambda v}\big)\big(1 - e^{-\mu v}\big)\,.$$

Thus, the distribution of V is continuous with density:

$$f_V(v) = \begin{cases} \lambda\, e^{-\lambda v} + \mu\, e^{-\mu v} - (\lambda + \mu)\, e^{-(\lambda+\mu)v} & ;\ v > 0 \\ 0 & ;\ \text{sicer} \end{cases}\,.$$

8. We can add the interventions arising from the redirected calls, assuming that their lengths are also distributed uniformly over the interval from half an hour to one hour and independent of each other as well as of the interventions of the original fire station and the emergency calls. As usual, denote by $T_1, T_2, \ldots$ the inter-arrival times between the emergency calls (either processed by the original fire station or redirected). Let U_k denote the length of the intervention following the k-th call.

Denoting the desired number of calls by K, this is the first natural number with $U_k > T_{k+1}$. Since $T_1, T_2, T_3, \ldots$ and $U_1, U_2, U_3, \ldots$ are all independent, so are the events $\{U_1 > T_2\}, \{U_2 > T_3\}, \ldots$. In addition, they are equiprobable, forming a Bernoulli sequence. Hence K follows the geometric distribution $\mathrm{Geom}\big(\mathbb{P}(U_1 > T_2)\big)$.

Finally, since $U_1 \sim \mathrm{Unif_c}\big((\tfrac{1}{2}, 1)\big)$ and $T_2 \sim \mathrm{Exp}\big(\tfrac{1}{2}\big)$ are independent, we have

$$\mathbb{P}(U_1 > T_2) = \int_{1/2}^{1} \int_0^u e^{-t/2}\, \mathrm{d}t\, \mathrm{d}u = 1 - 4\, e^{-1/4} + 4\, e^{-1/2} \doteq 0{\cdot}311\,.$$

9. a) For $0 \le s \le t$, we have:

$$\begin{aligned} F_{S_1 \mid N_t = 1}(s) &= \mathbb{P}(S_1 \le s \mid N_t = 1) = \\ &= \mathbb{P}(N_s \ge 1 \mid N_t = 1) = \\ &= \mathbb{P}(N_s = 1 \mid N_t = 1) = \\ &= \frac{\mathbb{P}(N_s = 1, N_t = 0)}{\mathbb{P}(N_t = 1)} = \\ &= \frac{\mathbb{P}(N_s = 1, N_t - N_s = 0)}{\mathbb{P}(N_t = 1)} = \\ &= \frac{\mathbb{P}(N_s = 1)\,\mathbb{P}(N_t - N_s = 0)}{\mathbb{P}(N_t = 1)} = \\ &= \frac{\lambda s\, e^{-\lambda s}\, e^{-\lambda(t-s)}}{\lambda t\, e^{-\lambda t}} = \\ &= \frac{s}{t}\,. \end{aligned}$$

Thus, the desired conditional distribution is exactly the uniform distribution over $(0, t)$.

b) Similarly as before, for $0 \le s \le t$, compute:

$$\begin{aligned} F_{S_2 \mid N_t = 2}(s) &= \mathbb{P}(N_s \ge 2 \mid N_t = 2) = \\ &= \frac{\mathbb{P}(N_s = 2, N_t - N_s = 0)}{\mathbb{P}(N_t = 2)} = \\ &= \frac{s^2}{t^2}\,, \end{aligned}$$

so that $f_{S_2 \mid N_t = 2}(s) = \dfrac{2s}{t^2}$ and $\mathbb{E}(S_2 \mid N_t = 2) = \dfrac{2t}{3}$.

Next,

$$\begin{aligned} F_{S_1 \mid N_t = 2}(s) &= \mathbb{P}(N_s \ge 1 \mid N_t = 2) = \\ &= \frac{\mathbb{P}(N_s = 1, N_t - N_s = 1) + \mathbb{P}(N_s = 2, N_t - N_s = 0)}{\mathbb{P}(N_t = 2)} = \\ &= \frac{2st - s^2}{t^2}\,, \end{aligned}$$

so that $f_{S_1 \mid N_t = 2}(s) = \dfrac{2(t-s)}{t^2}$ and $\mathbb{E}(S_1 \mid N_t = 2) = \dfrac{t}{3}$.

10. *First method.* Observe that:

$$W = (t - S_1) + (t - S_2) + \cdots + (t - S_{N_t}) = tN_t - S_1 - S_2 - \cdots - S_{N_t}\,.$$

Now recall that the conditional distribution of the sum $S_1 + \cdots + S_n$ given $N_t = n$ matches the unconditional distribution of the sum $U_{(1)} + \cdots + U_{(n)} = U_1 + \cdots + U_n$, where $U_1, \ldots, U_n$ are independent and distributed uniformly over $(0, t)$, and where $U_{(1)}, \dots, U_{(n)}$ are the underlying order statistics. Therefore, the arrival times $S_1, S_2, \dots$ can be replaced with random variables $U_1, U_2, \dots$, which are distributed uniformly over $(0, t)$ and independent of each other as well as of N_t. As a result,

$$\mathbb{E}(W) = \mathbb{E}\big(tN_t - U_1 - U_2 - \cdots - U_{N_t}\big) .$$

By Wald's equation, we have:

$$\mathbb{E}(W) = t\,\mathbb{E}(N_t) - \mathbb{E}(U_1)\,\mathbb{E}(N_t) = t \cdot \lambda t - \frac{t}{2} \cdot \lambda t = \frac{\lambda t^2}{2} .$$

Second method. Observe that $W = \int_0^t N_s \, \mathrm{d}s$. Taking the expectation, we find that:

$$\mathbb{E}(W) = \int_0^t \mathbb{E}(N_s)\,\mathrm{d}s = \int_0^t \lambda s \,\mathrm{d}s = \frac{\lambda t^2}{2} .$$

11. *First method.* Write:

$$X(t) = \sum_{i=1}^{N_t} e^{-\theta(t - S_i)} ,$$

where, as usual, $S_1, S_2, \dots$ denote the arrival times. Equivalently, describing our counting process by a random set $\mathcal{P}$, we may write:

$$X(t) = \sum_{s \in \mathcal{P}^{\rightarrow\!\!\!| t}} e^{-\theta(t-s)} .$$

However, from part c) of Exercise 9, it follows that:

$$\mathbb{E}\big[X(t) \mid N_t = n\big] = \mathbb{E}\left(\sum_{i=1}^{n} e^{-\theta(t - U_i)} \right) ,$$

where $U_1, U_2, \dots, U_n$ are independent and distributed uniformly over $[0, t]$. Therefore,

$$\mathbb{E}\big[X(t) \mid N_t = n\big] = n\,\mathbb{E}\big(e^{-\theta(t-U_1)}\big) = n\,\frac{1 - e^{-\theta t}}{\theta t} ,$$

or, alternatively,

$$\mathbb{E}\big[X(t) \mid N_t\big] = N_t\,\frac{1 - e^{-\theta t}}{\theta t} ,$$

leading to:

$$\mathbb{E}\big[X(t)\big] = \frac{\lambda(1 - e^{-\theta t})}{\theta} .$$

Second method. Observe that the total effect of all shocks can be written as a Riemann–Stieltjes integral:

$$X(t) = \int_0^t e^{-\theta(t-s)}\,\mathrm{d}N_s .$$

Integrating by parts, we obtain:

$$X(t) = N_t - \theta \int_0^t e^{-\theta(t-s)} N_s \, \mathrm{d}s \,,$$

which (noting that $\int e^{\theta s} \, \mathrm{d}s = \bigl(\frac{s}{\theta} - \frac{1}{\theta^2}\bigr) e^{\theta s} + C$) yields:

$$\mathbb{E}\bigl[X(t)\bigr] = \lambda t - \lambda\theta \int_0^t s \, e^{-\theta(t-s)} \, \mathrm{d}s = \frac{\lambda(1 - e^{-\theta t})}{\theta} \,.$$

12. *First method.* Denote the desired time by T and, as usual, by T_1 the first arrival time. Clearly, if $T_1 \ge c$, then $T = c$. Otherwise, we aply the strong time-homogeneous Markov property: loosely speaking, after the first arrival, everything starts at new. Therefore,

$$\begin{aligned} \mathbb{E}(T \mid T_1) &= \begin{cases} c & ;\ T_1 \ge c \\ T_1 + \mathbb{E}(T) & ;\ T_1 < c \end{cases} = \\ &= c \, \mathbb{1}(T_1 \ge c) + \bigl(T_1 + \mathbb{E}(T)\bigr) \, \mathbb{1}(T_1 < c) \,. \end{aligned}$$

Taking the expectation, we obtain:

$$\begin{aligned} \mathbb{E}(T) &= c \, \mathbb{P}(T_1 \ge c) + \mathbb{E}\bigl[T_1 \, \mathbb{1}(T_1 < c)\bigr] + \mathbb{E}(T) \, \mathbb{P}(T_1 < c) = \\ &= c\lambda \int_c^\infty e^{-\lambda t} \, \mathrm{d}t + \lambda \int_0^c t \, e^{-\lambda t} \, \mathrm{d}t + \mathbb{E}(T) \int_0^c e^{-\lambda t} \, \mathrm{d}t = \\ &= \frac{1 - e^{-\lambda c}}{\lambda} + \mathbb{E}(T)\bigl(1 - e^{-\lambda c}\bigr) \,, \end{aligned}$$

leading to:

$$\mathbb{E}(T) = \frac{e^{\lambda c} - 1}{\lambda} \,.$$

Second method. Number the arriving cars by $1, 2, 3, \ldots$ and denote by N the number of the (first) car which allows the hen to safely cross the road. Observe that:

$$\{N = n\} = \{T_1 < c,\ T_2 < c, \ldots, T_{n-1} < c,\ T_n \ge c\} \,. \qquad (*)$$

On the event $\{N = n\}$, we have $T = T_1 + T_2 + \cdots + T_{n-1} + c$. Next, by independence, we have:

$$\begin{aligned} \mathbb{E}(T_i \mid N = n) &= \mathbb{E}(T_i \mid T_i < c) = \frac{\mathbb{E}\bigl[T_i \, \mathbb{1}(T_i < c)\bigr]}{\mathbb{P}(T_i < c)} = \frac{1}{1 - e^{-\lambda c}} \int_0^c \lambda t \, e^{-\lambda t} \, \mathrm{d}t = \\ &= \frac{1}{\lambda} - \frac{c \, e^{-\lambda c}}{1 - e^{-\lambda c}} \,. \end{aligned}$$

for all $i = 1, 2, \ldots, n-1$. As a result,

$$\mathbb{E}(T \mid N) = (N - 1)\left(\frac{1}{\lambda} - \frac{c \, e^{-\lambda c}}{1 - e^{-\lambda c}}\right) + c \,.$$

Since N follows the geometric distribution $\mathrm{Geom}\big(\mathbb{P}(T_1 \ge c)\big) = \mathrm{Geom}\big(e^{-\lambda c}\big)$, its expectation equals $\mathbb{E}(N) = e^{\lambda c}$. Therefore,

$$\mathbb{E}(T) = \big(\mathbb{E}(N) - 1\big)\left(\frac{1}{\lambda} - \frac{c\,e^{-\lambda c}}{1 - e^{-\lambda c}}\right) + c = \frac{e^{\lambda c} - 1}{\lambda}\,.$$

Third method. Keeping the notation from the second method and recalling $(*)$, observe that N is a stopping time. Write:

$$T = T_1 + T_2 + \cdots + T_N + c - T_N\,.$$

By Wald's equation, we have:

$$\mathbb{E}(T_1 + T_2 + \cdots + T_N) = \mathbb{E}(T_1)\,\mathbb{E}(N) = \frac{e^{\lambda c}}{\lambda}\,.$$

Next, by $(*)$, independence and memorylessness of the exponential distribution, we have:

$$\mathbb{E}(T_N \mid N = n) = \mathbb{E}(T_n \mid T_n \ge c) = c + \frac{1}{\lambda}\,,$$

leading to $\mathbb{E}(T_N) = c + \frac{1}{\lambda}$. Collecting all together, we obtain $\mathbb{E}(T) = \frac{e^{\lambda c} - 1}{\lambda}$, which is, of course, the same as before.

13. *First method.* Denote by T_1 the time of the first sharing after the initial one. If $T_1 \le \delta$, Tony gets his prize in that moment, so that $T = T_1$. Otherwise, everything starts once again: from that moment on, Tony has to wait for time T' to get the prize; the conditional distribution of T' given the whole history matches the unconditional distribution of T. Therefore, we have:

$$T = T_1 + T'\,\mathbb{1}(T_1 > \delta)\,,$$

implying:

$$\mathbb{E}(T) = \mathbb{E}(T_1) + \mathbb{E}(T)\,\mathbb{P}(T_1 > \delta)\,.$$

Since $T_1 \sim \mathrm{Exp}(\lambda)$, we have $\mathbb{E}(T_1) = 1/\lambda$ and $\mathbb{P}(T_1 > \delta) = e^{-\lambda\delta}$, leading to:

$$\mathbb{E}(T) = \frac{1}{\lambda} + e^{-\lambda\delta}\,\mathbb{E}(T)\,.$$

As a result, we have:

$$\mathbb{E}(T) = \frac{1}{\lambda\big(1 - e^{-\lambda\delta}\big)}\,.$$

Second method. Enumerate the sharings according to their times, assigning 0 to the initial sharing. Denote by N the number of the sharing when Tony gets his prize. Next, denoting by $T_1, T_2, \ldots$ the inter-arrival times in the sharing process, observe that:

$$\{N = n\} = \{T_1 > \delta,\ T_2 > \delta, \ldots, T_{n-1} > \delta,\ T_n \le \delta\}\,.$$

In the event $\{N = n\}$, we have $T = T_1 + T_2 + \cdots + T_n$. Next, for $i = 1, 2, \ldots, n-1$, we have:

$$\mathbb{E}(T_i \mid N = n) = \mathbb{E}(T_i \mid T_i > \delta) = \frac{\mathbb{E}\big[T_i \, \mathbb{1}(T_i > \delta)\big]}{\mathbb{P}(T_i > \delta)} = \frac{1}{e^{-\lambda\delta}} \int_\delta^\infty \lambda t \, e^{-\lambda t} \, \mathrm{d}t = \frac{1}{\lambda} + \delta$$

(this follows from the memorylessness of the exponential distribution), while:

$$\mathbb{E}(T_n \mid N = n) = \mathbb{E}(T_n \mid T_n \le \delta) = \frac{\mathbb{E}\big[T_i \, \mathbb{1}(T_i < \delta)\big]}{\mathbb{P}(T_i \le \delta)} = \frac{1}{1 - e^{-\lambda\delta}} \int_0^\delta \lambda t \, e^{-\lambda t} \, \mathrm{d}t =$$
$$= \frac{1}{\lambda} - \frac{\delta \, e^{-\lambda\delta}}{1 - e^{-\lambda\delta}} \, .$$

Therefore,

$$\mathbb{E}(T \mid N) = \frac{N}{\lambda} + (N-1)\delta - \frac{\delta \, e^{-\lambda\delta}}{1 - e^{-\lambda\delta}} = N \left(\frac{1}{\lambda} + \delta \right) - \frac{\delta}{1 - e^{-\lambda\delta}} \, .$$

Since N follows the geometric distrubution $\mathrm{Geom}\big(\mathbb{P}(T_1 < \delta)\big) = \mathrm{Geom}\big(1 - e^{-\lambda\delta}\big)$, we have $\mathbb{E}(N) = 1 \,/\, \big(1 - e^{-\lambda\delta}\big)$. As a result,

$$\mathbb{E}(T) = \frac{\frac{1}{\lambda} + \delta}{1 - e^{-\lambda\delta}} - \frac{\delta}{1 - e^{-\lambda\delta}} = \frac{1}{\lambda\big(1 - e^{-\lambda\delta}\big)} \, .$$

Third method. With the notation from the second method, we have $T = T_1 + T_2 + \cdots + T_N$. Observe that N is a stopping time because the event $\{N = n\}$ is uniquely determined by $T_1, T_2, \ldots, T_n$ (see above). By Wald's equation, we have:

$$\mathbb{E}(T) = \mathbb{E}(N) \, \mathbb{E}(T_1) = \frac{1}{\lambda\big(1 - e^{-\lambda\delta}\big)} \, .$$

14. Describing the process by a random set $\mathcal{P}$, observe that the age is uniquely deternined by $\mathcal{P}^{\to\!\!\!\!/\, t}$ and the exceedance uniquely by $\mathcal{P}^{t\to}$. Since these two random sets are independent, so are the age and the exceedance. Moreover, E_t matches the first arrival time in the process $\mathcal{P}^{t\to}$, which follows the same distribution as $\mathcal{P}$. Therefore, $E_t \sim \mathrm{Exp}(\lambda)$. Next, for $0 \le s \le t$, we have:

$$\mathbb{P}(A_t < s) = \mathbb{P}(N_t - N_{t-s} \ge 1) = 1 - e^{-\lambda s} \, ,$$

and therefore, for $0 \le s < t$,

$$F_{A_t}(s) = \mathbb{P}(A_t \le s) = 1 - e^{-\lambda s} \, .$$

Since A_t is bounded from above by t, we have $F_{A_t}(s) = 1$ for all $s \ge t$. Thus, the random variable A_t is neither discrete nor continuous. However, it follows the same distribution as the random variable $\min\{\tilde{A}_t, t\}$, where $\tilde{A}_t \sim \mathrm{Exp}(\lambda)$ (imagine that the given homogeneous Poisson process is extended to negative time).

There are at leat two methods to find the distribution of $A_t + E_t$.

First method. The distribution of the sum $A_t + E_t$ matches the distribution of $\min\{\tilde{A}_t, t\} + E_t$, where $\tilde{A}_t$ and E_t are independent with the exponential distribution $\mathrm{Exp}(\lambda)$. Therefore, for $0 \le s < t$, we have:

$$\begin{aligned}
\mathbb{P}(A_t + E_t \le s) &= \mathbb{P}(\tilde{A}_t + E_t \le s) = \\
&= \int_0^s \int_0^{s-x} f_{\tilde{A}_t}(x)\, f_{E_t}(y)\, \mathrm{d}y\, \mathrm{d}x = \\
&= \lambda^2 \int_0^s e^{-\lambda x} \int_0^{s-x} e^{-\lambda y}\, \mathrm{d}y\, \mathrm{d}x = \\
&= \lambda \int_0^s \left(e^{-\lambda x} - e^{-\lambda s}\right) \mathrm{d}x = \\
&= 1 - e^{-\lambda s} - \lambda s\, e^{-\lambda s}
\end{aligned}$$

and, for $s \ge t$,

$$\begin{aligned}
\mathbb{P}(A_t + E_t \le s) &= \mathbb{P}(A_t = t, E_t \le s - t) + \mathbb{P}(A_t < t, A_t + E_t \le s) = \\
&= \mathbb{P}(\tilde{A}_t \ge t, E_t \le s - t) + \mathbb{P}(\tilde{A}_t \le t, \tilde{A}_t + E_t \le s) = \\
&= \lambda^2 \int_t^\infty e^{-\lambda x}\, \mathrm{d}x \int_0^{s-t} e^{-\lambda y}\, \mathrm{d}y + \lambda^2 \int_0^t e^{-\lambda x} \int_0^{s-x} e^{-\lambda y}\, \mathrm{d}y = \\
&= e^{-\lambda t}\left(1 - e^{-\lambda(s-t)}\right) + \lambda \int_0^t \left(e^{-\lambda x} - e^{-\lambda s}\right) \mathrm{d}x = \\
&= e^{-\lambda t}\left(1 - e^{-\lambda(s-t)}\right) + 1 - e^{-\lambda t} - \lambda t\, e^{-\lambda s} = \\
&= 1 - e^{-\lambda s} - \lambda t\, e^{-\lambda s}\,.
\end{aligned}$$

Summing up, we obtain:

$$F_{A_t+E_t}(s) = \begin{cases} 0 & ;\ s \le 0 \\ 1 - e^{-\lambda s} - \lambda s\, e^{-\lambda s} & ;\ 0 \le s < t \\ 1 - e^{-\lambda s} - \lambda t\, e^{-\lambda s} & ;\ s \ge t \end{cases}$$

Second method. Consider the conditional cumulative ditribution function of $A_t + E_t$ given E_t:

$$\begin{aligned}
F_{A_t+E_t \mid E_t}(s \mid y) &= \mathbb{P}(A_t + E_t \le s \mid E_t = y) = \mathbb{P}(A_t \le s - y \mid E_t = y) = \\
&= \mathbb{P}(A_t \le s - y)\,.
\end{aligned}$$

From F_{A_t}, we obtain:

$$\begin{aligned}
F_{A_t+E_t \mid E_t}(s \mid y) &= \begin{cases} 0 & ;\ s \le y \\ 1 - e^{-\lambda(s-y)} & ;\ y \le s \le y + t \\ 1 & ;\ s \ge y + t \end{cases} = \\
&= \begin{cases} 1 & ;\ y \le s - t \\ 1 - e^{-\lambda(s-y)} & ;\ s - t \le y \le s \\ 0 & ;\ y \ge s \end{cases}.
\end{aligned}$$

Now we apply $F_{A_t+E_t}(s) = \lambda \int_0^\infty F_{A_t+E_t|E_t}(s \mid y)\, e^{-\lambda y}\, \mathrm{d}y$. For $0 \le s \le t$, we have:

$$F_{A_t+E_t}(s) = \lambda \int_0^s \bigl(1 - e^{-\lambda(s-y)}\bigr) e^{-\lambda y}\, \mathrm{d}y = 1 - e^{-\lambda s} - \lambda s\, e^{\lambda s}$$

and for $s \ge t$, we have:

$$F_{A_t+E_t}(s) = \lambda \int_0^{s-t} e^{-\lambda y}\, \mathrm{d}y + \lambda \int_{s-t}^s \bigl(1 - e^{-\lambda(s-y)}\bigr) e^{-\lambda y}\, \mathrm{d}y = 1 - e^{-\lambda s} - \lambda t\, e^{\lambda s}\,.$$

Combining both formulas, we obtain the same cumulative distribution function as before.

Now observe that the desired cumulative distribution function is continuous, even more, absolutely continuous. Hence the random variable $A_t + E_t$ is also continuous in the sense that it has a density, which can be obtained by differentiation resulting in:

$$f_{A_t+E_t}(s) = \begin{cases} 0 & ;\ s < 0 \\ \lambda^2 s\, e^{-\lambda s} & ;\ 0 < s < t \\ (\lambda^2 t + \lambda)\, e^{-\lambda s} & ;\ s > t \end{cases}\,.$$

4 Marking, Thinning, Superposition

1. The described process is equivalent to the union of a homogeneous Poisson process of the lorries with intensity 4 lorries per hour and a homogeneous Poisson process of the cars with intensity 36 cars per hour.

a) $1 - e^{-4} \doteq 0{\cdot}982$.

b) Because of independence of the process of lorries and the process of cars, the conditional expectation equals the unconditional, i. e., 36.

c) Using the original description of the process and independence, we find that the desired probability equals $\binom{50}{5} \cdot 0{\cdot}1^5 \cdot 0{\cdot}9^{45} \doteq 0{\cdot}185$.

d) Denoting by A_t the number of cars until time t and by T the arrival time of the first lorry, the desired expectation is $\mathbb{E}(A_T)$. Conditioning on T, we have $\mathbb{E}(A_T \mid T) = 36T$ and therefore $\mathbb{E}(A_T) = 36\,\mathbb{E}(T) = 36/4 = 9$.

2. a) Consider the proces of all replacements and focus on the moment of the n-th replacement or the beginning if $n = 0$. From the strong time-homogeneous Markov property, it follows that the time until the next replacement along with the maintainer who performs it has the same distribution as in the following situation:

- Take independent random variables $U \sim \mathrm{Exp}(0{\cdot}005)$ and $V \sim \mathrm{Exp}(0{\cdot}01)$.
- The time until the next replacement equals $\min\{U, V\}$.
- The next replacement is performed by the first maintainer if $U < V$ and by the second one otherwise.

This is true regardless whether the current replacement is due to the first or to the second maintainer, or whether we are at the beginning.

Now observe that the above description is also valid in the case where there are two lamps, each of them with one bulb. Both bulbs are replaced as soon as they blow out and the life time of the first follows the exponential distribution $\mathrm{Exp}(0{\cdot}005)$, while for the second bulb, we have $\mathrm{Exp}(0{\cdot}01)$, assuming independence of all bulbs. This case is *not the same* as in the original problem, but it follows exactly the same distribution. Therefore, the actual changes form a homogeneous Poisson process with intensity $0{\cdot}015$ replacement a day, so that each bulb is replaced in $1/0{\cdot}015 \doteq 67$ days on average.

b) $\dfrac{0{\cdot}005}{0{\cdot}015} = \dfrac{1}{3}$.

3. *First method.* Denote by M the time needed to get the man, by Z the time needed to get the two women, and by T the time needed to get all desired actors. Clearly, $T = \max\{M, Z\}$. We have $M \sim \mathrm{Exp}(2)$ and $Z \sim \mathrm{Gama}(2, 1)$, so that for $t > 0$,

$$f_M(t) = 2\,e^{-2t}\,, \qquad F_M(t) = 1 - e^{-2t}\,,$$
$$f_Z(t) = t\,e^{-t}\,, \qquad F_Z(t) = 1 - (1+t)e^{-t}\,.$$

Thanks to the independence, we have:

$$F_T(t) = F_M(t)\,F_Z(t) = 1 - (1+t)e^{-t} - e^{-2t} + (1+t)e^{-3t}\,.$$

The expectation can be computed either by means of the desity:

$$f_T(t) = t\,e^{-t} + 2\,e^{-2t} - (2+3t)\,e^{-3t}\,,$$
$$\mathbb{E}(T) = \int_0^\infty \Big(t^2\,e^{-t} + 2t\,e^{-2t} - (2t+3t^2)\,e^{-3t}\Big)\,\mathrm{d}t = \frac{37}{18}$$

or by means of the survival function:

$$\mathbb{E}(T) = \int_0^\infty \big(1 - F_T(t)\big)\,\mathrm{d}t = \int_0^\infty \Big((1+t)e^{-t} + 2\,e^{-2t} - 3(1+t)\,e^{-3t}\Big)\,\mathrm{d}t = \frac{37}{18}\,.$$

Second method. Consider the united process of the arrivals of men and women. This is a homogeneous Poisson process with intensity 3, where each arrival represents a woman with probability 1/3 and a man with probability 2/3. Denote by N the first arrival after which it is true that at least two women and at least one man has arrived. Denoting by $T_1, T_2, \ldots$ the inter-arrival times in the united process, we have $T = T_1 + T_2 + \cdots + T_n$. By Wald's equation, we than have:

$$\mathbb{E}(T) = \mathbb{E}(N)\,\mathbb{E}(T_1) = \frac{1}{3}\,\mathbb{E}(N)\,.$$

Thus, the problem has been reduced to the computation of the expectation:

$$\mathbb{E}(N) = \sum_{n=0}^\infty \mathbb{P}(N > n)\,.$$

Clearly, $\mathbb{P}(N > 0) = \mathbb{P}(N > 1) = \mathbb{P}(N > 2) = 1$. For $n \ge 2$, $\{N > n\}$ matches the event that among the people that arrived up to the n-th arrival, there are either only women or only men or exactly one woman and $n-1$ men. Consequently,

$$\mathbb{P}(N > n) = \left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n + n\left(\frac{2}{3}\right)^{n-1}$$

(for $n = 0$ and $n = 1$, this calculation is invalid because the underlying subevents are not paiwise disjoint). Therefore,

$$\mathbb{E}(N) = 2 + \sum_{n=2}^\infty \left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n + n\left(\frac{2}{3}\right)^{n-1}\right].$$

Making use of the formulas:

$$\sum_{n=0}^\infty q^n = \frac{1}{1-q}\,, \qquad \sum_{n=0}^\infty n q^{n-1} = \frac{1}{(1-q)^2}\,; \qquad -1 < q < 1$$

we finally obtain $\mathbb{E}(N) = 37/6$ and consequently $\mathbb{E}(T) = 37/18$, which is the same as beofore.

4. *First method.* Denote by Z_3 the time when the wife is ready to buy a car and by M_2 the time when the husband is ready. These two random variables are independent with $Z_3 \sim \text{Gama}(3, \lambda)$ and $M_2 \sim \text{Gama}(2, \mu)$.

a) The desired probability equals:

$$\begin{aligned}\mathbb{P}(Z_3 < M_2) &= \int_0^\infty \int_x^\infty f_{Z_3}(x) f_{M_2}(y)\,\mathrm{d}y\,\mathrm{d}x = \\ &= \frac{\lambda^3\mu^2}{2} \int_0^\infty x^2\, e^{-\lambda x} \int_x^\infty y\, e^{-\mu y}\,\mathrm{d}y\,\mathrm{d}x = \\ &= \frac{\lambda^3}{2} \int_0^\infty x^2(\mu x + 1)\, e^{-(\lambda+\mu)x}\,\mathrm{d}x = \\ &= \frac{\lambda^3(\lambda + 4\mu)}{(\lambda+\mu)^4}\,.\end{aligned}$$

b) The time needed to buy a car is the minimum of the random variables Z_3 and M_2. We make use of the survival function:

$$\mathbb{P}(M_2 > t) = (\mu t + 1)e^{-\mu t}\,, \qquad \mathbb{P}(Z_3 > t) = \frac{1}{2}\,(\lambda^2 t^2 + 2\lambda t + 2)e^{-\lambda t}\,.$$

As a result, we have:

$$\begin{aligned}\mathbb{P}(T > t) &= \mathbb{P}(Z_3 > t, M_2 > t) = \mathbb{P}(Z_3 > t)\,\mathbb{P}(M_2 > t) = \\ &= \frac{1}{2}\big(\lambda^2\mu t^3 + \lambda(\lambda + 2\mu)t^2 + 2(\lambda+\mu)t + 2\big)e^{-(\lambda+\mu)t}\,.\end{aligned}$$

which (see Problem 2) yields:

$$\begin{aligned}\mathbb{E}(T) &= \int_0^\infty \mathbb{P}(T > t)\,\mathrm{d}t = \frac{3\lambda^2\mu}{(\lambda+\mu)^4} + \frac{\lambda(\lambda+2\mu)}{(\lambda+\mu)^3} + \frac{1}{\lambda+\mu} + \frac{1}{\lambda+\mu} = \\ &= \frac{3\lambda^3 + 12\lambda^2\mu + 8\lambda\mu^2 + 2\mu^3}{(\lambda+\mu)^4}\,.\end{aligned}$$

Second method. We apply the fact that the marks (i. e., brands) corresponding to particular arrivals (i. e., offers) are independent and identically distributed: each offer corresponds to wife's brand with probability $\lambda/(\lambda+\mu)$ and to husband's brand with probability $\mu/(\lambda+\mu)$.

a) The event that they buy a car according to wife's choice corresponds to the following beginnings of the mark sequence:

$$\text{WWW}\,,\ \text{HWWW}\,,\ \text{WHWW}\,,\ \text{WWHW}\,.$$

Therefore, the desired probability equals:

$$\left(\frac{\lambda}{\lambda+\mu}\right)^3 + 3\left(\frac{\lambda}{\lambda+\mu}\right)^3 \frac{\mu}{\lambda+\mu} = \frac{\lambda^3(\lambda+4\mu)}{(\lambda+\mu)^4}\,.$$

b) We make use of Wald's equation. Consider the times between particular offers of either brand, i. e., the inter-arrival times of the united process. As usual, denote them by $T_1, T_2, T_3, \ldots$. Then we may write $T = T_1 + T_2 + \cdots + T_N$, where N denotes the number of the offer where a car is eventually bought. For each time interval, we know its length (T_n) and the brand of the offer appearing at the end of the interval. With respect to this information, N is a stoping time. As the united offers form a homogeneous Poisson process with intensity $\lambda + \mu$, we have $\mathbb{E}(T_n) = 1/(\lambda + \mu)$. By Wald's equation, it suffices to compute $\mathbb{E}(N)$.

The possible sequences of the brands of the offers along with their probabilities and lengths (N) are given in the table below:

Sequence	WWW	HWWW	WHWW	WWHW
Length	3	4	4	4
Probability	$\frac{\lambda^3}{(\lambda+\mu)^3}$	$\frac{\lambda^3\mu}{(\lambda+\mu)^4}$	$\frac{\lambda^3\mu}{(\lambda+\mu)^4}$	$\frac{\lambda^3\mu}{(\lambda+\mu)^4}$

Sequence	HH	WHH	HWH	WWHH	WHWH	HWWH
Length	2	3	3	4	4	4
Probability	$\frac{\mu^2}{(\lambda+\mu)^2}$	$\frac{\lambda\mu^2}{(\lambda+\mu)^3}$	$\frac{\lambda\mu^2}{(\lambda+\mu)^3}$	$\frac{\lambda^2\mu^2}{(\lambda+\mu)^4}$	$\frac{\lambda^2\mu^2}{(\lambda+\mu)^4}$	$\frac{\lambda^2\mu^2}{(\lambda+\mu)^4}$

This, $N \sim \begin{pmatrix} 2 & 3 & 4 \\ \frac{\mu^2}{(\lambda+\mu)^2} & \frac{\lambda^3+2\lambda\mu^2}{(\lambda+\mu)^3} & \frac{3\lambda^2\mu}{(\lambda+\mu)^3} \end{pmatrix}$, which gives:

$$\mathbb{E}(N) = \frac{3\lambda^3 + 12\lambda^2\mu + 8\lambda\mu^2 + 2\mu^3}{(\lambda + \mu)^3}$$

and, finally, by Wald's equation,

$$\mathbb{E}(T) = \frac{3\lambda^3 + 12\lambda^2\mu + 8\lambda\mu^2 + 2\mu^3}{(\lambda + \mu)^4},$$

which is the same as before.

5. *First method.* The event that exactly exactly one arrival in the second process occurs before the first arrival in the first process matches the event that among all arrivals, the first one is due to the second process and the second one to the first process. The probability of this event equals $\dfrac{\lambda\mu}{(\lambda + \mu)^2}$.

More generally, all arrivals form a Bernoulli sequence of trials, where successes are identified with the arrivals due to the first process. Denoting by X the number of arrivals in the second process before the first arrival in the first process, $X + 1$ then denotes the number of all trials up to and including the first successful one. Therefore, $X + 1 \sim \mathrm{Geom}\left(\dfrac{\lambda}{\lambda + \mu}\right)$, which yields $\mathbb{E}(X) = \dfrac{\mu}{\lambda}$.

Second method. Denoting by T the first arrival time in the first process, the conditional distribution of X given T is Poisson $\mathrm{Pois}(\mu T)$, so that:

$$\mathbb{P}(X = 1 \mid T) = \mu T\, e^{-\mu T} \quad \text{and} \quad \mathbb{E}(X \mid T) = \mu T\,.$$

Combining this with the fact that $T \sim \mathrm{Exp}(\mu)$, we obtain:

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{E}\big[\mu T\, e^{-\mu T}\big] = \mu \int_0^\infty t\, e^{-\mu t}\, \lambda e^{-\lambda t}\, \mathrm{d}t = \frac{\lambda\mu}{(\lambda+\mu)^2}\,, \\ \mathbb{E}(X) &= \mu\, \mathbb{E}(T) = \frac{\mu}{\lambda}\,. \end{aligned}$$

6. Denote by H the event that exactly one student has arrived during the first half an hour. This event is a disjoint union of two events: the event H_F that this only student arises from financial mathematics and the event H_S that he/she arises from general mathematics. From the theory of marking, it follows that:

$$\mathbb{P}(H_F \mid H) = \frac{4}{6} = \frac{2}{3} \quad \text{and} \quad \mathbb{P}(H_S \mid H) = \frac{2}{6} = \frac{1}{3}\,.$$

Denote by T_F the arrival time of the first student of financial mathematics. Given H_F, this random variable is uniformly distributed over the interval from 0 to 1/2 (measured in hours), while given H_S, the random variable $T_F - 1/2$ follows the exponential distribution Exp(4). Therefore:

$$\mathbb{E}(T_F \mid H_F) = \frac{1}{4} \quad \text{in} \quad \mathbb{E}(T_F \mid H_S) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}\,.$$

Finally, the desired conditional expectation equals:

$$\begin{aligned} \mathbb{E}(T_F \mid H) &= \frac{\mathbb{E}\big(T_F\, \mathbb{1}(H)\big)}{\mathbb{P}(H)} = \\ &= \frac{\mathbb{E}\big(T_F\, \mathbb{1}(H_F)\big) + \mathbb{E}\big(T_F\, \mathbb{1}(H_S)\big)}{\mathbb{P}(H)} = \\ &= \frac{\mathbb{P}(H_F)\, \mathbb{E}(T_F \mid H_F) + \mathbb{P}(H_S)\, \mathbb{E}(T_F \mid H_S)}{\mathbb{P}(H)} = \\ &= \mathbb{P}(H_F \mid H)\, \mathbb{E}(T_F \mid H_F) + \mathbb{P}(H_S \mid H)\, \mathbb{E}(T_F \mid H_S) = \\ &= \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \\ &= \frac{5}{12}\,, \end{aligned}$$

or, equivalently, 25 minutes.

7. In the union of the two processes, each arrival arises from the first process with probability $\lambda/(\lambda+\mu)$ and from the second process with probability $\mu/(\lambda+\mu)$. Saying that there were exactly k arrivals in the first process before the n-th arrival in the second process is equivalent to saying that among the first $n+k-1$ arrivals in the united process, there were k arrivals in the first process and $n-1$ arrivals in the second process and that the $(n+k)$-th process in the united process arises from the second process. Denoting by X the number of arrivals in the first process before the n-th arrival in the second process, we have:

$$\mathbb{P}(X = k) = \frac{(n+k-1)!}{k!\,(n-1)!} \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{n-1} \quad ; \quad k = 0, 1, 2, \ldots$$

In other words, the random variable X follows the negative binomial distribution $\mathrm{NegBin}\bigl(n, \frac{\lambda}{\lambda+\mu}\bigr)$, shifted n to the left.

8. The given walk reaches the point (i, j) if and only if among the first $i + j$ total arrivals, there are exactly i due to the the first and exactly j due to the second process. The probability of this event equals:

$$\frac{(i+j)!}{i!\,j!} \frac{\lambda^i \mu^j}{(\lambda+\mu)^{i+j}} .$$

9. The event that the bank passes all the tests matches the event that the number of the tests the bank fails equals zero. The tests the bank fails correspond to the pairs (t, s), where $s > at$. Thus, the desired probability equals $e^{-\theta}$, where:

$$\theta = \frac{4\lambda}{\pi} \iint_{0<at<s} \frac{1}{(1+s^2)^2} \,\mathrm{d}t\,\mathrm{d}s = \frac{4\lambda}{\pi a} \int_0^\infty \frac{s}{(1+s^2)^2} \,\mathrm{d}s = \frac{2\lambda}{\pi a} .$$

5 General Poisson Process

1. a) Pois(24)

b) $e^{-4} \doteq 0\text{˙}0183$.

c) For $10 \le s \le 12$, observe that:

$$\begin{aligned} F_{S_2 \mid N_{12}=2}(s) &= \mathbb{P}(N_s \ge 2 \mid N_{12} = 2) = \\ &= \frac{\mathbb{P}(N_s = 2, N_{12} - N_s = 0)}{\mathbb{P}(N_{12} = 2)} = \\ &= \frac{(s-10)^4}{16} \,. \end{aligned}$$

In further computation, it is easier to work with $S_2 - 10$. For $0 \le t \le 2$, we have:

$$F_{S_2 - 10 \mid N_{12}=2}(t) = \frac{t^2}{16} \,, \qquad f_{S_2 - 10 \mid N_2 = 2}(t) = \frac{t^3}{4} \,,$$

leading to $\mathbb{E}(S_2 - 10 \mid N_2 = 2) = \frac{8}{5} = 1\frac{3}{5}$ and therefore $\mathbb{E}(S_2 \mid N_2 = 2) = 11{:}36$. Next,

$$\begin{aligned} F_{S_1 \mid N_{12}=2}(s) &= \mathbb{P}(N_s \ge 1 \mid N_{12} = 2) = \\ &= \frac{\mathbb{P}(N_s = 1, N_{12} - N_s = 1) + \mathbb{P}(N_s = 2, N_{12} - N_s = 0)}{\mathbb{P}(N_{12} = 2)} = \\ &= \frac{(s-10)^2}{2} - \frac{(s-10)^4}{16} \,, \end{aligned}$$

which gives:

$$F_{S_1 - 10 \mid N_{12}=2}(t) = \frac{t^2}{2} - \frac{t^4}{16} \,, \qquad f_{S_1 - 10 \mid N_{12} = 2}(t) = t - \frac{t^3}{4}$$

and $\mathbb{E}(S_1 - 10 \mid N_2 = 2) = \frac{16}{15} = 1\frac{1}{15}$, so that $\mathbb{E}(S_1 \mid N_2 = 2) = 11{:}04$.

2. Noting that $N_t \sim \text{Pois}\left(\int_0^t \frac{a}{1+s}\,\mathrm{d}s\right) = \text{Pois}\big(a \ln(1+t)\big)$, we obtain:

$$F_{T_1}(t) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-a\ln(1+t)} = 1 - \frac{1}{(1+t)^a} \,.$$

The expectation can be computed either by means of the density:

$$f_{T_1}(t) = \frac{a}{(1+t)^{a+1}} \,, \qquad \mathbb{E}(T_1) = a \int_0^\infty \frac{t\,\mathrm{d}t}{(1+t)^{a+1}} = \frac{1}{a-1} \,; \qquad a > 1$$

or by means of the survival function:

$$\mathbb{E}(T) = \int_0^\infty \big(1 - F_T(t)\big)\,\mathrm{d}t = a \int_0^\infty \frac{\mathrm{d}t}{(1+t)^{a+1}} = \frac{1}{a-1} \,; \qquad a > 1 \,.$$

3. a) Denoting by N_t the number of latecomers appearing up to time t, $N_s - N_t$ then represents the number of latecomers arriving with delay from t to s, provided that $t \le s$. Observe that:

$$\begin{aligned} N_t &\sim \mathrm{Pois}\left(\int_0^t e^{-u}\,\mathrm{d}u\right) = \mathrm{Pois}\big(1 - e^{-t}\big)\,, \\ N_s - N_t &\sim \mathrm{Pois}\left(\int_t^s e^{-u}\,\mathrm{d}u\right) = \mathrm{Pois}\big(e^{-t} - e^{-s}\big)\,. \end{aligned}$$

In addition, N_t and $N_s - N_t$ are independent. Extending the definition of N_s to $s = \infty$ (the total number of latecomers), $N_\infty - N_t$ denotes the number of latecomers which arrive with delay more than t; note that $N_\infty - N_t \sim \mathrm{Exp}(e^{-t})$.

The event that exactly one latecomer appears, arriving with delay more than two months, can be expressed as:

$$A := \{N_2 = 0, N_\infty - N_2 = 1\}\,.$$

Thanks to the independence, its probability equals:

$$\mathbb{P}(A) = \mathbb{P}(N_2 = 0)\,\mathbb{P}(N_\infty - N_2 = 1) = e^{e^{-2}-1}\,e^{-2}\,e^{-e^{-2}} = e^{-3} \doteq 0{\cdot}0498\,.$$

b) What we have to compute is $\mathbb{E}(S_1 \mid A)$.

First method. Consider the conditional cumulative distribution function:

$$\begin{aligned} F_{S_1|A}(t) &= \mathbb{P}(S_1 \le t \mid A) = \\ &= \mathbb{P}(N_t \ge 1 \mid A) = \\ &= \frac{\mathbb{P}(N_t \ge 1, N_2 = 0, N_\infty - N_2 = 1)}{\mathbb{P}(N_2 = 0, N_\infty - N_2 = 1)}\,. \end{aligned}$$

Clearly, $F_{S_1|A}(t) = 0$ for $t \le 2$; for $t \ge 2$, we have:

$$\begin{aligned} F_{S_1|A}(t) &= \frac{\mathbb{P}(N_2 = 0, N_t - N_2 = 1, N_\infty - N_t = 0)}{\mathbb{P}(N_2 = 0, N_\infty - N_2 = 1)} = \\ &= \frac{\mathbb{P}(N_2 = 0)\,\mathbb{P}(N_t - N_2 = 1)\,\mathbb{P}(N_\infty - N_t = 0)}{\mathbb{P}(N_2 = 0)\,\mathbb{P}(N_\infty - N_2 = 1)} = \\ &= 1 - e^{2-t}\,. \end{aligned}$$

The desired conditional expecation can be derived directly from the survival function:

$$\mathbb{E}(S_1 \mid A) = \int_0^\infty \big(1 - F_{S_1|A}(t)\big)\,\mathrm{d}t = 2 + \int_2^\infty e^{2-t}\,\mathrm{d}t = 3\,.$$

Second method. Thanks to the independence of the restrictions of the process to time intervals up to two months and more than two months of delay, the conditional distribution of the arrival time of the only latecomer given A matches the conditional distribution of the arrival time of the only latecomer with a delay of more than two months, given that there was exactly one latecomer with more than two months of delay (regardless what happened less than two months from the deadline). More precisely, denoting by $S_{1;(2,\infty)}$ the arrival time of the first latecomer with delay of more than two months, observe that

$$\begin{aligned}\mathbb{P}(T_1 > t \mid A) &= \mathbb{P}(N_t = 0 \mid N_2 = 0, N_\infty - N_2 = 1) = \\ &= \frac{\mathbb{P}(N_t = 0, N_2 = 0, N_\infty - N_2 = 1)}{\mathbb{P}(N_2 = 0, N_\infty - N_2 = 1)} = \\ &= \frac{\mathbb{P}(N_2 = 0, N_t - N_2 = 0, N_\infty - N_2 = 1)}{\mathbb{P}(N_2 = 0, N_\infty - N_2 = 1)} = \\ &= \frac{\mathbb{P}(N_2 = 0)\, \mathbb{P}(N_t - N_2 = 0, N_\infty - N_2 = 1)}{\mathbb{P}(N_2 = 0)\, \mathbb{P}(N_\infty - N_2 = 1)} = \\ &= \frac{\mathbb{P}(N_t - N_2 = 0, N_\infty - N_2 = 1)}{\mathbb{P}(N_\infty - N_2 = 1)} = \\ &= \mathbb{P}(N_t - N_2 = 0 \mid N_\infty - N_2 = 1) = \\ &= \mathbb{P}(S_{1;(2,\infty)} > t \mid N_\infty - N_2 = 1)\,.\end{aligned}$$

for all $t > 2$. As a result, the corresponding conditional density equals:

$$f_{S_1 \mid A}(t) = f_{S_{1;(2,\infty)} \mid N_\infty - N_2 = 1}(t) = \frac{e^{-t}}{\int_2^\infty e^{-s}\, \mathrm{d}s} = e^{2-t}\,.$$

Integration yields:

$$\mathbb{E}(S_1 \mid A) = \int_2^\infty t\, e^{2-t}\, \mathrm{d}t = 3\,.$$

Alternatively, one can observe that the conditional distribution matches the exponential distribution Exp(1) shifted by 2 to the right, which has expectation 3, as before.

4. Denote by X the number of arrivals with the specified property and by N the total number of arrivals. Recall that given $\{N = n\}$, the set of arrivals follows the same distribution as the set $U_1, U_2, \ldots, U_n$, where $U_1, \ldots, U_n$ are independent with the exponential distribution $\mathrm{Exp}(\lambda)$. Therefore, the conditional distribution of X given $\{N = n\}$ matches the (unconditional) distribution of the sum:

$$\sum_{i=1}^n \mathbb{1}\big(\{U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n\} \cap (U_i, U_i + \delta] = \emptyset\big)$$

so that:

$$\begin{aligned}\mathbb{E}(X \mid N = n) &= \sum_{i=1}^n \mathbb{P}\big(\{U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n\} \cap (U_i, U_i + \delta] = \emptyset\big) = \\ &= n\, \mathbb{P}\big(\{U_1, U_2, \ldots, U_{n-1}\} \cap (U_n, U_n + \delta] = \emptyset\big)\,.\end{aligned}$$

The latter probabilities can be computed by conditioning on U_i. First, using the independence of the random variables $U_1, U_2, \dots, U_{n-1}$, observe that

$$\begin{aligned}&\mathbb{P}\bigl(\{U_1, U_2, \dots, U_{n-1}\} \cap (u, u+\delta] = \emptyset\bigr) = \\ &\quad = \mathbb{P}\bigl(U_1 \notin (u, u+\delta],\ U_2 \notin (u, u+\delta],\ \dots,\ U_{n-1} \notin (u, u+\delta]\bigr) = \\ &\quad = \mathbb{P}\bigl(U_1 \notin (u, u+\delta]\bigr)\,\mathbb{P}\bigl(U_2 \notin (u, u+\delta]\bigr) \cdots \mathbb{P}\bigl(U_{n-1} \notin (u, u+\delta]\bigr) = \\ &\quad = \Bigl(\mathbb{P}\bigl(U_1 \notin (u, u+\delta]\bigr)\Bigr)^n .\end{aligned}$$

for all $u \ge 0$. Next, compute

$$\mathbb{P}\bigl(U_j \notin (u, u+\delta]\bigr) = 1 - \lambda \int_u^{u+\delta} e^{-\lambda u}\,\mathrm{d}u = 1 - e^{-\lambda u}\bigl(1 - e^{-\lambda\delta}\bigr),$$

so that

$$\mathbb{P}\bigl(\{U_1, U_2, \dots, U_{n-1}\} \cap (u, u+\delta] = \emptyset\bigr) = \Bigl(1 - e^{-\lambda u}\bigl(1 - e^{-\lambda\delta}\bigr)\Bigr)^{n-1}.$$

However, since U_n is independent of the vector $(U_1, \dots, U_{n-1})$, we also have:

$$\mathbb{P}\bigl(\{U_1, U_2, \dots, U_{n-1}\} \cap (U_n, U_n+\delta] = \emptyset \mid U_n\bigr) = \Bigl(1 - e^{-\lambda U_n}\bigl(1 - e^{-\lambda\delta}\bigr)\Bigr)^{n-1},$$

leading to

$$\begin{aligned}&\mathbb{P}\bigl(\{U_1, U_2, \dots, U_{n-1}\} \cap (U_n, U_n+\delta] = \emptyset \mid U_n\bigr) = \\ &\quad = \mathbb{E}\left[\Bigl(1 - e^{-\lambda U_n}\bigl(1 - e^{-\lambda\delta}\bigr)\Bigr)^{n-1}\right] = \\ &\quad = \lambda \int_0^\infty \Bigl(1 - e^{-\lambda u}\bigl(1 - e^{-\lambda\delta}\bigr)\Bigr)^{n-1} e^{-\lambda u}\,\mathrm{d}u = \\ &\quad = \frac{1 - e^{-n\lambda\delta}}{n\bigl(1 - e^{-\lambda\delta}\bigr)} .\end{aligned}$$

Therefore,

$$\mathbb{E}(X \mid N = n) = \frac{1 - e^{-n\lambda\delta}}{1 - e^{-\lambda\delta}} .$$

Since $N \sim \mathrm{Pois}\bigl(\frac{a}{\lambda}\bigr)$, we finally find that:

$$\mathbb{E}(X) = \frac{e^{-a/\lambda}}{1 - e^{-\lambda\delta}} \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{a}{\lambda}\right)^n \bigl(1 - e^{-n\lambda\delta}\bigr) = \frac{1 - e^{-a(1-e^{-\lambda\delta})/\lambda}}{1 - e^{-\lambda\delta}} .$$

5. Denoting by N_t the number of arrivals up to time t, observe that:

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \exp\left(-\int_0^t \frac{\mathrm{d}u}{1+u}\right) = \frac{1}{1+t} .$$

Differentiating, we obtain the probability density function:

$$f_{T_1}(t) = \frac{1}{(1+t)^2} \,.$$

Now we turn to the distribution of T_2, which will again initially be represented by the probabilities $\mathbb{P}(T_2 > s)$. First, we shall compute the conditional probabilities

$$\mathbb{P}(T_2 > s \mid T_1) \,,$$

where we shall take advantage of the above-mentioned characterization of the conditional distribution given T_1 (and $\mathcal{P}^{\rightarrow\!|T_1}$). Like in T_1, we could use the basic equivalence

$$\{T_2 > s\} = \{S_2 > T_1 + s\} = \{N_{T_1+s} > 2\} \,.$$

However, it is more beneficial to express this event just in terms of $\mathcal{P}^{T\rightarrow}$ or $\mathcal{P} \cap (T_1, \infty)$. Such an expression is

$$\{T_2 > s\} = \{N_{T_1+s} - N_{T_1} = 0\} \,.$$

Given T_1, the process $\mathcal{P} \cap (T_1, \infty)$ is a Poisson process with intensity function $t \mapsto \frac{1}{1+t} \mathbb{1}(t > T_1)$. Therefore, given T_1, the random variable $N_{T_1+s} - N_{T_1}$ follows the Poisson distribution with parameter:

$$\int_{T_1}^{T_1+s} \frac{1}{1+t} \,\mathrm{d}t = \ln \frac{1+T_1+s}{1+T_1} \,.$$

As a result, we have:

$$\mathbb{P}(T_2 > s \mid T_1) = \mathbb{P}(N_{T_1+s} - N_{T_1} = 0 \mid T_1) = \exp\left(-\ln \frac{1+T_1+s}{1+T_1}\right) = \frac{1+T_1}{1+T_1+s} \,.$$

Integrating, we obtain the unconditional survival function:

$$\mathbb{P}(T_2 > s) = \int_0^\infty \frac{1+t}{1+t+s} f_{T_1}(t) \,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(1+t)(1+t+s)} = \frac{\ln(1+s)}{s}$$

and differentiation again yields the probability density function:

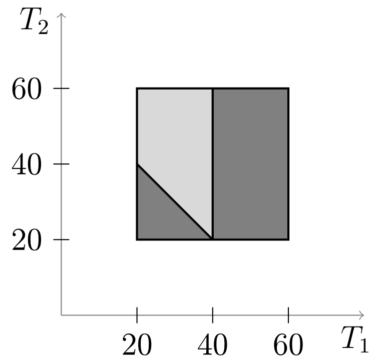
$$f_{T_2}(s) = \frac{(1+s)\ln(1+s) - s}{s^2(1+s)} \,.$$

6 Renewal Processes

1. a) The arrivals of the bus form a renewal process. Denoting the inter-arrival times by $T_1, T_2, \ldots$, the long-term number of arrivals per hour equals:

$$\frac{1}{\mathbb{E}(T_1)} = \frac{3}{2}\,.$$

b) The event that the Simpsons will wait for less than 20 minutes can be expressed as $\{T_1 \geq 40 \text{ min}\} \cup \{T_1 + T_2 < 1 \text{ h}\}$. This can be represented by the following diagram:



The ratio of the areas yields the desired probability 5/8.

2. Denote by N_t the number of all calls received up to time t and by $\tilde{N}_t$ the number of the calls it can respond. Since all calls form a homogeneous Poisson process, they satisfy the strong law of large numbers:

$$\frac{N_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{2}\,.$$

Now we turn to the calls to which the station can respond. We may assume that the whole process starts with such a call. In this case, these calls form a renewal process with expected inter-arrival time $\frac{3}{4} + \frac{1}{1/2} = \frac{11}{4}$. By the strong law of large numbers for renewal processes, we have:

$$\frac{\tilde{N}_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{4}{11}\,.$$

This allows us to derive the long-term proportion of redirected calls:

$$1 - \frac{\tilde{N}_t}{N_t} \xrightarrow[t\to\infty]{\text{a.s.}} 1 - \frac{8}{11} = \frac{3}{11}\,.$$

3. Taking fines for rewards, we get a renewal–reward process. Since Ben has to pay a fine with probability 1/3 each time, the expected amount of fine per inspection cycle is 35 euros. As the expected length of the inspection cycle is 7/4 years, the long-term amount of fine per year equals:

$$\frac{35}{7/4} = 20 \text{ euros.}$$

4. This can be regarded as a renewal–reward process: suppose that the process starts in State 1. For the arrivals, consider the jumps from State 2 to State 1; the reward attached to the i-th arrival (jump) is the length of the stay in State 1 in the inter-arrival interval finishing with that arrival/jump; the reward is being received evenly during the time spent in State 1. Then W_t is equal to the total time spent in State 1, while W_t/t is equal to the proportion of the time spent in this state.

Noting that $\mathbb{E}(T_1) = \mu_1 + \mu_2$ and $\mathbb{E}(R_1) = \mu_1$ and applying the strong law of large numbers, the long-term proportion of those stays equals $\dfrac{\mu_1}{\mu_1 + \mu_2}$.

A similar reasoning leads to the same result in the case when the process starts in State 2: then define the arrivals to be the jumps from State 1 to State 2.

5. This is a renewal–reward process, where the inter-arrival times are the durations of the calls, while the reward R_i equals one if Monica manages to persuade the customer in her i-th call and zero otherwise. Clearly, $\tau \le 1/2$. Therefore,

$$F_{T_i}(t) = \begin{cases} 0 & ; t \le 0 \\ 3(t - t^2) & ; 0 \le t < \tau \\ 1 & ; t \ge \tau \end{cases} .$$

Observe that the random variable T_i is neither discrete nor continuous. However, there are several ways to compute the expectation.

First method: by means of the Riemann–Stieltjes integral, splitting into continuous and discrete part:

$$\begin{aligned} \mathbb{E}(T_i) = \int_0^\infty t \,\mathrm{d}F_{T_i}(t) &= \int_0^\tau t\, F'_{T_i}(t)\,\mathrm{d}t + \tau\big(F_{T_i}(\tau) - F_{T_i}(\tau^-)\big) = \\ &= \int_0^\tau (3t - 6t^2)\,\mathrm{d}t + \tau(1 - 3\tau + 3\tau^2) = \\ &= \tau - \frac{3\tau^3}{2} + \tau^3 . \end{aligned}$$

Second method: choose a random variable $\tilde{T}$ with cumulative distribution function which is absolute continuous and matches the cumulative distribution function of T_i

on the interval $[0, \tau)$. Then T_i follows the same distribution as $\min\{\tilde{T}, \tau\}$. Therefore,

$$\begin{aligned}\mathbb{E}(T_i) &= \mathbb{E}\big[\min\{\tilde{T}, \tau\}\big] = \\ &= \int_{-\infty}^{\infty} \min\{t, \tau\}\, f_{\tilde{T}}(t)\,\mathrm{d}t = \\ &= \int_{-\infty}^{\tau} t\, f_{\tilde{T}}(t)\,\mathrm{d}t + \tau \int_{\tau}^{\infty} f_{\tilde{T}}(t)\,\mathrm{d}t = \\ &= \int_{0}^{\tau} t\, F'_{\tilde{T}}(t)\,\mathrm{d}t + \tau\big(1 - F_{\tilde{T}}(\tau)\big) = \\ &= \int_{0}^{\tau} t\, F'_{T_i}(t)\,\mathrm{d}t + \tau\big(F_{T_i}(\tau) - F_{T_i}(\tau^-)\big)\,,\end{aligned}$$

which is the same as before.

Third method: use the fact that, since $T_i \geq 0$,

$$\mathbb{E}(T_i) = \int_0^{\infty} \mathbb{P}(T_i > t)\,\mathrm{d}t = \int_0^{\tau} (1 - 3t + 3t^2)\,\mathrm{d}t = \tau - \frac{3\tau^3}{2} + \tau^3\,,$$

which is again the same as before.

The event $\{R_i = 1\}$ matches the event that Monica succeeds to persuade the customer up to time τ. Hence,

$$\mathbb{E}(R_i) = \mathbb{P}(R_i = 1) = 3(\tau - \tau^2)\,.$$

Denoting by W_t the number of sold articles up to time t, we almost surely have:

$$\lim_{t\to\infty} \frac{W_t}{t} = \frac{\mathbb{E}(R_1)}{\mathbb{E}(T_1)} = \frac{6(1-\tau)}{2 - 3\tau + 2\tau^2} =: h(\tau)\,.$$

From:

$$h'(\tau) = \frac{6(2\tau^2 - 4\tau + 1)}{(\tau^2 - 3\tau + 2)^2}$$

we find that on the interval $[0, 1/2]$, the function h attains its maximum at $\tau = 1 - \frac{\sqrt{2}}{2} \doteq 0{\cdot}293$. If the time is measured in hours, this means that it pays to hang up after 17 minutes and 34 seconds.

6. *First method.* Consider the moment of the first sharing after the one we know Tony has missed. Denote this moment by T_1. If $T_1 \leq 30$, Tony obtains this prize, so that $T = T_1$. Otherwise, everything starts at new: Tony has to wait for a time T', which, given the history, follows exactly the same distribution as T. Therefore,

$$T = T_1 + T'\, \mathbb{1}(T_1 > 30)\,,$$

implying:

$$\mathbb{E}(T) = \mathbb{E}(T_1) + \mathbb{E}(T)\,\mathbb{P}(T_1 > 30) = 30 + \frac{1}{2}\,\mathbb{E}(T)\,,$$

which yields $\mathbb{E}(T) = 60$. In other words, Tony should expect to wait for an hour to obtain a prize.

Second method. Denote by N the number of the sharing at which Tony obtains his prize (the sharing we know Tony has missed is excluded). Denoting by $T_1, T_2, \ldots$ the times between the sharings, observe that:

$$\{N = n\} = \{T_1 > 30, T_2 > 30, \ldots, T_{n-1} > 30, T_n \le 30\}\,.$$

In the event $\{N = n\}$, we have $T = T_1 + T_2 + \cdots + T_n$. Next, given $\{N = n\}$ the conditional distribution of the times $T_1, \ldots, T_{n-1}$ is uniform over the interval from 30 to 40 minutes (so that $\mathbb{E}(T_k \mid N = n) = 35$ for $k = 1, \ldots, n-1$), while T_n is uniformly distributed over the interval from 20 to 30 minutes (so that $\mathbb{E}(T_n \mid N = n) = 25$). Therefore,

$$\mathbb{E}(T \mid N) = 35(N-1) + 25\,.$$

Since N follows the geometric distribution Geom$(1/2)$, we have $\mathbb{E}(N) = 2$ and consequently,

$$\mathbb{E}(T) = \mathbb{E}\big[35(N-1) + 25\big] = 60\,.$$

Third method. Keeping the notation used in the second method, observe that $T = T_1 + T_2 + \cdots + T_N$. Moreover, N is a stopping time, for the event $\{N = n\}$ is uniquely determined by $T_1, T_2, \ldots, T_n$ (see above). Applying Wald's equation, we find that $\mathbb{E}(T) = \mathbb{E}(N)\,\mathbb{E}(T_1) = 2 \cdot 30 = 60$.

Fourth method. As before, denote by $T_1, T_2, \ldots$ the times between the prize sharings. Imagine that Tony waits for the prizes infinitely long (staying on the spot each time after receiving a prize). Then the moments when Tony gets a prize can be considered as arrivals in another renewal process. Denote its inter-arrival times by $\tilde{T}_1, \tilde{T}_2, \ldots$ (so that $T = \tilde{T}_1$).

Denote by W_t the number of prizes Tony that obtains until time t. This can be expressed in two ways. First, in terms of the renewal process of the moments when Tony obtains a prize, and second, in terms of the renewal–reward process of all sharings combined with rewards $R_1, R_2, \ldots$, where $R_n = 1$ if Tony obtains a prize at the n-th sharing and $R_n = 0$ otherwise. Observe that $\mathbb{P}(R_n = 0) = \mathbb{P}(R_n = 1) = 1/2$. Applying the strong law of large numbers for both characterizations, we find that:

$$\lim_{t\to\infty} \frac{W_t}{t} = \frac{1}{\mathbb{E}(\tilde{T}_1)} = \frac{1}{\mathbb{E}(T)} = \frac{\mathbb{E}(R_1)}{\mathbb{E}(T_1)}$$

almost surely,. As a result,

$$\mathbb{E}(T) = \frac{\mathbb{E}(T_1)}{\mathbb{E}(R_1)} = 60\,.$$

7. The inter-arrival distribution of such a process is Gama$(2, \lambda)$, with cumulative distribution function:

$$F(t) = \lambda^2 \int_0^t s\, e^{-\lambda s}\,\mathrm{d}s\,,$$

and Laplace–Stieltjes transform:

$$\hat{F}(z) = \frac{\lambda^2}{(z+\lambda)^2}$$

(which can be, by a convolution argument, also derived from the Laplace transform of the exponential distribution). Thus, the Laplace–Stieltjes transform of the renewal function is:

$$\hat{M}(z) = \frac{\lambda^2}{z(z+2\lambda)} = \frac{\lambda}{2}\left[\frac{1}{z} - \frac{1}{z+2\lambda}\right]$$

resulting in the renewal function:

$$M(t) = \frac{\lambda}{2}\int_0^t \left(1 - e^{-2\lambda s}\right) \mathrm{d}s = \frac{\lambda t}{2} - \frac{1 - e^{-2\lambda t}}{4}\,.$$

8. From the Laplace transform of the inter-arrival distribution:

$$\hat{F}(z) = p + \frac{(1-p)\lambda}{\lambda + z}$$

we obtain the Laplace–Stieltjes transform of the renewal function:

$$\hat{M}(z) = \frac{p}{1-p} + \frac{\lambda}{(1-p)z}\,.$$

Therefore, the renewal function equals:

$$M(t) = \frac{p}{1-p} + \frac{\lambda}{1-p}\,t\,.$$

9. Let $M(t) := \mathbb{E}(N_t)$. This is a delayed renewal process with the first (inter-)arrival time following the gamma $\mathrm{Gama}(2, \lambda)$ distribution with Laplace transform:

$$\hat{G}(z) = \frac{\lambda^2}{(z+\lambda)^2}$$

and with subsequent inter-arrival times following the exponential $\mathrm{Exp}(\lambda)$ distribution with Laplace transform:

$$\hat{F}(z) = \frac{\lambda}{z+\lambda}\,.$$

Thus, the Laplace–Stieltjes transform of the renewal measure equals:

$$\hat{M}(z) = \frac{\hat{G}(z)}{1-\hat{F}(z)} = \frac{\lambda^2}{z(z+\lambda)} = \frac{\lambda}{z} - \frac{\lambda}{z+\lambda}\,,$$

while the renewal measure itself equals:

$$M(t) = \lambda\int_0^t \mathrm{d}s - \lambda\int_0^s e^{-\lambda s}\,\mathrm{d}s = \lambda t - 1 + e^{-\lambda t}\,.$$

10. a) The first arrival time follows the exponentional distribution $\mathrm{Exp}(\mu)$, so that its Laplace transform equals:

$$G(z) = \frac{\mu}{z+\mu}\,.$$

By the strong time-homogeneous Markov property, all further arrivals can be expressed as sums of two independent random variables, one of them following the exponential distribution $\mathrm{Exp}(\lambda)$ (service time), while the other following $\mathrm{Exp}(\mu)$ (idle time). Thus, the Laplace transform of the further inter-arrival times equals:

$$\hat{F}(z) = \frac{\lambda\mu}{(z+\lambda)(z+\mu)}\,.$$

Thus, the Laplace–Stieltjes transform od the renewal function equals:

$$\hat{M}(z) = \frac{\mu(z+\lambda)}{z(z+\lambda+\mu)} = \frac{\lambda\mu}{\lambda+\mu}\,\frac{1}{z} + \frac{\mu^2}{\lambda+\mu}\,\frac{1}{z+\lambda+\mu}\,.$$

and finally, the renewal function is:

$$M(t) = \frac{\lambda\mu}{\lambda+\mu}\int_0^t \mathrm{d}s + \frac{\mu^2}{\lambda+\mu}\int_0^t e^{-(\lambda+\mu)s}\,\mathrm{d}s = \frac{\lambda\mu}{\lambda+\mu}\,t + \frac{\mu^2}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)t}\right).$$

b) *First method.* By linearity of the expectation,

$$\mathbb{E}(T_2) = \frac{1}{\lambda} + \frac{1}{\mu} = \frac{\lambda+\mu}{\lambda\mu}\,,$$

Plugging this into the strong law of large numbers, we obtain the limiting intensity $\dfrac{1}{\mathbb{E}(T_2)} = \dfrac{\lambda\mu}{\lambda+\mu}$.

Second method. By convergence of the renewal measure, we have:

$$\lim_{t\to\infty}\frac{M(t)}{t} = \frac{\lambda\mu}{\lambda+\mu}\,.$$

c) $\dfrac{\lambda\mu}{\lambda+\mu}\Big/\mu = \dfrac{\lambda}{\lambda+\mu}$.

11. Observe that each time Roy meets a supervisor, he continues in place A with probability $1/2$ and in place B with probability $1/2$, regardless of the history. In addition, from the strong time-homogeneous Markov property, it follows that the inter-arrival times are independent. The distribution of the first arrival time is exponential $\mathrm{Exp}(1)$, while the distributions of the subsequent inter-arrival times are mixtures of half $\mathrm{Exp}(1)$ and half $\mathrm{Exp}(1/2)$. Therefore, the arrivals of the supervisors indeed form a delayed renewal process.

Denoting by G the cumulative distribution function of the first arrival time and by F the cumulative distribution function of the subsequent inter-arrival times, that is:

$$G(t) = 1 - e^{-t}\,, \qquad F(t) = \tfrac{1}{2}\left(1 - e^{-t}\right) + \tfrac{1}{2}\left(1 - e^{-t/2}\right).$$

The Laplace transforms are:

$$\hat{G}(z) = \frac{1}{z+1}\,, \qquad \hat{F}(z) = \frac{1}{2}\left[\frac{1}{2z+1} + \frac{1}{z+1}\right] = \frac{3z+2}{2(2z+1)(z+1)}$$

and the Laplace–Stieltjes transform of the renewal measure is:

$$\hat{M}(z) = \frac{2(2z+1)}{z(4z+3)} = \frac{2}{3z} + \frac{4}{3(4z+3)}\,,$$

resulting in the renewal measure:

$$M(t) = \frac{2t}{3} + \frac{4\bigl(1 - e^{-3t/4}\bigr)}{9}\,.$$

12. *First method*: by renewal equation. From the Laplace transforms of both distributions:

$$\hat{G}(z) = \frac{1 - e^{-az}}{az}\,, \qquad \hat{F}(z) = \frac{\lambda}{z+\lambda}$$

we obtain the Laplace–Stieltjes transform of the renewal measure:

$$\hat{M}(z) = \frac{\bigl(1 - e^{-az}\bigr)(z+\lambda)}{az^2}\,.$$

Now write:

$$\hat{M}_1(z) := \frac{z+\lambda}{az^2}\,, \quad \hat{M}_2(z) := e^{-az}\hat{M}_1(z)\,, \quad \hat{M}(z) = \hat{M}_1(z) - \hat{M}_2(z)\,.$$

The functions $\hat{M}_1$ in $\hat{M}_2$ are the Laplace–Stieltjes transforms of the functions:

$$M_1(t) = \frac{t}{a} + \frac{\lambda t^2}{2a}\,, \quad M_2(t) = \begin{cases} 0 & ;\ t < a \\ M_1(t-a) & ;\ t \ge a \end{cases}\,,$$

so that the renewal measure equals:

$$M(t) = M_1(t) - M_2(t) = \begin{cases} \dfrac{t}{a} + \dfrac{\lambda t^2}{2a} & ;\ t \le a \\ 1 + \lambda t - \dfrac{\lambda a}{2} & ;\ t \ge a \end{cases}\,.$$

Second method: directly from the homogeneous Poisson process: conditioning on the first arrival time, we obtain:

$$\mathbb{E}(N_t \mid T_1) = \bigl(1 + \lambda(t - T_1)\bigr)\,\mathbb{1}(t \ge T_1)\,.$$

Integrating, we find that:

$$\mathbb{E}(N_t) = \frac{1}{a}\int_0^a \bigl(1 + \lambda(t-s)\bigr)\,\mathrm{d}s = 1 + \lambda t - \frac{\lambda a}{2}$$

for $t \ge a$, while:

$$\mathbb{E}(N_t) = \frac{1}{a}\int_0^t \bigl(1 + \lambda(t-s)\bigr)\,\mathrm{d}s = \frac{t}{a} + \frac{\lambda t^2}{2a}$$

for $t \le a$. This is the same as before.

13. Consider the number of arrivals between times t and $t+s$. Clearly, its expected value equals $M(t+s) - M(t)$. However, this is also the number of arrivals up to time s in the process $\mathcal{P}$. Thanks to stationarity, this follows the same distribution as the number of arrivals up to time s in the original process, which has expectation $M(s)$. As a result, the renewal function is *additive*:

$$M(t+s) = M(t) + M(s)\,.$$

Since it is increasing, there is no other way but $M(t) = ct$ for some constant $c \ge 0$ (details are omitted). Plugging this into the renewal equation, we obtain:

$$\begin{aligned}G(t) &= c\left[t - \int_{[0,t]} (t-s)\,\mathrm{d}F(s)\right] = \\ &= c\Big(t - \mathbb{E}\big[(t-T_2)\,\mathbb{1}(T_2 \le t)\big]\Big) = \\ &= c\,\mathbb{E}\big[t\,\mathbb{1}(T_2 > t) + T_2\,\mathbb{1}(T_2 \le t)\big] = \\ &= c\,\mathbb{E}\big[\min\{T_2, t\}\big]\,,\end{aligned}$$

provided that $t \ge 0$; clearly, $G(t) = 0$ for $t < 0$. Taking the limit as t tends to infinity, we find that $c = 1/\,\mathbb{E}(T_2)$. Hence, finally,

$$G(t) = \frac{\mathbb{E}\big[\max\{T_2, t\}\big]}{\mathbb{E}(T_2)}\,.$$

Alternatively, the expression arising from the renewal equation can be integrated by parts. Strictly speaking, we make use of the Fubini theorem. More precisely, from:

$$\min\{s,t\} = \int_{\substack{u \ge 0 \\ u < s \\ u \le t}} \mathrm{d}u$$

we derive the following alternative form:

$$\begin{aligned}G(t) &= \frac{1}{\mathbb{E}(T_2)} \iint_{\substack{u \ge 0 \\ u < s \\ u \le t}} \mathrm{d}F(s)\,\mathrm{d}u = \\ &= \frac{1}{\mathbb{E}(T_2)} \int_{[0,t]} \int_{(u,\infty]} \mathrm{d}F(s)\,\mathrm{d}u = \\ &= \frac{1}{\mathbb{E}(T_2)} \int_0^t \big(1 - F(u)\big)\,\mathrm{d}u = \\ &= \frac{\displaystyle\int_0^t \big(1-F(u)\big)\,\mathrm{d}u}{\displaystyle\int_0^\infty \big(1-F(u)\big)\,\mathrm{d}u}\,.\end{aligned}$$

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