Well-dominated graphs without cycles of lengths 4 and 5

David Tankus

Ariel University, ISRAEL

Joint work with

Vadim Levit

Ariel University, ISRAEL

Dominating sets

- A set S of vertices of G is dominating if every vertex of G is either in S or a neighbor of a vertex in S.
- A dominating set is minimal if it does not contain another dominating set.
- A dominating set is minimum if the graph does not admit a dominating set with a smaller cardinality.

Independent sets

- An independent set is a set of pairwise non-adjacent vertices.
- An independent set is maximal if it is not contained in any larger independent set.
- An independent set is maximum if the graph does not admit an independent set with a higher cardinality.

Definitions

- $\gamma(G)$ = the cardinality of a minimum dominating set.
- $\Gamma(G)$ = the maximal cardinality of a minimal dominating set.
- $i(G)$ = the minimal cardinality of a maximal independent set.
- $\alpha(G)$ = the cardinality of a maximum independent set.
- $\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$

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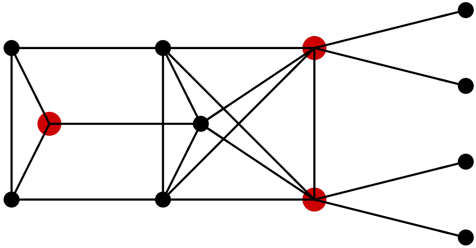

- $\gamma(G) = 3$
- $i(G) = 4$
- $\alpha(G) = 6$
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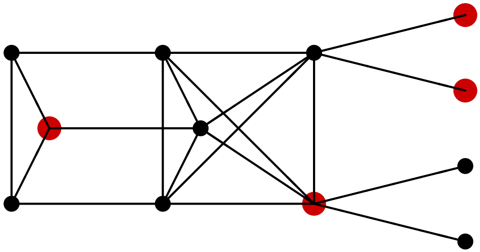

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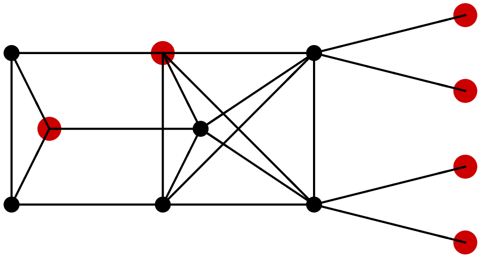

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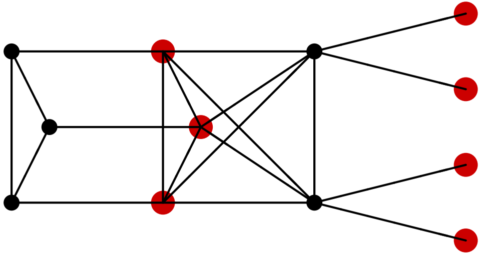

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Definitions

- $\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$
- $i(G) = \alpha(G) \iff G$ is **well-covered**
- $\gamma(G) = \Gamma(G) \iff G$ is **well-dominated**

Theorem (Finbow, Hartnell, Nowakowski, 1998)

Every well-dominated graph is well-covered.

There exist well-covered graphs which are not well-dominated:

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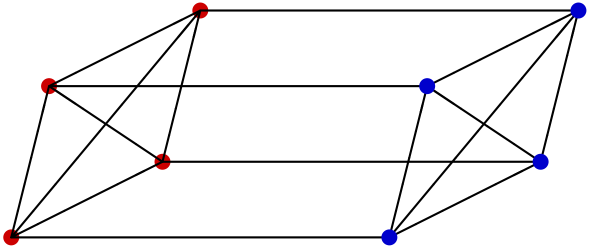

Known Results

The complexity status of recognizing well-dominated graphs is not known. It is even not known whether the problem is in **NP**.

Theorem (Finbow, Hartnell, Nowakowski, 1998)

Recognizing well-dominated graphs with girth at least 6 can be done polynomially.

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A graph G is in the family F if there exists $\{x_1, ..., x_k\} \subseteq V(G)$ such that x_i is simplicial for each $1 \leq i \leq k$, and $\{N[x_i] : 1 \leq i \leq k\}$ is a partition of $V(G)$.

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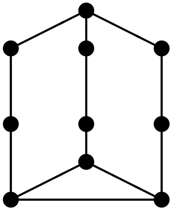

Definitions

Definition

$\mathcal{G}(\widehat{C_{i_1}}, ..., \widehat{C_{i_k}})$ is the family of all graphs which do not contain cycles of lengths $i_1, ..., i_k$.
The forbidden cycles are not necessarily induced.

$K_{10} \notin \mathcal{G}(\widehat{C_4})$

Main result 1

Theorem (Finbow, Hartnell, Nowakowski, 1994)

Let $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5})$ be a connected graph. Then G is well-covered if and only if one of the following holds:

1. *G is isomorphic to either C_7 or T_{10}.*
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Weighted graphs

Definition

Let G be a graph and let $w : V(G) \longrightarrow \mathbb{R}$. Then

- $mDS_w(G)$ is the minimum weight of a dominating set.
- $MDS_w(G)$ is the maximum weight of a minimal dominating set.
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Let G be a graph and let $w : V(G) \longrightarrow \mathbb{R}$. Then

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- G is w-**well-dominated** if $mDS_w(G) = MDS_w(G)$.

If G is w-well-dominated then G is w-well-covered.

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Theorem

Let G be a graph. Then the set of weight functions $w : V(G) \longrightarrow \mathbb{R}$ such that G is w-well-dominated is a vector space.

Proof:

$\forall v \in V \quad w(v) = w_1(v) + \lambda w_2(v)$

$$w(S) = \sum_{s \in S} w(s) = \sum_{s \in S} (w_1(s) + \lambda w_2(s)) = \sum_{s \in S} w_1(s) + \lambda \sum_{s \in S} w_2(s)$$

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Vector space

Definition

The vector space of weight functions $w : V(G) \longrightarrow \mathbb{R}$ such that G is w-well-covered is denoted by $WCW(G)$.

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The vector space of weight functions $w : V(G) \longrightarrow \mathbb{R}$ such that G is w-well-dominated is denoted by $WWD(G)$.

Theorem

$WWD(G)$ is a subspace of $WCW(G)$.

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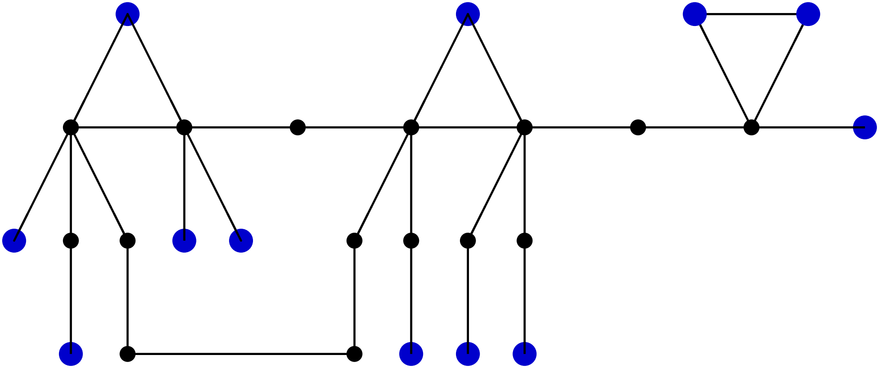
Definition

$L(G)$ is the set of all vertices $v \in V(G)$ such that either $d(v) = 1$ or v is on a triangle and $d(v) = 2$.



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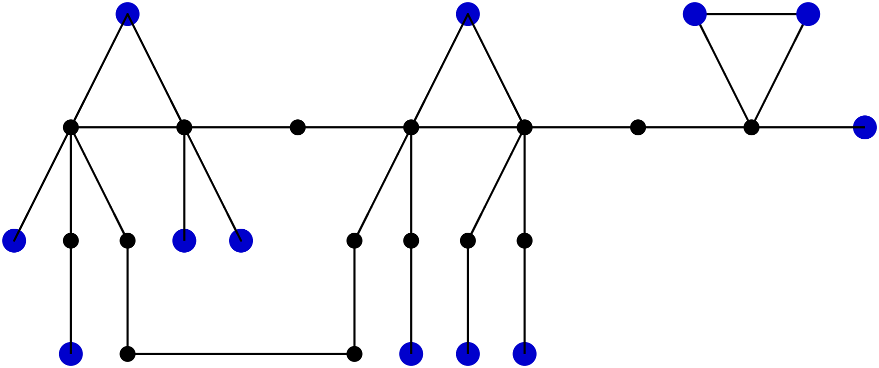
$D(v) = N(v) \setminus N(N_2(v)) = N(v) \cap L(G)$.
$M(v)$ is a maximal independent set of $D(v)$.



$WCW(G)$ **when** $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$

Theorem (Levit, Tankus, 2015)

Let $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$ be a graph, and let $w : V(G) \longrightarrow \mathbb{R}$. Then G is w-well-covered if and only if one of the following holds:

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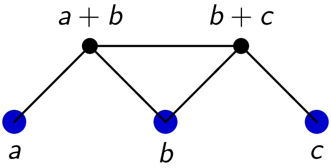
   - *For every $v \in V(G) \setminus L(G)$ it holds that $w(v) = w(M(v))$ for some maximal independent set $M(v)$ of $D(v)$.*

$WWD(G) \subsetneq WCW(G)$



$w \in WCW(G)$

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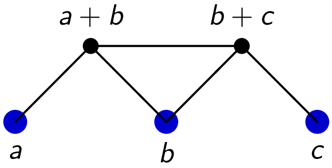

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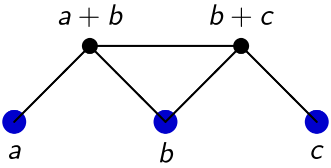

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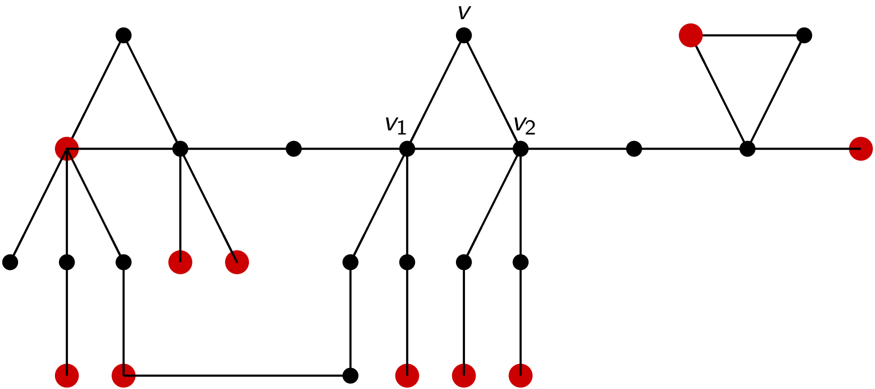
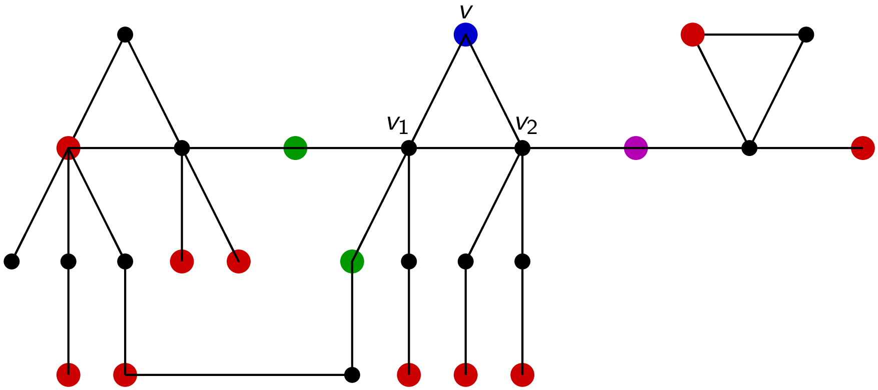


S is a maximal independent set of $G \setminus N_2[v]$.
S dominates neither $N(v_1) \cap N_2(v)$ nor $N(v_2) \cap N_2(v)$.



S is a maximal independent set of $G \setminus N_2[v]$.
S_1 is a maximal independent set of $(N(v_1) \cap N_2(v)) \setminus N(S)$.
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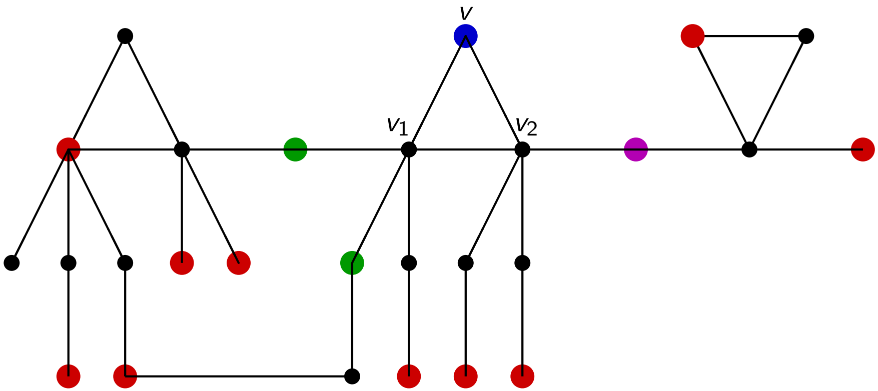

$T = S \cup S_1 \cup S_2 \cup \{v\}$

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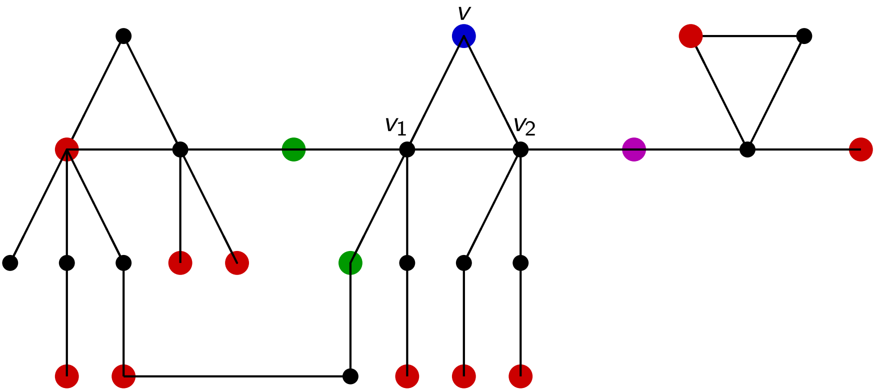

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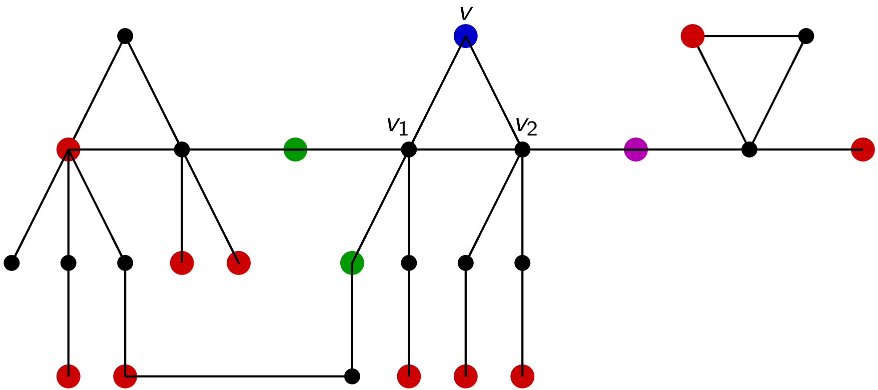

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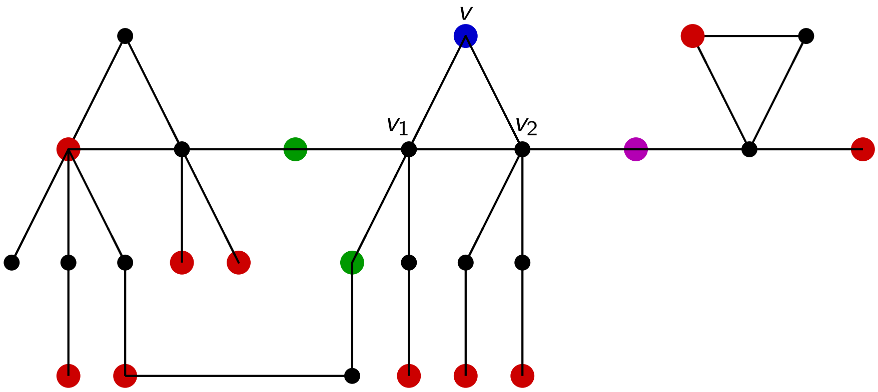

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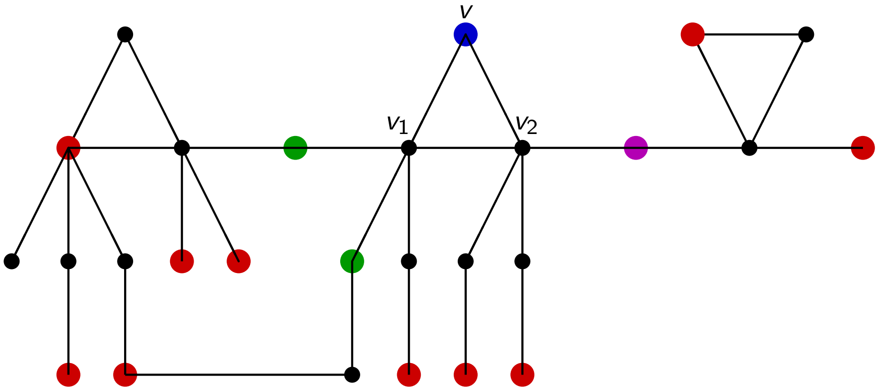

$w(v) = 0$

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Definition

$L^*(G)$ is the set of all vertices $v \in V(G)$ such that one of the following holds:

- $d(v) = 1$.
- The following conditions hold:
  - $d(v) = 2$.
  - v is on a triangle, (v, v_1, v_2).
  - Every maximal independent set of $V(G) \setminus N_2[v]$ dominates at least one of $N(v_1) \cap N_2(v)$ and $N(v_2) \cap N_2(v)$.

$L^*(G) \subseteq L(G)$.

$v \in L(G) \setminus L^*(G)$ if and only if the following conditions hold:

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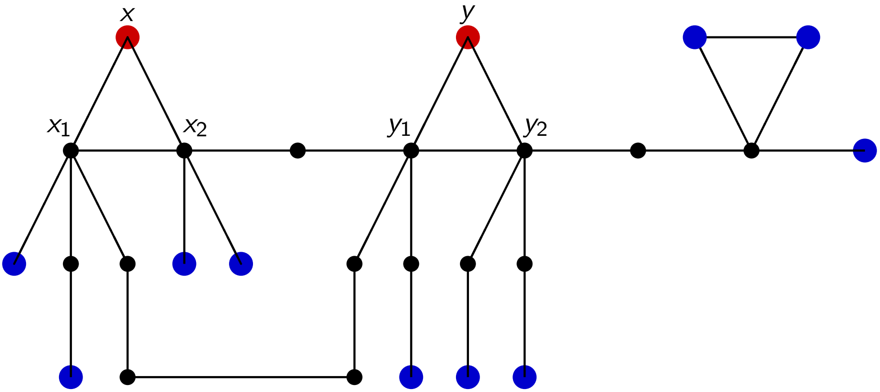


$L(G) \setminus L^*(G) = \{x, y\}$

$WWD(G)$ when $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$

Theorem

Let $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$ be a connected graph, and let $w : V(G) \longrightarrow \mathbb{R}$. Then G is w-well-dominated if and only if one of the following holds:

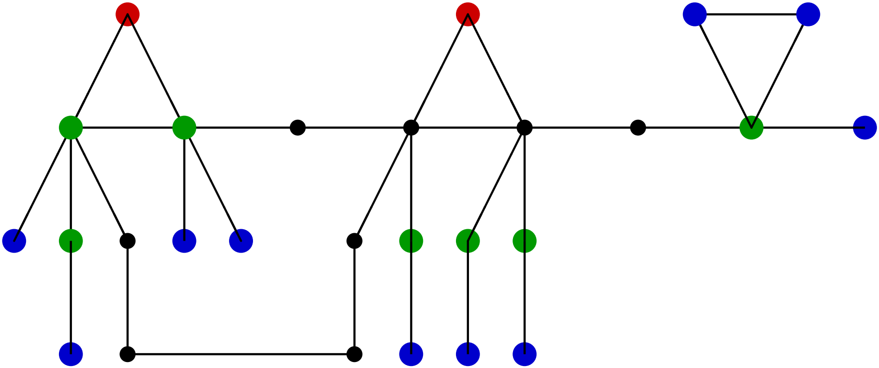
1. *G is isomorphic to either C_7 or T_{10}, and there exists a constant $k \in \mathbb{R}$ such that $w \equiv k$.*
2. *The following conditions hold:*
   1. *G is isomorphic to neither C_7 nor T_{10}.*
   2. *For every two vertices, l_1 and l_2, in the same component of $L(G)$ it holds that $w(l_1) = w(l_2)$.*
   3. *$w(v) = 0$ for every vertex $v \in L(G) \setminus L^*(G)$.*
   4. *For every $v \in V(G) \setminus L(G)$ it holds that $w(v) = w(M(v))$ for some maximal independent set $M(v)$ of $D(v)$.*

Proof

Let $N(L^*(G)) = \{v_1, ..., v_k\}$.

For each $1 \leq i \leq k$ let $T_i = \{v_i\} \cup D(v_i)$.

$S \cap T_i \in \{\{v_i\}, M(v_i)\} \implies w(S \cap T_i) = w(v_i)$.



Proof

$$w(S) = w(S \setminus (\bigcup_{1 \le i \le k} T_i)) + \sum_{1 \le i \le k} w(S \cap T_i) - \sum_{1 \le i < j \le k} w(S \cap T_i \cap T_j) =$$

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$$w(S) = w(S \setminus (\bigcup_{1 \leq i \leq k} T_i)) + \sum_{1 \leq i \leq k} w(S \cap T_i) - \sum_{1 \leq i < j \leq k} w(S \cap T_i \cap T_j) =$$

$$= 0 + \sum_{1 \leq i \leq k} w(v_i) - 0 = \sum_{1 \leq i \leq k} w(v_i).$$

$$mDS_w(G) = MDS_w(G) = \sum_{1 \leq i \leq k} w(v_i)$$

Corollary

Assume $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$. *Then*

- $dim(WWD(G)) = \alpha(G[L^*(G)])$
- $L^*(G) = L(G) \iff WWD(G) = WCW(G)$.

Future Work

Problem

Discover more cases, where recognizing well-dominated graphs and/or finding $WWD(G)$ can be done polynomially.

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Characterize all graphs, where the equality $WCW(G) = WWD(G)$ holds.

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The end

Thank you