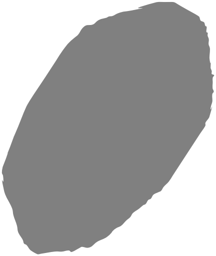
# Convex partitions of graphs

Maya Stein
Universidad de Chile

with Luciano Grippo, Martín Matamala, Martín Safe

Koper, June 2015

# Euclidean convexity



Convex

Not convex

Convex set C: has to contain all points on shortest lines between points in C.

# Euclidean convexity

Convex | Not convex

Convex set C: has to contain all points on shortest lines between points in C.

# Convexity spaces

A convexity $\mathcal{C}$ on a nonempty set $V$ is a collection of subsets of $V$, which we call convex sets, such that:

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex.
- Every nested union of convex sets is convex.

A convexity space is an ordered pair $(V, \mathcal{C})$, where $V$ is a nonempty set and $\mathcal{C}$ is a convexity on $V$.

## Convexity space $(V, \mathcal{C})$:

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex.
- Every nested union of convex sets is convex.

EXAMPLES:

- Standard convexity in a real vector space $V$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \forall t \in [0,1] : t \cdot x + (1-t) \cdot y \in C$.
- Order convexity in a poset $(V, \leq)$:
  $C \subseteq V$ is order convex iff $\forall x, y \in C$ : if $x \leq z \leq y$ then $z \in C$.
- Metric convexity in a metric space $(V, d)$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \{z \in V : d(x,z) + d(z,y) = d(x,y)\} \subseteq C$.

## Convexity space $(V, \mathcal{C})$:

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex.
- Every nested union of convex sets is convex.

EXAMPLES:

- Standard convexity in a real vector space $V$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \forall t \in [0,1] : t \cdot x + (1-t) \cdot y \in C$.
- Order convexity in a poset $(V, \leq)$:
  $C \subseteq V$ is order convex iff $\forall x, y \in C$ : if $x \leq z \leq y$ then $z \in C$.
- Metric convexity in a metric space $(V, d)$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \{z \in V : d(x,z) + d(z,y) = d(x,y)\} \subseteq C$.

## Convexity space $(V, \mathcal{C})$:

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex.
- Every nested union of convex sets is convex.

EXAMPLES:

- Standard convexity in a real vector space $V$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \forall t \in [0,1] : t \cdot x + (1-t) \cdot y \in C$.
- Order convexity in a poset $(V, \leq)$:
  $C \subseteq V$ is order convex iff $\forall x, y \in C$ : if $x \leq z \leq y$ then $z \in C$.
- Metric convexity in a metric space $(V, d)$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \{z \in V : d(x,z) + d(z,y) = d(x,y)\} \subseteq C$.

## Convexity space $(V, \mathcal{C})$:

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex.
- Every nested union of convex sets is convex.

EXAMPLES:

- Standard convexity in a real vector space $V$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \forall t \in [0, 1] : t \cdot x + (1 - t) \cdot y \in C$.
- Order convexity in a poset $(V, \leq)$:
  $C \subseteq V$ is order convex iff $\forall x, y \in C$ : if $x \leq z \leq y$ then $z \in C$.
- Metric convexity in a metric space $(V, d)$:
  $C \subseteq V$ is convex iff
  $\forall x, y \in C, \{z \in V : d(x, z) + d(z, y) = d(x, y)\} \subseteq C$.

# Graph convexities

- Geodesic convexity (or shortest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.
  (Feldman Högaasen 1969, Harary, Nieminen 1981)
- Monophonic convexity (or induced path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on induced $x$-$y$ paths lie in $C$.
  (Farber, Jamison 1986)
- Detour convexity (or longest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on longest $x$-$y$ paths lie in $C$.
  (Chartrand, Garry, Zhang 2003)
- $P_3$-convexity, triangle-path convexity, ...
- Survey P. Duchet 1987, Book I. Pelayo 2013

# Graph convexities

- Geodesic convexity (or shortest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.
  (Feldman Högaasen 1969, Harary, Nieminen 1981)
- Monophonic convexity (or induced path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on induced $x$-$y$ paths lie in $C$.
  (Farber, Jamison 1986)
- Detour convexity (or longest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on longest $x$-$y$ paths lie in $C$.
  (Chartrand, Garry, Zhang 2003)
- $P_3$-convexity, triangle-path convexity, ...
- Survey P. Duchet 1987, Book I. Pelayo 2013

# Graph convexities

- Geodesic convexity (or shortest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.
  (Feldman Högaasen 1969, Harary, Nieminen 1981)
- Monophonic convexity (or induced path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on induced $x$-$y$ paths lie in $C$.
  (Farber, Jamison 1986)
- Detour convexity (or longest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on longest $x$-$y$ paths lie in $C$.
  (Chartrand, Garry, Zhang 2003)
- $P_3$-convexity, triangle-path convexity, ...
- Survey P. Duchet 1987, Book I. Pelayo 2013

# Graph convexities

- Geodesic convexity (or shortest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.
  (Feldman Högaasen 1969, Harary, Nieminen 1981)
- Monophonic convexity (or induced path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on induced $x$-$y$ paths lie in $C$.
  (Farber, Jamison 1986)
- Detour convexity (or longest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on longest $x$-$y$ paths lie in $C$.
  (Chartrand, Garry, Zhang 2003)
- $P_3$-convexity, triangle-path convexity, ...
- Survey P. Duchet 1987, Book I. Pelayo 2013

# Graph convexities

- Geodesic convexity (or shortest path convexity):
  $C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.
  (Feldman Högaasen 1969, Harary, Nieminen 1981)
-
-
-
- Survey P. Duchet 1987, Book I. Pelayo 2013

We will stick to:

Geodesic convexity (or shortest path convexity):

$C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.

- In a connected graph, convex sets are connected.
- Cliques are convex sets.
- Shortest cycles are convex.
- Unions of convex sets might fail to be convex.

We will stick to:

Geodesic convexity (or shortest path convexity):

$C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.

- In a connected graph, convex sets are connected.
- Cliques are convex sets.
- Shortest cycles are convex.
- Unions of convex sets might fail to be convex.

We will stick to:

Geodesic convexity (or shortest path convexity):

$C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.

- In a connected graph, convex sets are connected.
- Cliques are convex sets.
- Shortest cycles are convex.
- Unions of convex sets might fail to be convex.

We will stick to:

Geodesic convexity (or shortest path convexity):

$C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.

- In a connected graph, convex sets are connected.
- Cliques are convex sets.
- Shortest cycles are convex.
- Unions of convex sets might fail to be convex.

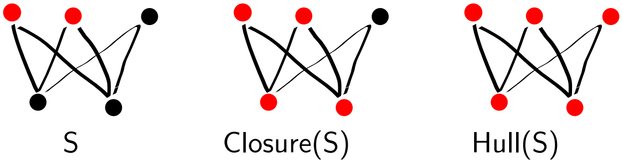
We will stick to:

Geodesic convexity (or shortest path convexity):

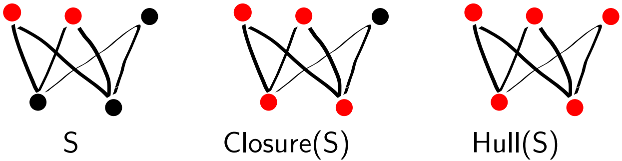
$C \subseteq V(G)$ is convex iff $\forall x, y \in C$, all vertices on shortest $x$-$y$ paths lie in $C$.

- In a connected graph, convex sets are connected.
- Cliques are convex sets.
- Shortest cycles are convex.
- Unions of convex sets might fail to be convex.

- Geodetic closure of a set $S$: obtained by adding all vertices on shortest paths between vertices of $S$.
- Convex hull of a set $S$: smallest convex set containing $S$.



- Geodetic closure of a set $S$: obtained by adding all vertices on shortest paths between vertices of $S$.
- Convex hull of a set $S$: smallest convex set containing $S$.



# Invariants and their complexity

The geodetic number $g(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose closure is $V(G)$. (Harary, Loukakis, Tsouros 1993)

Geodetic Number Problem: Given $G$ and $k$, determine whether $g(G) \leq k$.

- The Geodetic Number Problem is NP-complete.(Atici 2003)
- Remains NP-complete for bipartite and for chordal graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2010)
- Polynomial for cographs...

# Invariants and their complexity

The geodetic number $g(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose closure is $V(G)$. (Harary, Loukakis, Tsouros 1993)

Geodetic Number Problem: Given $G$ and $k$, determine whether $g(G) \leq k$.

- The Geodetic Number Problem is NP-complete.(Atici 2003)
- Remains NP-complete for bipartite and for chordal graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2010)
- Polynomial for cographs...

# Invariants and their complexity

The geodetic number $g(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose closure is $V(G)$. (Harary, Loukakis, Tsouros 1993)

Geodetic Number Problem: Given $G$ and $k$, determine whether $g(G) \leq k$.

- The Geodetic Number Problem is NP-complete.(Atici 2003)
- Remains NP-complete for bipartite and for chordal graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2010)
- Polynomial for cographs...

# Invariants and their complexity

The geodetic number $g(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose closure is $V(G)$. (Harary, Loukakis, Tsouros 1993)

Geodetic Number Problem: Given $G$ and $k$, determine whether $g(G) \leq k$.

- The Geodetic Number Problem is NP-complete.(Atici 2003)
- Remains NP-complete for bipartite and for chordal graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2010)
- Polynomial for cographs...

# Invariants and their complexity

The hull number $h(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose hull is $V(G)$.
(Everett, Seidman 1985)

Hull Number Problem: Given $G$ and $k$, determine whether $h(G) \leq k$.

- The Hull Number Problem is NP-complete. (Dourado, Gimbel, Kratochvíl, Protti, Szwarcfiter 2009)
- Remains NP-complete for bipartite graphs.(Araújo, Campos, Giroire, Nisse, Sampaio, Pardo Soares 2011)
- Polynomial for cographs, proper interval graphs, split graphs...

# Invariants and their complexity

The hull number $h(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose hull is $V(G)$.
(Everett, Seidman 1985)

Hull Number Problem: Given $G$ and $k$, determine whether $h(G) \leq k$.

- The Hull Number Problem is NP-complete. (Dourado, Gimbel, Kratochvíl, Protti, Szwarcfiter 2009)
- Remains NP-complete for bipartite graphs.(Araújo, Campos, Giroire, Nisse, Sampaio, Pardo Soares 2011)
- Polynomial for cographs, proper interval graphs, split graphs...

# Invariants and their complexity

The hull number $h(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose hull is $V(G)$.
(Everett, Seidman 1985)

Hull Number Problem: Given $G$ and $k$, determine whether $h(G) \leq k$.

- The Hull Number Problem is NP-complete. (Dourado, Gimbel, Kratochvíl, Protti, Szwarcfiter 2009)
- Remains NP-complete for bipartite graphs.(Araújo, Campos, Giroire, Nisse, Sampaio, Pardo Soares 2011)
- Polynomial for cographs, proper interval graphs, split graphs...

# Invariants and their complexity

The hull number $h(G)$ of a connected graph $G$ is the minimum cardinality of a set $S \subseteq V(G)$ whose hull is $V(G)$.
(Everett, Seidman 1985)

Hull Number Problem: Given $G$ and $k$, determine whether $h(G) \leq k$.

- The Hull Number Problem is NP-complete. (Dourado, Gimbel, Kratochvíl, Protti, Szwarcfiter 2009)
- Remains NP-complete for bipartite graphs.(Araújo, Campos, Giroire, Nisse, Sampaio, Pardo Soares 2011)
- Polynomial for cographs, proper interval graphs, split graphs...

# Invariants and their complexity

The convexity number $con(G)$ of a connected graph $G$ is the maximum cardinality of a proper convex set of $G$.
(Chartrand, Wall, and Zhang 2002)

Convexity Number Problem: Given $G$ and $k$, determine whether $con(G) \geq k$.

- The Convexity Number Problem is NP-complete.(Gimbel 2003)
- Remains NP-complete for bipartite graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2012)
- Linear for cographs...

# Invariants and their complexity

The convexity number $con(G)$ of a connected graph $G$ is the maximum cardinality of a proper convex set of $G$.
(Chartrand, Wall, and Zhang 2002)

Convexity Number Problem: Given $G$ and $k$, determine whether $con(G) \geq k$.

- The Convexity Number Problem is NP-complete.(Gimbel 2003)
- Remains NP-complete for bipartite graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2012)
- Linear for cographs...

# Invariants and their complexity

The convexity number $con(G)$ of a connected graph $G$ is the maximum cardinality of a proper convex set of $G$.
(Chartrand, Wall, and Zhang 2002)

Convexity Number Problem: Given $G$ and $k$, determine whether $con(G) \geq k$.

- The Convexity Number Problem is NP-complete.(Gimbel 2003)
- Remains NP-complete for bipartite graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2012)
- Linear for cographs...

# Invariants and their complexity

The convexity number $con(G)$ of a connected graph $G$ is the maximum cardinality of a proper convex set of $G$.
(Chartrand, Wall, and Zhang 2002)

Convexity Number Problem: Given $G$ and $k$, determine whether $con(G) \geq k$.

- The Convexity Number Problem is NP-complete.(Gimbel 2003)
- Remains NP-complete for bipartite graphs. (Dourado, Protti, Rautenbach, Szwarcfiter 2012)
- Linear for cographs...

# Convex partitions

A convex $p$-partition of a graph $G$ is a partition of $V(G)$ into $p$ convex sets.

- Every graph has a convex 1-partition, and a convex $|V(G)|$-partition.
- If $G$ has a matching of size $m$, then $G$ has a convex $(|V(G)| - m)$-partition.

Convex Partition Problem
Given $G$ and $p$, determine whether $G$ has a convex $p$-partition.

Generalization of the Clique Partition Problem.

# Convex partitions

A convex $p$-partition of a graph $G$ is a partition of $V(G)$ into $p$ convex sets.

- Every graph has a convex 1-partition, and a convex $|V(G)|$-partition.
- If $G$ has a matching of size $m$, then $G$ has a convex $(|V(G)| - m)$-partition.

**Convex Partition Problem**
Given $G$ and $p$, determine whether $G$ has a convex $p$-partition.

Generalization of the Clique Partition Problem.

# Convex partitions

A convex $p$-partition of a graph $G$ is a partition of $V(G)$ into $p$ convex sets.

- Every graph has a convex 1-partition, and a convex $|V(G)|$-partition.
- If $G$ has a matching of size $m$, then $G$ has a convex $(|V(G)| - m)$-partition.

Convex Partition Problem
Given $G$ and $p$, determine whether $G$ has a convex $p$-partition.

Generalization of the Clique Partition Problem.

# Convex partitions

A convex $p$-partition of a graph $G$ is a partition of $V(G)$ into $p$ convex sets.

- Every graph has a convex 1-partition, and a convex $|V(G)|$-partition.
- If $G$ has a matching of size $m$, then $G$ has a convex $(|V(G)| - m)$-partition.

Convex Partition Problem
Given $G$ and $p$, determine whether $G$ has a convex $p$-partition.

Generalization of the Clique Partition Problem.

# Convex partitions

A convex $p$-partition of a graph $G$ is a partition of $V(G)$ into $p$ convex sets.

- Every graph has a convex 1-partition, and a convex $|V(G)|$-partition.
- If $G$ has a matching of size $m$, then $G$ has a convex $(|V(G)| - m)$-partition.

Convex Partition Problem
Given $G$ and $p$, determine whether $G$ has a convex $p$-partition.

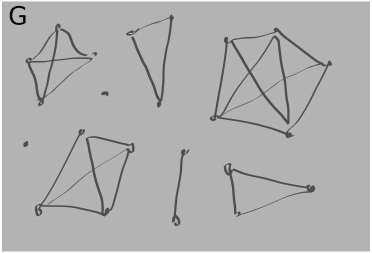
Generalization of the Clique Partition Problem.

# Clique partitions

A clique partition of a graph $G$ is a partition of $V(G)$ into $p$ cliques.

Clique Partition Problem. Given $G$ and $k$, determine whether $G$ has a partition into $k$ cliques.

- One of Karp's 21 NP-complete problems.
- Equivalent to $k$-colouring (of the complement of $G$).

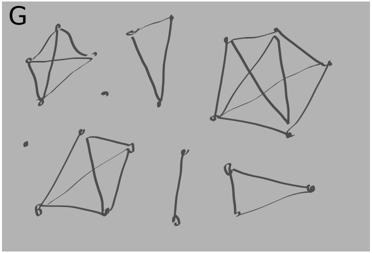


# Clique partitions

A clique partition of a graph $G$ is a partition of $V(G)$ into $p$ cliques.

Clique Partition Problem. Given $G$ and $k$, determine whether $G$ has a partition into $k$ cliques.

- One of Karp's 21 NP-complete problems.
- Equivalent to $k$-colouring (of the complement of $G$).

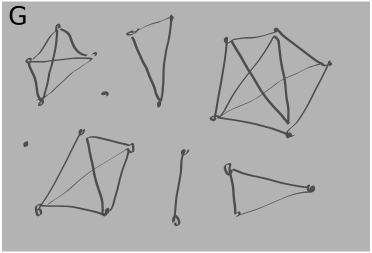


# Clique partitions

A clique partition of a graph $G$ is a partition of $V(G)$ into $p$ cliques.

Clique Partition Problem. Given $G$ and $k$, determine whether $G$ has a partition into $k$ cliques.

- One of Karp's 21 NP-complete problems.
- Equivalent to $k$-colouring (of the complement of $G$).

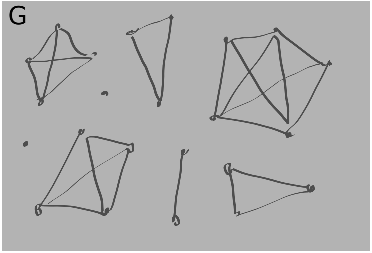


# Clique partitions

A clique partition of a graph $G$ is a partition of $V(G)$ into $p$ cliques.

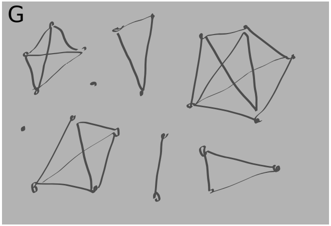
Clique Partition Problem. Given $G$ and $k$, determine whether $G$ has a partition into $k$ cliques.

- One of Karp's 21 NP-complete problems.
- Equivalent to $k$-colouring (of the complement of $G$).

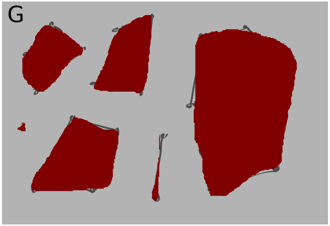


# Clique partitions vs Convex partitions

Clique Partition Problem

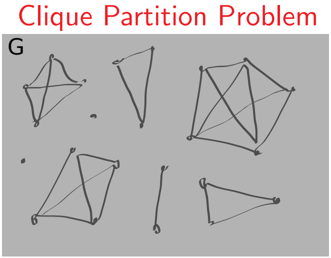



Convex Partition Problem




- $G$ has clique $(k-1)$-partition $\Rightarrow$ $G$ has clique $k$-partition
- $G$ has convex $(p-1)$-partition $\not\Rightarrow$ $G$ has convex $p$-partition

# Clique partitions vs Convex partitions

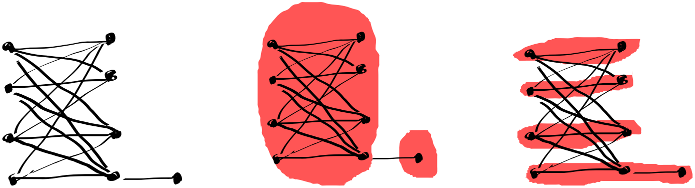




- $G$ has clique $(k-1)$-partition $\Rightarrow$ $G$ has clique $k$-partition
- $G$ has convex $(p-1)$-partition $\not\Rightarrow$ $G$ has convex $p$-partition

# Convex partitions

- $G$ has convex $(p-1)$-partition $\not\Rightarrow$ $G$ has convex $p$-partition
- $G$ has convex $(p+1)$-partition $\not\Rightarrow$ $G$ has convex $p$-partition

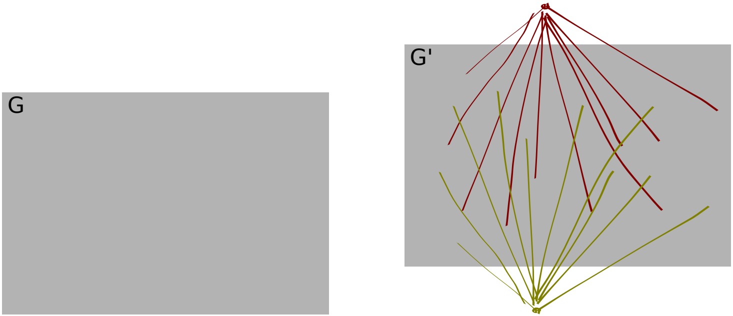


# Complexity of convex partitions

**Theorem (Artigas, Dantas, Dourado, Szwarcfiter 2011)**

*The Convex p-Partition Problem is NP-complete for $p \geq 2$.*

Follows from NP-completeness of the Clique Partition Problem, if $p \geq 3$:




Obtain $G'$ from $G$ by adding two universal vertices.

Observe: In $G'$, every convex set $\neq V(G')$ is a clique.

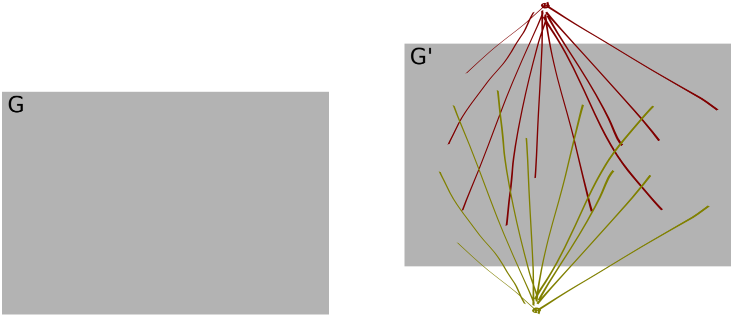
For $p = 2$, reduction to 1-in-3 Problem. $\square$

# Complexity of convex partitions

**Theorem (Artigas, Dantas, Dourado, Szwarcfiter 2011)**

*The Convex p-Partition Problem is NP-complete for $p \geq 2$.*

Follows from NP-completeness of the Clique Partition Problem, if $p \geq 3$:




Obtain $G'$ from $G$ by adding two universal vertices.

Observe: In $G'$, every convex set $\neq V(G')$ is a clique.

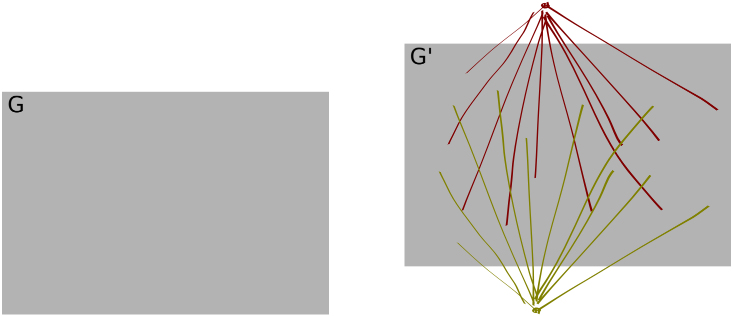
For $p = 2$, reduction to 1-in-3 Positive

# Complexity of convex partitions

**Theorem (Artigas, Dantas, Dourado, Szwarcfiter 2011)**

*The Convex p-Partition Problem is NP-complete for $p \geq 2$.*

Follows from NP-completeness of the Clique Partition Problem, if $p \geq 3$:




Obtain $G'$ from $G$ by adding two universal vertices.

Observe: In $G'$, every convex set $\neq V(G')$ is a clique.

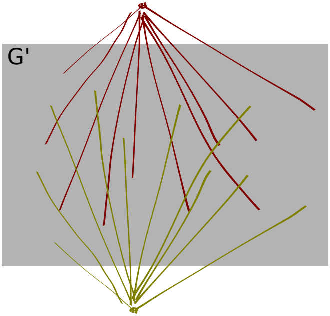
For $p = 2$, reduction to 1-in-3 Problem □

# Complexity of convex partitions

**Theorem (Artigas, Dantas, Dourado, Szwarcfiter 2011)**

*The Convex p-Partition Problem is NP-complete for $p \geq 2$.*

Follows from NP-completeness of the Clique Partition Problem, if $p \geq 3$:

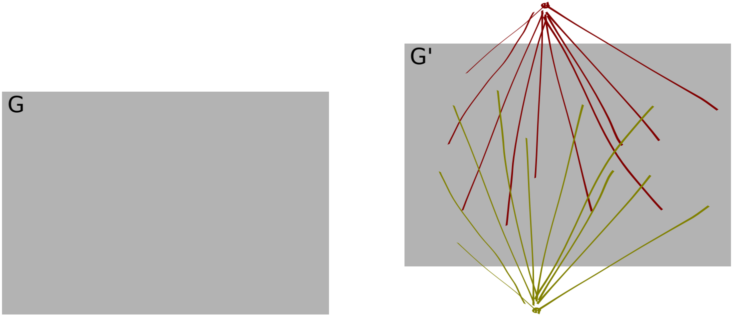

Obtain $G'$ from $G$ by adding two universal vertices.
Observe: In $G'$, every convex set $\neq V(G')$ is a clique.
For $p = 2$, reduction to 1-in-3 Problem □

# Complexity of convex partitions

**Theorem (Artigas, Dantas, Dourado, Szwarcfiter 2011)**

*The Convex p-Partition Problem is NP-complete for $p \geq 2$.*

Follows from NP-completeness of the Clique Partition Problem, if $p \geq 3$:




Obtain $G'$ from $G$ by adding two universal vertices.
Observe: In $G'$, every convex set $\neq V(G')$ is a clique.
For $p = 2$, reduction to 1-in-3 Problem $\square$

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

The Convex $p$-Partition Problem is ...

- NP-complete in general
- polynomial for cographs. (Artigas, Dantas, Dourado, Szwarcfiter 2011)
- polynomial for planar graphs, if $p = 2$. (Glantz, Mayerhenke 2013)
  (They use work of Chepoi et al. on the links between alternating and convex cuts of plane graphs.)

Also, it is known all chordal graphs allow convex $p$-partitions, for all $1 \leq p \leq n$.

And, all $C_n^k$ have convex 2-partitions.

# Complexity of convex partitions

**Conjecture (Pelayo 2013)**

The Convex $p$-Partition Problem is NP-complete, even when restricted to bipartite graphs.

special case $p = 2$:
byproduct of Glantz, Meyerhenke 2013: The Convex 2-Partition Problem is polynomial for bipartite graphs.

**Theorem (Grippo, Matamala, Safe, St 2015)**

*The Convex $p$-Partition Problem is polynomial for bipartite graphs, for all $p \geq 2$.*

# Complexity of convex partitions

**Conjecture (Pelayo 2013)**

The Convex $p$-Partition Problem is NP-complete, even when restricted to bipartite graphs.

special case $p = 2$:
byproduct of Glantz, Mayerhenke 2013: The Convex 2-Partition Problem is polynomial for bipartite graphs.

**Theorem (Grippo, Matamala, Safe, St 2015)**

*The Convex p-Partition Problem is polynomial for bipartite graphs, for all $p \geq 2$.*

# Complexity of convex partitions

**Conjecture (Pelayo 2013)**

The Convex $p$-Partition Problem is NP-complete, even when restricted to bipartite graphs.

special case $p = 2$:
byproduct of Glantz, Mayerhenke 2013: The Convex 2-Partition Problem is polynomial for bipartite graphs.

**Theorem (Grippo, Matamala, Safe, St 2015)**

*The Convex p-Partition Problem is polynomial for bipartite graphs, for all $p \geq 2$.*

# Complexity of convex partitions

**Conjecture (Pelayo 2013)**

The Convex $p$-Partition Problem is NP-complete, even when restricted to bipartite graphs.

special case $p = 2$:

byproduct of Glantz, Mayerhenke 2013: The Convex 2-Partition Problem is polynomial for bipartite graphs.

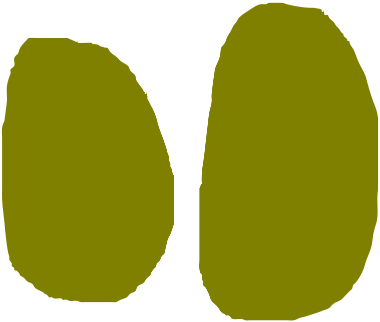
**Theorem (Grippo, Matamala, Safe, St 2015)**

*The Convex $p$-Partition Problem is polynomial for bipartite graphs, for all $p \geq 2$.*

**Theorem (Grippo, Matamala, Safe, St 2015)**

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:

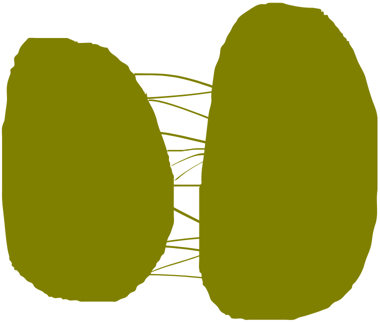


Consider a convex 2-partition.

**Theorem (Grippo, Matamala, Safe, St 2015)**

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:

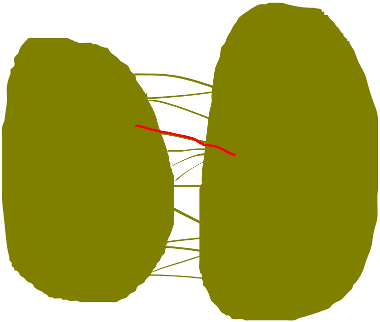


Consider a convex 2-partition.

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:

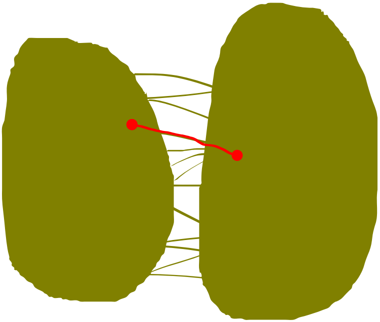


Consider a convex 2-partition.

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:

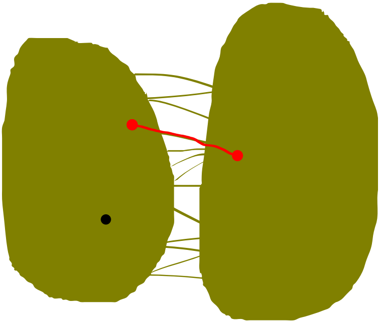


Consider a convex 2-partition.

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:

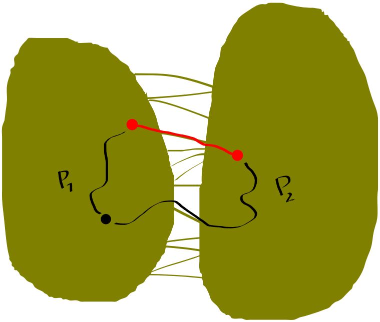


Consider a convex 2-partition.

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:



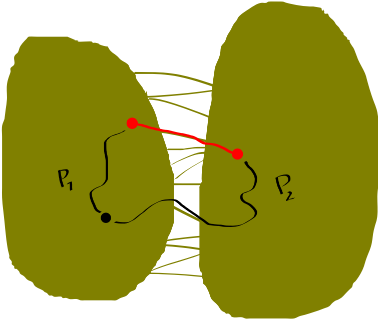

$|P_1| < |P_2|$

Consider a convex 2-partition.

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

Observation:




$|P_1| < |P_2|$

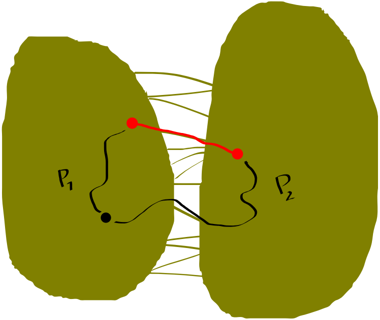
Fix partition $(X, Y)$, fix $X$-$Y$ edge $xy$. Then:

$(X, Y)$ is a convex 2-partition $\Rightarrow$ the v's in $X$ are closer to $x$ than to $y$ and vice versa

## Theorem (Grippo, Matamala, Safe, St 2015)

*Convex p-Partition is polynomial for bipartite graphs, for all $p \geq 2$.*

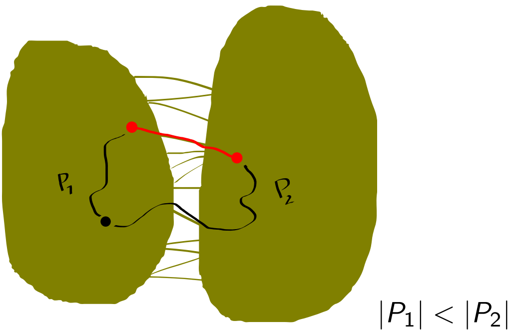
Observation:




$|P_1| < |P_2|$

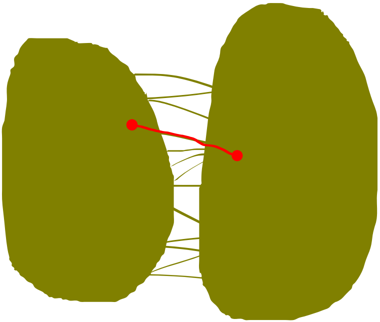
Fix $(X, Y)$, fix edge $xy$. Let $V_{xy} = \{v : d(x, v) < d(y, v)\}$ and $V_{yx} = \{v : d(x, v) > d(y, v)\}$. Then:

$(X, Y)$ is a convex 2-partition $\Rightarrow X = V_{xy}$ and $Y = V_{yx}$



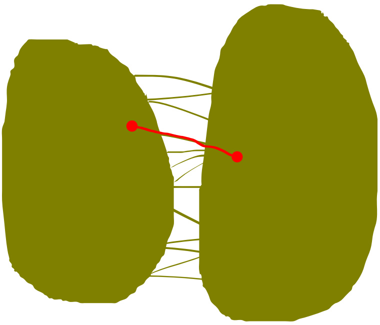

Fix $(X, Y)$, fix edge $xy$. Then:

$(X, Y)$ is a convex 2-partition $\Rightarrow X = V_{xy}$ and $Y = V_{yx}$



Fix $(X, Y)$, fix edge $xy$. Then:

$(X, Y)$ is a convex 2-partition $\Rightarrow X = V_{xy}$ and $Y = V_{yx}$



Fix $(X, Y)$, fix edge $xy$. Then:

$(X, Y)$ is a convex 2-partition $\Rightarrow X = V_{xy}$ and $Y = V_{yx}$

In order to decide whether bipartite $G$ has a convex 2-partition, it suffices to check for all edges $xy$ whether $V_{xy}$ and $V_{yx}$ are convex.

# Some side remarks

## Djoković 1973

$G$ embeds isometrically into the $r$-dimensional cube, for some $r \Leftrightarrow G$ is bipartite and for every edge $xy$ of $G$, the sets $V_{xy}$ and $V_{yx}$ are convex.

Define relation on $E(G)$ ($G$ connected):
$xy \sim vw$ iff $d(x,v) + d(y,w) \neq d(x,w) + d(y,v)$

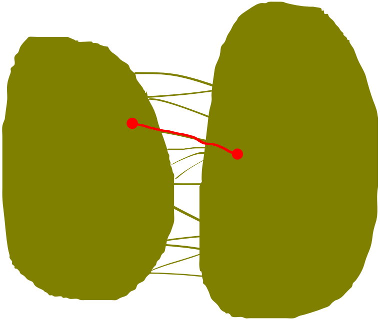
## Winkler 1984

$G$ embeds isometrically into the $r$-dimensional hypercube, for some $r \Leftrightarrow G$ is bipartite and $\sim$ is transitive on $E(G)$.

## Imrich, Klavžar 1997

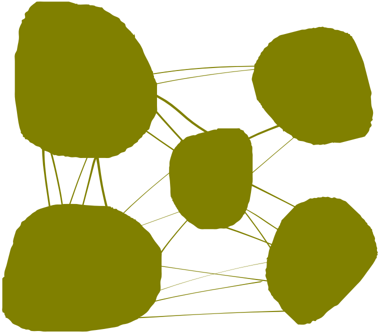
Let $G$ be bipartite, and $C \subseteq V(G)$ connected and induced. Then $C$ is convex $\Leftrightarrow$ for all edges $e \in E(G[C])$, $f \in E(C, G-C)$ we have $e \not\sim f$.

# Go back to our proof...



In order to decide whether bipartite $G$ has a convex 2-partition, it suffices to check for all edges $xy$ whether $V_{xy}$ and $V_{yx}$ are convex.

# For $p \geq 3$

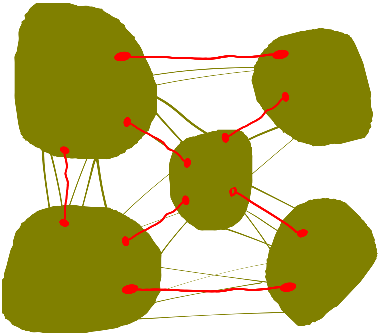


Convex $p$-partition $P$ of $G$.

Skeleton $(F; \phi)$ of $P$: edge set $F$, map $\phi$ from $V(F)$ to $[p]$.

IDEA: Check for all sets of $\leq \binom{p}{2}$ edges whether they can be the skeleton of some convex $p$-partition of $G$.

# For $p \geq 3$

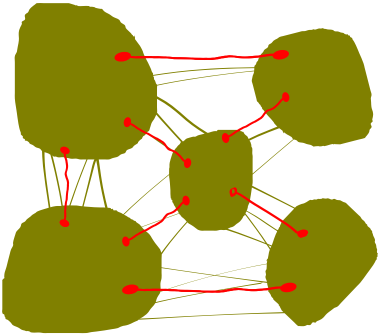


Convex $p$-partition $P$ of $G$.

Skeleton $(F; \phi)$ of $P$: edge set $F$, map $\phi$ from $V(F)$ to $[p]$.

IDEA: Check for all sets of $\leq \binom{p}{2}$ edges whether they can be the skeleton of some convex $p$-partition of $G$.

# For $p \geq 3$

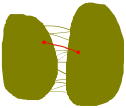


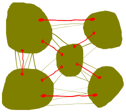
Convex $p$-partition $P$ of $G$.

Skeleton $(F; \phi)$ of $P$: edge set $F$, map $\phi$ from $V(F)$ to $[p]$.

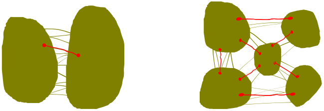
IDEA: Check for all sets of $\leq \binom{p}{2}$ edges whether they can be the skeleton of some convex $p$-partition of $G$.

# From $p = 2$ to $p \geq 3$

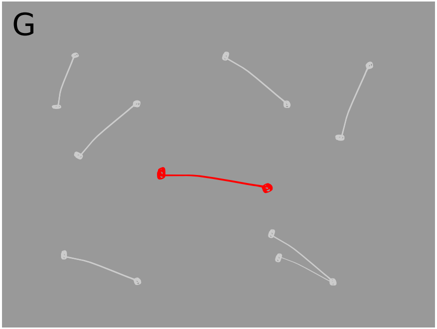


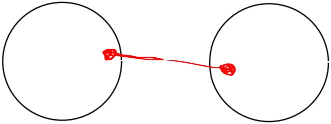


# From $p = 2$ to $p \geq 3$



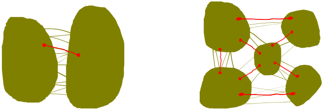
Remember for $p = 2$



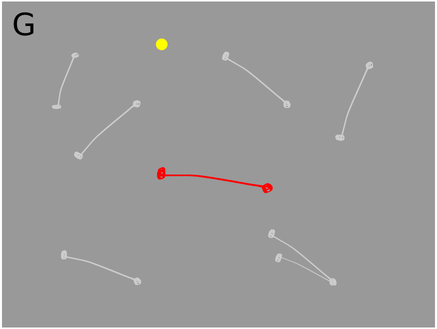



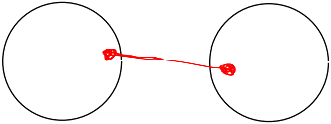
... we had to check $\forall xy$ whether $V_{xy}$ and $V_{yx}$ are convex

# From $p = 2$ to $p \geq 3$



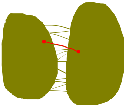
Remember for $p = 2$

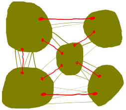


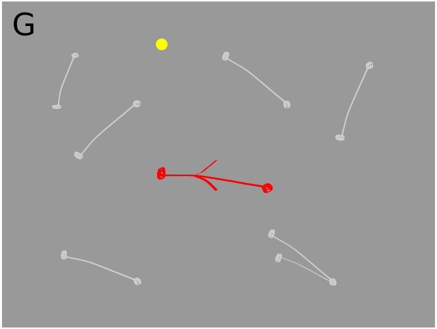
... we had to check $\forall xy$ whether $V_{xy}$ and $V_{yx}$ are convex

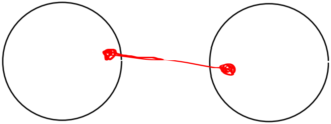
# From $p = 2$ to $p \geq 3$





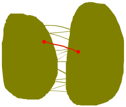
Remember for $p = 2$

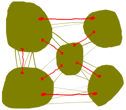


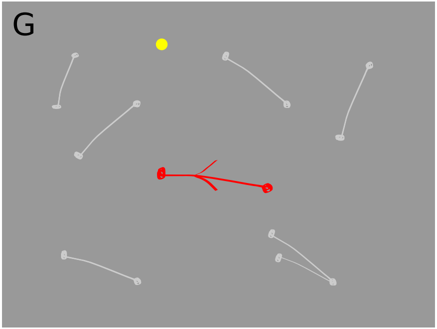
... we had to check $\forall xy$ whether $V_{xy}$ and $V_{yx}$ are convex

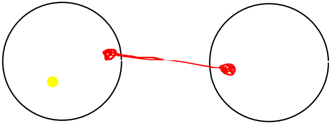
# From $p = 2$ to $p \geq 3$





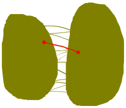
Remember for $p = 2$

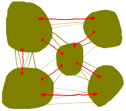


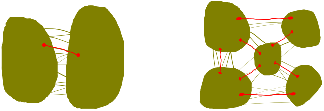
... we had to check $\forall xy$ whether $V_{xy}$ and $V_{yx}$ are convex

# From $p = 2$ to $p \geq 3$

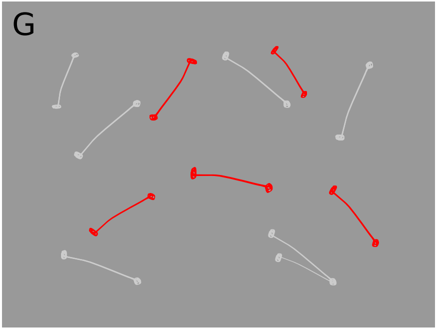


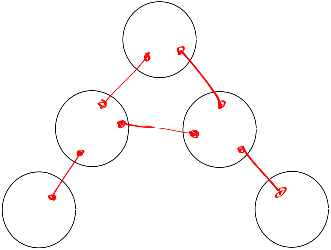


# From $p = 2$ to $p \geq 3$



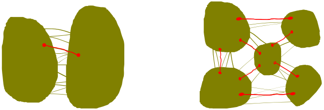
Naive idea for $p \geq 3$:



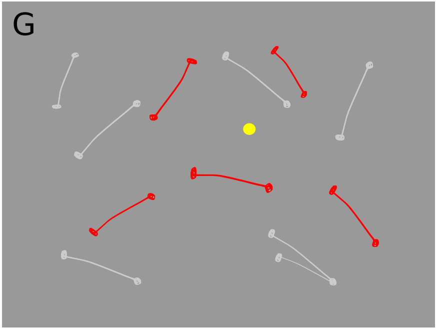



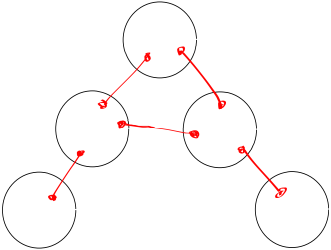
check $\forall F$ (with $|F| \leq \binom{p}{2}$) whether $F$ generates convex 'sink sets'.

# From $p = 2$ to $p \geq 3$



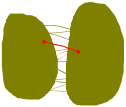
Naive idea for $p \geq 3$:

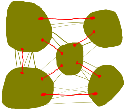


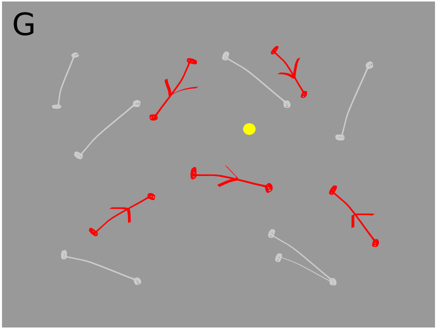
check $\forall F$ (with $|F| \leq \binom{p}{2}$) whether $F$ generates convex 'sink sets'.

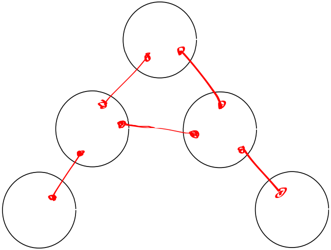
# From $p = 2$ to $p \geq 3$





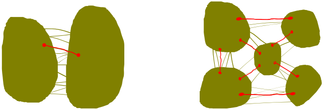
Naive idea for $p \geq 3$:



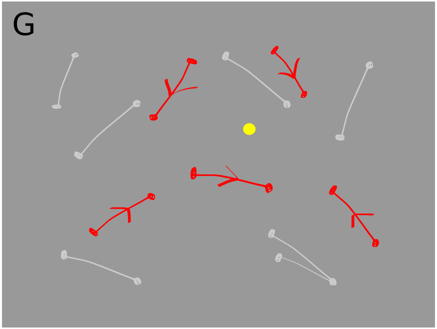



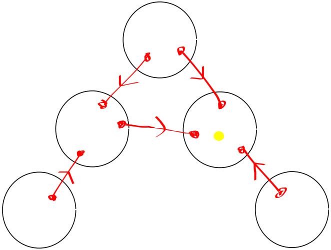
check $\forall F$ (with $|F| \leq \binom{p}{2}$) whether $F$ generates convex 'sink sets'.

# From $p = 2$ to $p \geq 3$



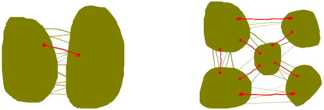
Naive idea for $p \geq 3$:



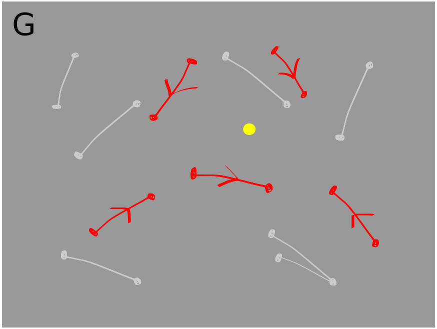



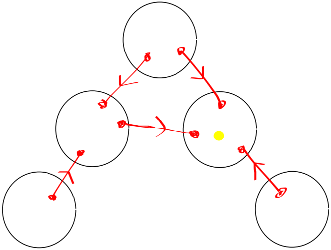
check $\forall F$ (with $|F| \leq \binom{p}{2}$) whether $F$ generates convex 'sink sets'.

# From $p = 2$ to $p \geq 3$



Naive idea for $p \geq 3$:

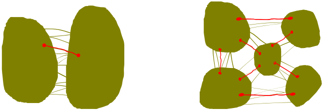


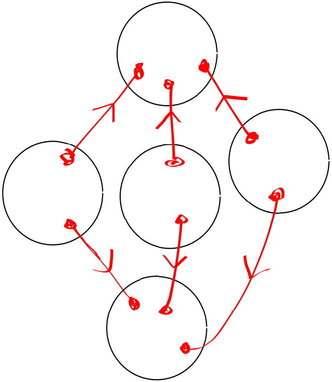
check $\forall F$ (with $|F| \leq \binom{p}{2}$) whether $F$ generates convex 'sink sets'.

This works if $|F| = \binom{p}{2}$. In other cases, there might be more than one sink.

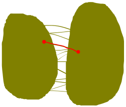
# From $p = 2$ to $p \geq 3$

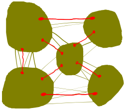


Example:

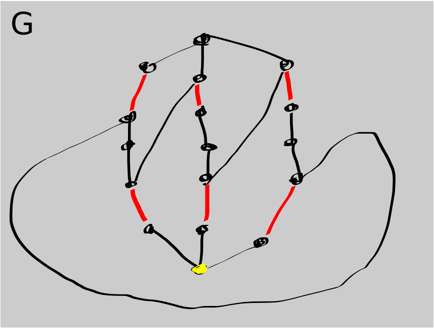


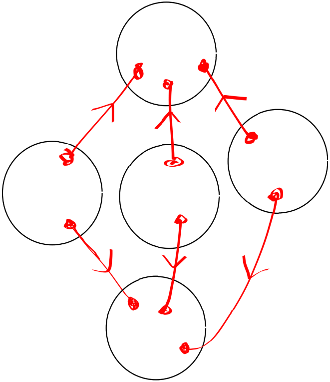
# From $p = 2$ to $p \geq 3$



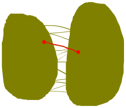


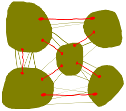
Example:



# From $p = 2$ to $p \geq 3$





But with some more analysis, we prove that

Let $G$ be a connected bipartite graph, let $F \subseteq E(G)$ and let $\phi : V(F) \to [p]$. Then there is at most one convex $p$-partition of $G$ with skeleton $(F; \phi)$. We can find such partition or show it does not exist in polynomial time.

Some questions:

- Is $p$-partition polynomial for planar graphs, for $p \geq 3$?

- Characterization of graphs with convex $p$-partitions?

- Other graph convexities?

Some questions:

- Is $p$-partition polynomial for planar graphs, for $p \geq 3$?

- Characterization of graphs with convex $p$-partitions?

- Other graph convexities?

Some questions:

- Is $p$-partition polynomial for planar graphs, for $p \geq 3$?

- Characterization of graphs with convex $p$-partitions?

- Other graph convexities?

Thank you!