On the Complexity of Computing the k-Metric Dimension of Graphs

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Joint work with Alejandro Estrada-Moreno and Juan A. Rodríguez-Velázquez

Outline

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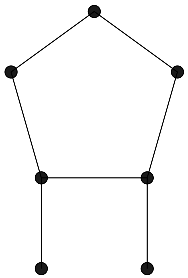

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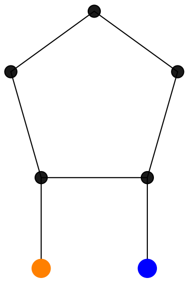

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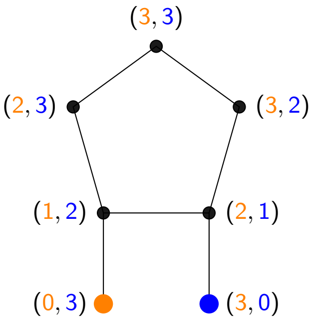

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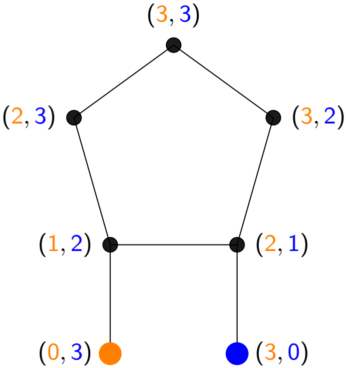
There is a remarkable weakness in the metric generator:

the uniqueness of some vertices identifying some pairs.

The weakness



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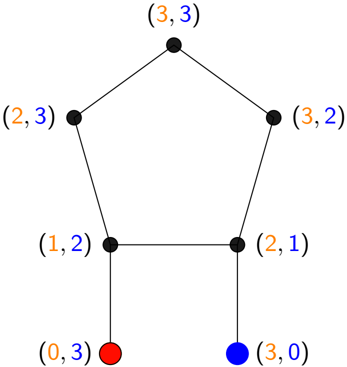

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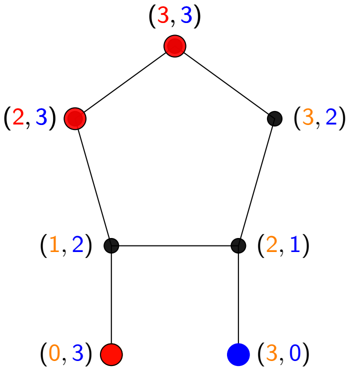

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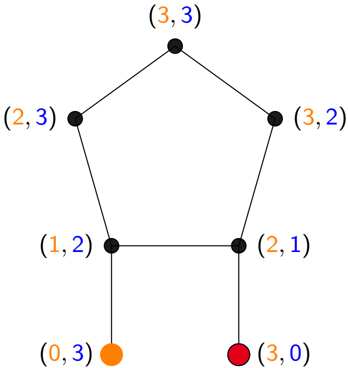

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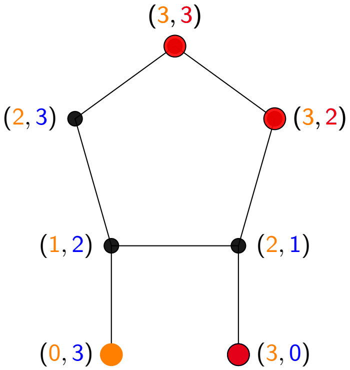

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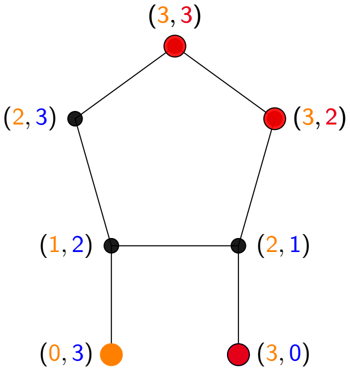

One possible solution: include more vertices so that each two vertices is identified by more than one vertex.

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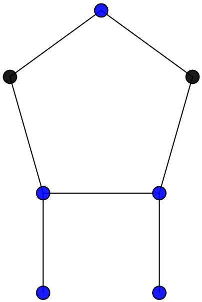

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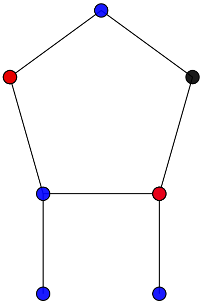

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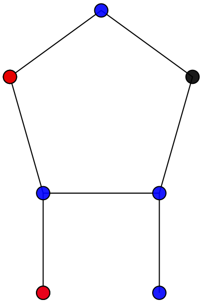

A 2-metric generator (in blue) (not a 3-metric generator)

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The complexity

The complexity of computing the value k is $O(n^3)$

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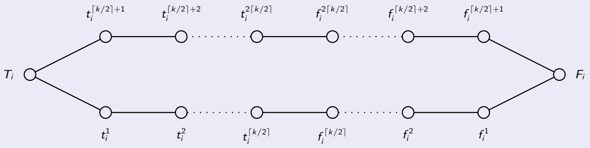

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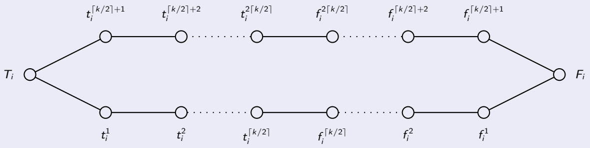
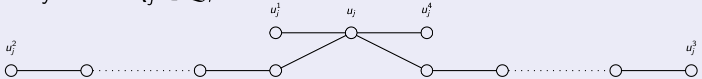


- For every clause $Q_j \in \mathcal{Q}$,



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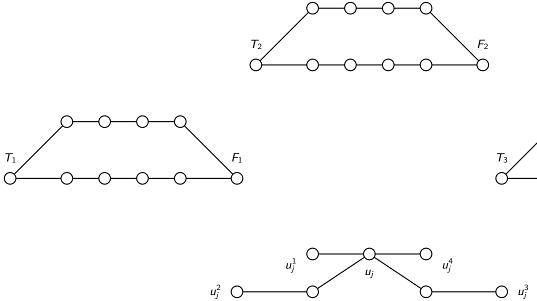

The subgraph associated to the clause $Q_j = (x_1 \vee \overline{x_2} \vee \overline{x_3})$ (taking $k = 4$).

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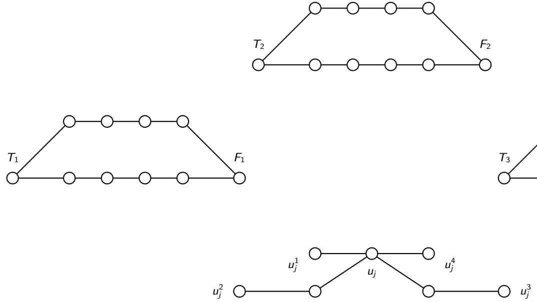

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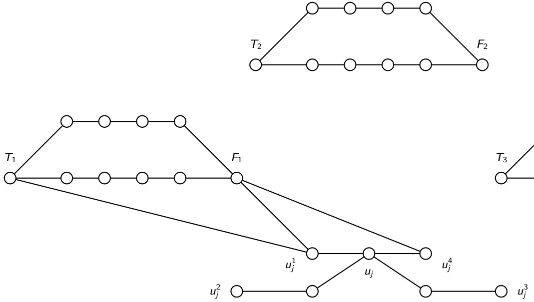

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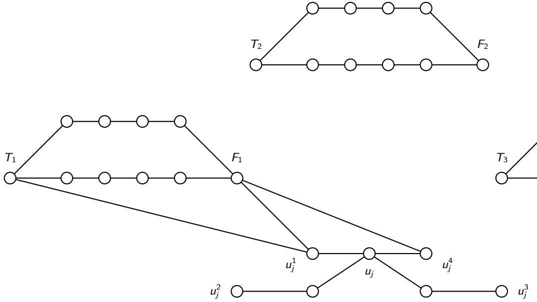

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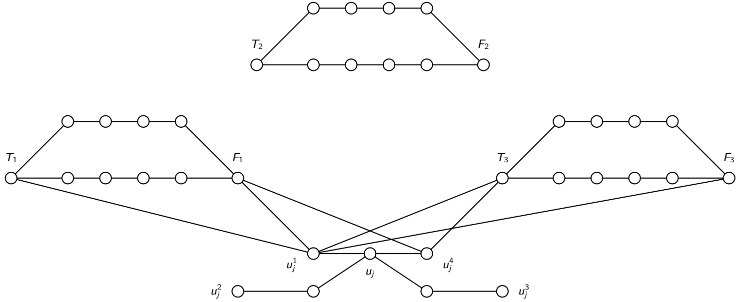

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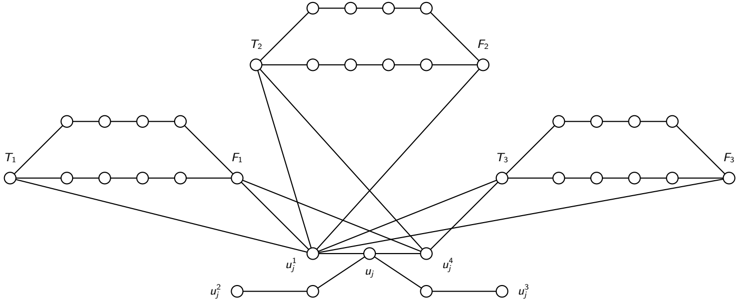

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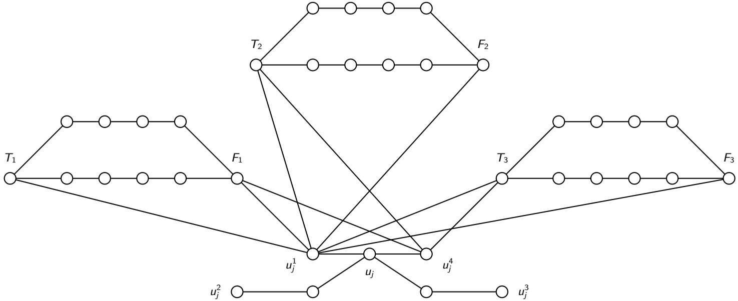

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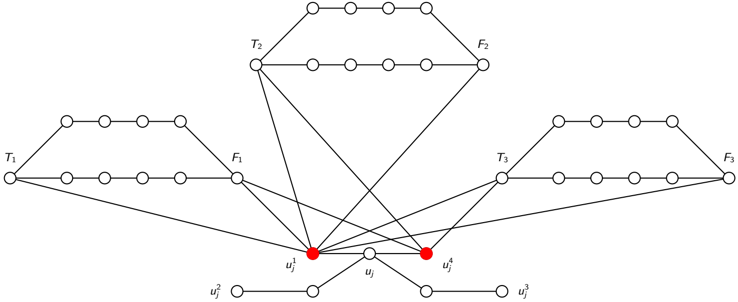

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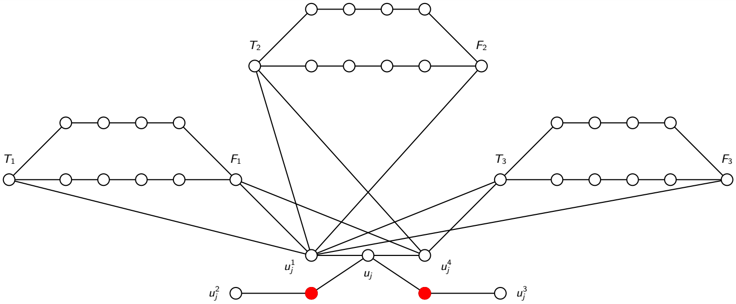

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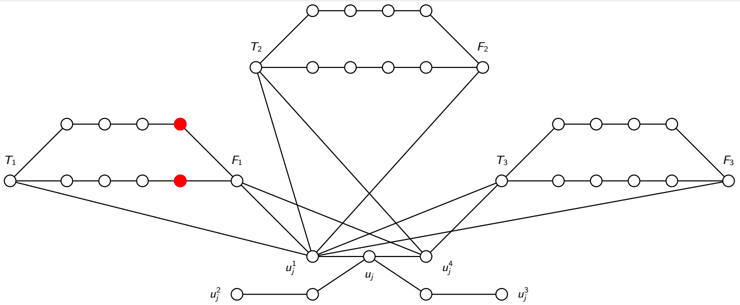

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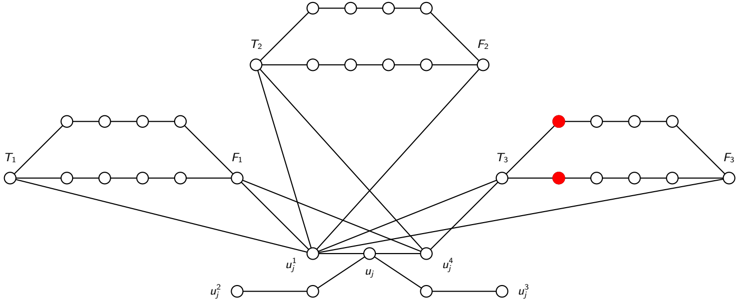

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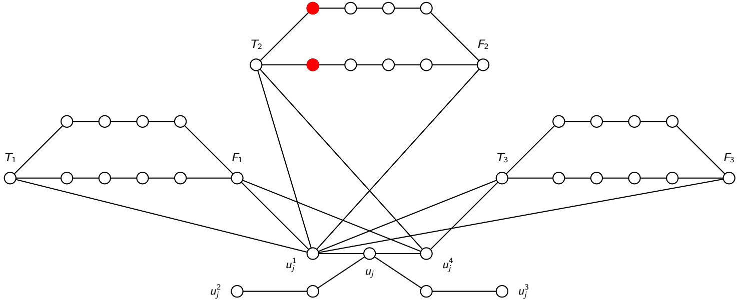

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$\mathcal{F}$ is satisfiable if and only if $dim_k(G_F) = k(n+r)$.

Outline

The case of trees

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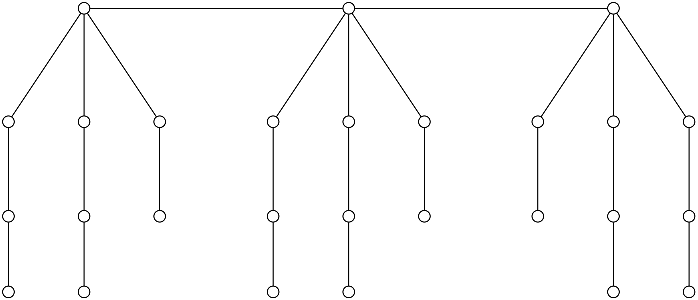

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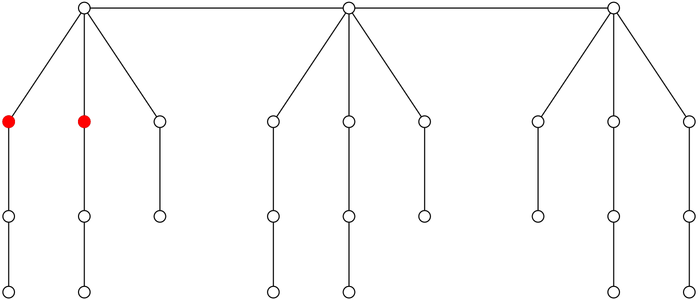

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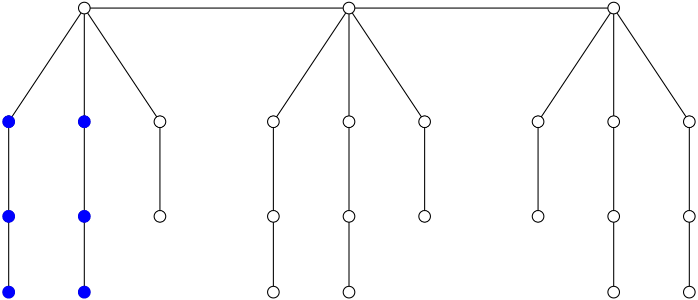

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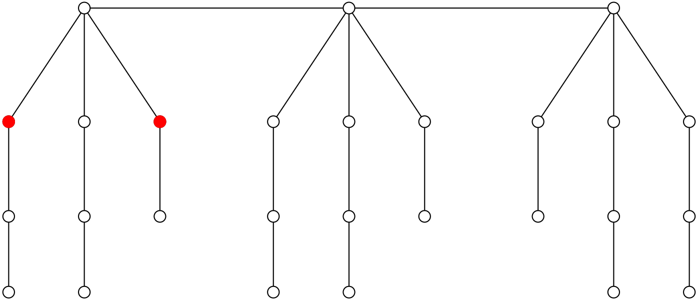

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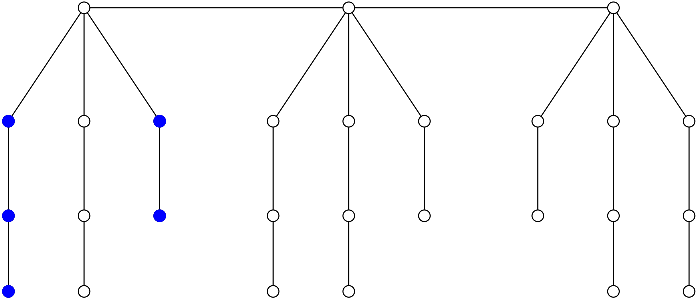

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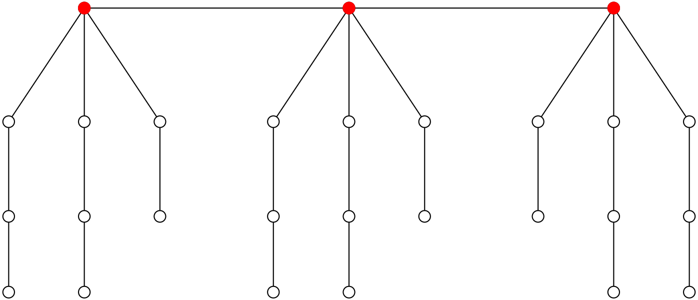

The tree T is 5-metric dimensional.

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A formula for the r-metric dimension of trees

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For an exterior major vertex w_i having terminal degree greater than one,

$$I_r(w_i) = \begin{cases} (ter(w_i) - 1)\,(r - I(w_i)) + I(w_i), & \text{if } I(w_i) \leq \lfloor \frac{r}{2} \rfloor, \\ (ter(w_i) - 1)\,\lceil \frac{r}{2} \rceil + \lfloor \frac{r}{2} \rfloor, & \text{otherwise.} \end{cases}$$

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the r-metric dimension of a tree T (not a path), is

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Complexity

Computing the k-metric dimension of a tree T is of order $O(n)$, where n is the number of vertices of T

THANKS !