Distances in Sierpiński Triangle Graphs

Sara Sabrina Zemljič

joint work with Andreas M. Hinz

June 18th 2015
Sierpiński triangle introduced by Wacław Sierpiński in 1915.
Motivation
Motivation
Motivation
Motivation

- **Sierpiński triangle**
  introduced by Wacław Sierpiński in 1915.

- **Sierpiński graphs**
  introduced by Klavžar and Milutinović in 1997,
  connected to the Tower of Hanoi puzzle – state graphs for the Switching Tower of Hanoi puzzle.
Motivation

Graphs $ST_3^3$ (left) and $S_3^3$ (right).
Motivation

- **Sierpiński triangle**
  introduced by Wacław Sierpiński in 1915.

- **Sierpiński graphs**
  introduced by Klavžar and Milutinović in 1997,
  connected to the Tower of Hanoi puzzle – state graphs for the Switching Tower of Hanoi puzzle.

- **Applications outside mathematics**
  - Physics – spectral theory (Laplace operator), spanning trees (Kirchhoff’s Theorem),
  - Psychology – ”state graphs” of the Tower of Hanoi puzzle.
Notations

- $[n] := \{1, \ldots, n\}$,
- $[n]_0 := \{0, \ldots, n - 1\}$,
- $T := [3]_0 = \{0, 1, 2\}$,
- $\hat{T} := \{\hat{0}, \hat{1}, \hat{2}\}$,
- $P := [p]_0 = \{0, \ldots, p - 1\}$,
- $\hat{P} := \{\hat{k} \mid k \in P\}$. 
Definition (Idle peg labeling)

Let $n \in \mathbb{N}$.

Sierpiński triangle graphs $ST^n_3$ ...

... are the graphs defined as follows:
Definition (Idle peg labeling)

Let $n \in \mathbb{N}$.

Sierpiński triangle graphs $ST_3^n$ ...

... are the graphs defined as follows:

$ST_3^0 \cong K_3$

$V(ST_3^0) = \hat{T}$

• vertices $\hat{0}$, $\hat{1}$, and $\hat{2}$ are primitive vertices
Definition (Idle peg labeling)

Let $n \in \mathbb{N}$.

Sierpiński triangle graphs $ST^n_3$ ...

... are the graphs defined as follows:

$$V(\text{ST}^n_3) = \hat{T} \cup \{s \in T^\nu \mid \nu \in [n]\},$$

$$E(\text{ST}^n_3) = \left\{ \{\hat{k}, k^{n-1}j\} \mid k \in T, j \in T \setminus \{k\} \right\} \cup \left\{ \{sk, sj\} \mid s \in T^{n-1}, \{j, k\} \in \binom{T}{2} \right\}$$

$$\cup \left\{ \{s(3 - i - j)i^{n-1-\nu}k, sj\} \mid s \in T^{\nu-1}, \nu \in [n], i \in T, j, k \in T \setminus \{i\} \right\}$$
Example – Idle peg labeling
Example – Idle peg labeling
Example – Idle peg labeling
Example – Idle peg labeling
Example – Idle peg labeling
Example – Idle peg labeling
Example – Idle peg labeling
Contraction labeling

Let $n \in \mathbb{N}$.

Contraction labeling of Sierpiński triangle graphs $ST_3^n$

$ST_3^0 \cong K_3$

$V(ST_3^0) = \hat{T}$
Let \( n \in \mathbb{N} \).

**Contraction labeling of Sierpiński triangle graphs** \( ST_3^n \)

\[
V(ST_3^n) = \hat{T} \cup \left\{ s\{i, j\} \mid s \in T^{\nu-1}, \nu \in [n], \{i, j\} \in \binom{T}{2} \right\},
\]

\[
E(ST_3^n) = \left\{ \{\hat{k}, k^{\nu-1}\{j, k\}\} \mid k \in T, j \in T \setminus \{k\} \right\} \cup \\
\left\{ s\{i, j\}, s\{i, k\} \mid s \in T^{\nu-1}, i \in T, \{j, k\} \in \binom{T \setminus \{i\}}{2} \right\} \cup \\
\left\{ s\{i, j\}, s\{i, k\} \mid s \in T^{\nu-1}, \nu \in [n-1], i \in T, \{j, k\} \in T \setminus \{i\} \right\}
\]
Basic properties

- $|ST_3^n| = \frac{3}{2}(3^n + 1)$
- $\|ST_3^n\| = 3^{n+1}$
- Graphs $ST_3^n$ are connected
Distance to a primitive vertex

Lemma.

If $n \in \mathbb{N}$ and $\nu \in [n]_0$, then for any $s, t \in V(ST^\nu_3)$

$$d_n(s, t) = 2^{n-\nu} d_\nu(s, t).$$
Distance to a primitive vertex

\[ \hat{0} \]

\[ \hat{1} \]

\[ \hat{2} \]
Distance to a primitive vertex

**Lemma.**
If \( n \in \mathbb{N} \) and \( \nu \in [n]_0 \), then for any \( s, t \in V(ST_3^\nu) \)

\[
d_n(s, t) = 2^{n-\nu} d_\nu(s, t).
\]

**Proposition.**
If \( \nu \in \mathbb{N} \) and \( s \in T^\nu \), then \( d_0(\hat{k}, \hat{\ell}) = (k \neq \ell) \), and

\[
d_\nu(s, \hat{\ell}) = 1 + (s_1 = \ell) + \sum_{d=2}^{\nu} (s_d \neq \ell) \cdot 2^{d-1}.
\]

There are \( 1 + (s_1 = \ell) \) shortest paths between \( s \) and \( \hat{\ell} \).

\(^a\)Here (X) is Iverson convention, which is 1 if X is true and 0 if X is false.
Distances – special case

Let \( \{i, j, k\} = T, n \in \mathbb{N} \) and \( s \in T^n \).

\[
\begin{align*}
d_{n+1}(is, j) &= d_n(s, \hat{k}) \\
d_{n+1}(is, i) &= \min\{d_n(s, \hat{k}) \mid k \in T \setminus \{i\}\} + 2^n
\end{align*}
\]

If \( s = i^\kappa s_{n-\kappa} \hat{s}, \kappa \in [n-1]_0 \), then \( d_{n+1}(is, i) = d_n(s, s_{n-\kappa}) + 2^n \) and the shortest path goes through vertex \( 3 - i - s_{n-\kappa} \).

- two shortest paths between \( is \) and \( i \) iff \( is = i^{\nu+1}, \nu \in [n] \)
- two shortest paths between \( is \) and \( j \) iff \( is = ik^\nu, \nu \in [n] \)
Distances – general formula

Theorem.

If \( n \in \mathbb{N} \) and \( \nu \in [n]_0 \), then for any \( s \in V(ST^n_3), t \in V(ST^\nu_3) \), and \( \{i, j, k\} = T \),

\[
d_{n+1}(is, jt) = \min\{d_n(s, \hat{j}) + 2^n - \nu d_\nu(t, \hat{i}); d_n(s, \hat{k}) + 2^n + 2^{n-\nu} d_\nu(t, \hat{k})\}.
\]

Problem of two shortest paths: shortest path either goes directly from \( i \)-subgraph to \( j \)-subgraph, or it goes through \( k \)-subgraph. It can also happen that there are two shortest paths.
Comparison with metric properties of Sierpiński graphs

<table>
<thead>
<tr>
<th>S$^n_3$</th>
<th>ST$^n_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d($s\bar{s}, s\bar{t}$) = d($s, t$)</td>
<td>d$_n$(s, t) = $2^{n-v}$d$_v$(s, t)</td>
</tr>
<tr>
<td>d(s,$j^n$) = $\sum_{d=1}^{n} (s_d \neq j)2^{d-1}$</td>
<td>d$_v$(s, $\hat{\ell}$) = 1 + (s$<em>1$ = $\ell$) + $\sum</em>{d=2}^{v} (s_d \neq \ell)2^{d-1}$</td>
</tr>
<tr>
<td>d(is, jt) = min{d$<em>{dir}$(is, jt), d$</em>{indir}$(is, jt)}</td>
<td></td>
</tr>
<tr>
<td>diam(S$^n_3$) = $2^n - 1$</td>
<td>diam(ST$^n_3$) = $2^n$</td>
</tr>
<tr>
<td>$\sum_{i \in T} d(s, i^n) = 2^{n+1} - 2$</td>
<td>$\sum_{i \in T} d(s, i^n) = 2^{n+1}$</td>
</tr>
</tbody>
</table>
Example

\[ d_4(002\{0, 2\}, 112\{1, 2\}) = 16 \]
(direct)

\[ d_4(020\{1, 2\}, 12\{0, 2\}) = 13 \]
(two shortest paths)

\[ d_4(022\{0, 1\}, 12\{0, 2\}) = 12 \]
(indirect)

Sierpiński triangle graphs $ST^n_p$ (n ∈ $\mathbb{N}$) ...

... are the graphs defined by:

$V(ST^n_p) = \hat{P} \cup \left\{ s\{i,j\} \mid s \in P^{v-1}, \, v \in [n], \, \{i,j\} \in \binom{P}{2} \right\}$,

$E(ST^n_p) = \left\{ k, k^{n-1}\{j,k\} \right\} \cup\left\{ s\{i,j\}, s\{i,k\} \mid s \in P^{n-1}, \, i \in P, \, \{j,k\} \in \binom{P \{i\}}{2} \right\} \cup\left\{ s k^{n-1-v}\{i,j\}, s\{i,k\} \mid s \in P^{v-1}, \, v \in [n-1], \, i \in P, \, \{j,k\} \in P \{i\} \right\}$.

As before, $ST^0_p \cong K_p$ and $V(ST^0_p) = \hat{P}$. 
Example $ST_T^{1/4}$
Distances in \( ST^n_p \)

\[
d_v(s\{i,j\}, \hat{\ell}) = 1 + (i \neq \ell)(j \neq \ell) + \sum_{d=1}^{v-1} (s_d \neq \ell) \cdot 2^d \in [2^v]
\]

- there are \( 1 + (p - 2)(i \neq \ell)(j \neq \ell) \ s\{i,j\}, \hat{\ell}\)-shortest paths
- \( \text{diam}(ST^n_p) = 2^n \)
- \( \forall s \in V(ST^n_p) : \sum_{\ell=0}^{p-1} d_n(s, \hat{\ell}) = (p - 1) \cdot 2^n \)

\[
d_{n+1}(is, jt) = \min\{d_n(s, \hat{j}) + 2^{n-v}d_v(t, \hat{i}) ;
\]
\[
d_n(s, \hat{k}) + 2^{n-v}d_v(t, \hat{k}) + 2^n \mid k \in P \setminus \{i,j\}\}.
\]
Open Problems

- explicit formula for average distance
- other metric properties which are known for Sierpiński graphs $S_p^n$
- we are currently working on the decision automaton for $p > 3$
THANK YOU!