Perfect codes in direct graph bundles

Janez Žerovnik

Institute of Mathematics, Physics and Mechanics, Ljubljana
and
FME, University of Ljubljana, Ljubljana

based on joint work with **Irena Hrastnik**, University of Maribor

Outline of the presentation

- Introduction
- The result
- Basic notions "Preliminaries"
- Proof sketch ... ?

Introduction

- general motivation - error correcting codes
- related work
  - Biggs [1973]
  - some later references: Livingston and Stout [90], Kratochvíl [91,94], Hedetniemi, McRae and Parks [98], Cull and Nelson [99], Jha [02,03], Klavžar, Milutinović and Petr [02], Jerebic, Klavžar and Špacapan [05], Klavžar, Špacapan, and Žerovnik [2006]

Introduction

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- my motivation - CAN "a complete description of perfect codes in the direct product of cycles" [—, Advances in applied math. 2008] BE GENERALIZED ... TO GRAPH BUNDLES ?

The result for product of cycles:

Theorem [Jha02-03 ($n = 2$), [JKŠ05] ($n = 3$), [KŠŽ06], [Z08] (general case)]:
Let $G = \times_{i=1}^{n} C_{\ell_i}$ be a direct product of cycles. For any $r \geq 1$, and any $n \geq 2$, each connected component of G contains a so–called canonical r-perfect code provided that each ℓ_i is a multiple of $r^n + (r+1)^n$. (And, for other ℓ_i no perfect codes are possible).

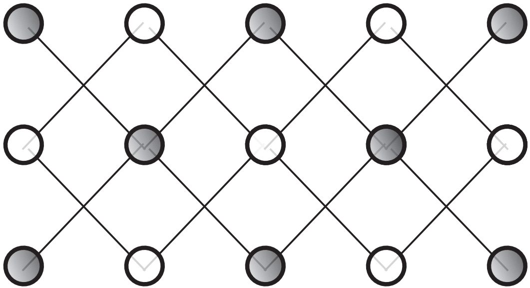
New result for product of bundles :

Theorem [Hrastnik and —, to appear] : Let $r \geq 1$, $m, n \geq 3$, and $t = (r+1)^2 + r^2$. Let $X = C_m \times^{\sigma_\ell} C_n$ be a direct graph bundle with fibre C_n and base C_m. Then each connected component of X contains an r-perfect code if and only if n is a multiple of t, $m > r$, and ℓ has a form of $\ell = (\alpha t \pm ms) \mod n$ for some $\alpha \in \mathbb{Z}$.

Theorem [Hrastnik and —, to appear] : There is no r-perfect code of (a connected component of) direct graph bundle $C_m \times^\alpha C_n$ where α is reflection.

Preliminaries

- **direct product** of graphs, $G \times H$
- nodes: $V(G) \times V(H)$
- edges: $(a, b) \sim ((c, d) \iff (a \sim c \text{ and } b \sim d)$
  example;



Preliminaries, conectivity

- **connectivity of direct product of cycles**
- $G = \times_{i=1}^{n} C_{\ell_i}$ is connected $\iff$ at most one of the ℓ_i's is even.

- Connectivity of direct graph bundles (of cycles over cycles) : DEPENDS on
  - parity of ℓ_1 and ℓ_2 AND
  - the automorphism ($\Rightarrow$ next slide)

Preliminaries, connectivity cont.

Table: Connected direct graph bundles $C_m \times^{\alpha} C_n$

<table>
<tr><td>n odd</td><td colspan="2">for any automorphism α of C_n</td></tr>
<tr><td rowspan="2">n even</td><td>m odd</td><td>$\alpha = id$
$\alpha = \sigma_\ell$, ℓ is even
$\alpha = \rho_2$</td></tr>
<tr><td>m even</td><td>$\alpha = \sigma_\ell$, ℓ is odd
$\alpha = \rho_0$</td></tr>
</table>

Preliminaries, connectivity cont.

Automorphisms of a cycle are of two types:

- A cyclic shift of the cycle by ℓ elements, denoted by σ_ℓ, $0 \leq \ell < n$, maps u_i to $u_{i+\ell}$ (indices are modulo n).
- (As a special case we have the identity ($\ell = 0$).)
- Other automorphisms of cycles are reflections. Depending on parity of n the reflection of a cycle may have one, two or no fixed points.

NOTATION : $C_m \times^\alpha C_n$

Graph bundles

Definition

Let B and F be graphs. A graph G is a direct graph bundle with fibre F over the base graph B if there is a graph map $p : G \to B$ *such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \times K_2$.*

Graph bundle - another definition

- Start with graph B.

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- Replace each vertex of B with a copy of F.

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Another point of view: assign authomorphisms to (directed) edges of B ...

Example

Graph bundles also appear as computer topologies.
(This one is CARTESIAN graph bundle, but very famous.)
A well known example is the twisted torus - *ILIAC IV architecture* on the Figure.

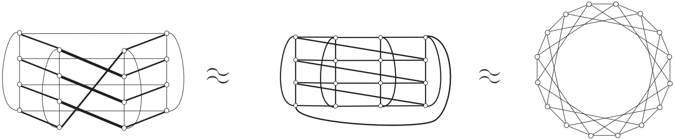


Figure: Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4

Cartesian graph bundle with fibre C_4 over base C_4 is the ILLIAC IV architecture, a famous supercomputer that inspired some modern multicomputer architectures.

- see G.H. Barnes, R.M. Brown, M. Kato, D.J. Kuck, D.L. Slotnick, R.A. Stokes, The ILLIAC IV Computer, *IEEE Transactions on Computers*,

Preliminaries - Perfect codes

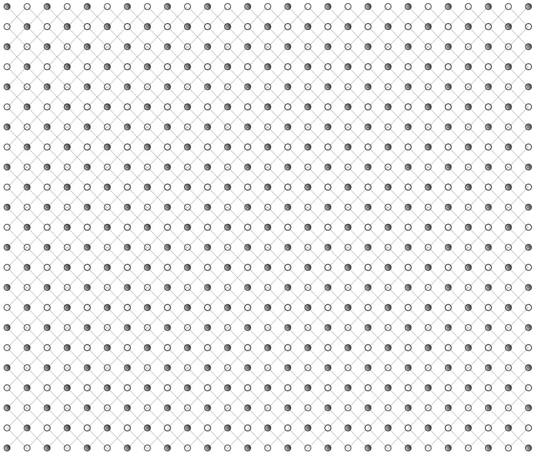
- set $C \subseteq V(G)$ is an r-**code** in G if $d(u, v) \geq 2r + 1$ for any two distinct vertices $u, v \in C$.

- C is an r-**perfect code** if for any $u \in V(G)$ there is exactly one $v \in C$ such that $d(u, v) \leq r$.

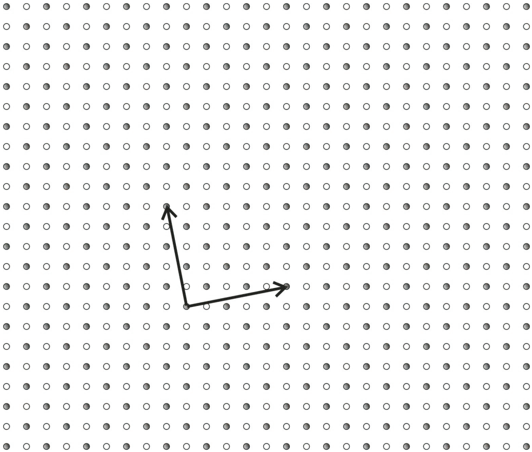
- **OR:** $C \subset V(G)$ is an r-perfect code if and only if the r-balls $B(u, r)$, where $u \in C$, form a partition of $V(G)$.

- **Abbreviation:** $s = 2r + 1$

direct grid - with edges

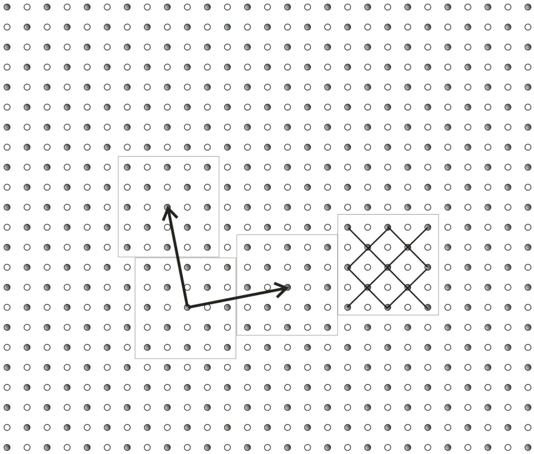


direct grid - base vectors



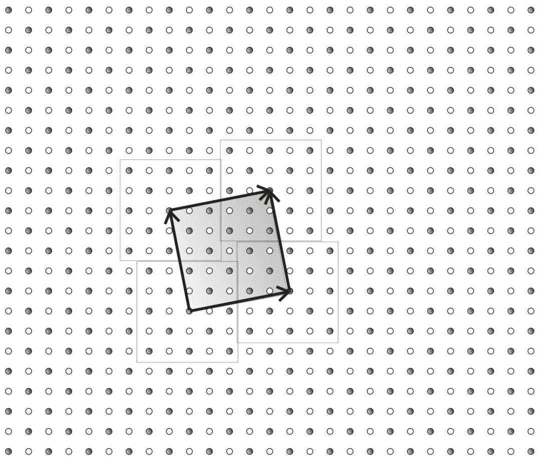
canonical local structure : $(s,1), (-1,s)$ (here $s = 2r+1$, $r = 2$)

direct grid - neighborhoods



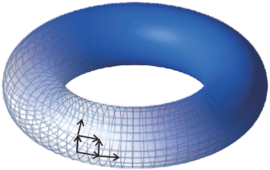
2-neighborhood has 1+4+8= 13 vertices

direct grid - the tilling



Preliminaries, cont.

- 2D product :
  - embedding of the product into a torus
  - area of the torus
  - the tilling covers the area twice



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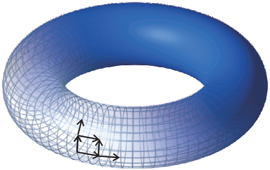

- bundle:
  - "twisted" torus
  - check the "JOIN"
  - TAKE CARE !!! double cover

Idea of proof

- cut the torus
- recall that a perfect code is determined by any two of its elements
- check if it extends consistently

for details see [Information Processing Letters., to appear]

References

- N. Biggs, Perfect codes in graphs, J. Combin. Theory Ser. B 15 (1973) 289–296.
- M. Cesati, Perfect Code is W[1]-complete, Inf. Process. Lett. 81 (2002) 163–168.
- R. Erveš, J. Žerovnik, Mixed fault diameter of Cartesian graph bundles, Discrete Appl. Math. 161(12) (2013) 1726–1733.
- I. Hrastnik Ladinek, J. Žerovnik, Perfect codes in direct graph bundles, Information Processing Letters, to appear. doi:10.1016/j.ipl.2015.03.010
- J. Jerebic, S. Klavžar, S. Špacapan, Characterizing r-perfect codes in direct products of two and three cycles, Inf. Process. Lett. 94(1) (2005) 1–6.
- P. K. Jha, Perfect r-domination in the Kronecker product of three cycles, IEEE Trans. Circuits Systems-I: Fundamental Theory Appl. 49 (2002) 89–92.
- P. K. Jha, Perfect r-domination in the Kronecker product of two cycles, with an application to diagonal toroidal mesh, Inf. Process. Lett. 87 (2003) 163–168.
- S. Klavžar, S. Špacapan, J. Žerovnik, An almost complete description of perfect codes in direct products of cycles, Adv. in Appl. Math. 37 (2006) 2–18.
- J. Žerovnik, Perfect codes in direct products of cycles - a complete characterization, Adv. in Appl. Math. 41 (2008) 197–205.

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