Graphs of Edge-Intersecting and Non-Splitting Paths

A. Boyacı B.U Istanbul
T. Ekim B.U, Istanbul
M. Shalom Tel-Hai College
S. Zaks Technion

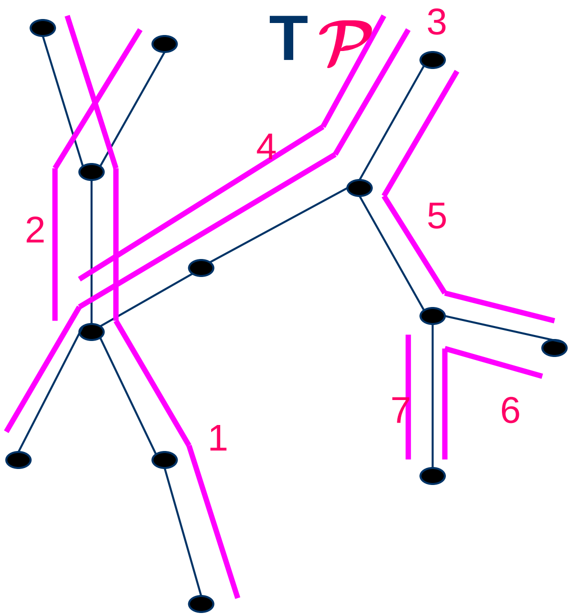
EPT and EPG Graphs

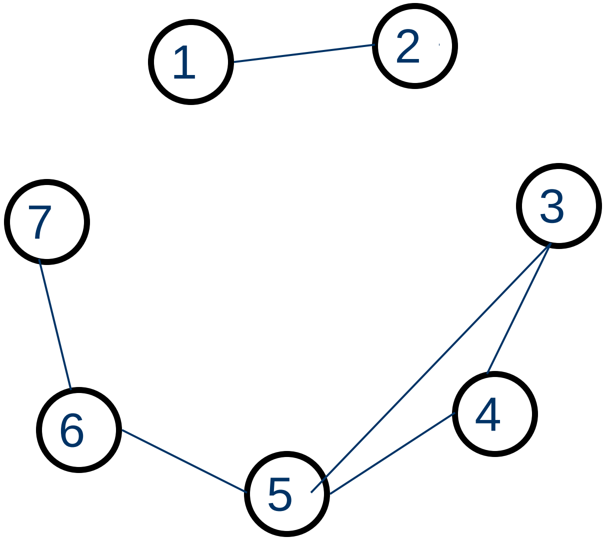
[1] Golumbic, M. C. & Jamison, R. E. (1985), 'The edge intersection graphs of paths in a tree', *Journal of Combinatorial Theory, Series B* **38**(1), 8 - 22.

[2] Golumbic, M. C.; Lipshteyn, M. & Stern, M. (2009), 'Edge intersection graphs of single bend paths on a grid', *Networks* **54**(3), 130-138.

[3] Heldt, D.; Knauer, K. & Ueckerdt, T. (2013), 'Edge-intersection graphs of grid paths: the bend-number', *Discrete Applied Mathematics*.

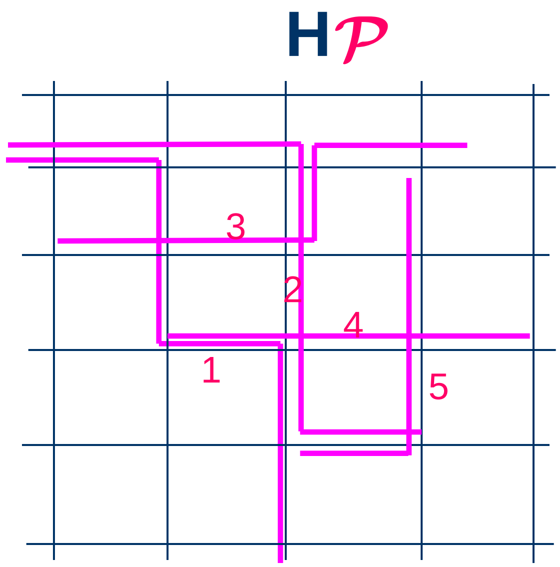
The EPT Graph EPT($\mathcal{P}$)

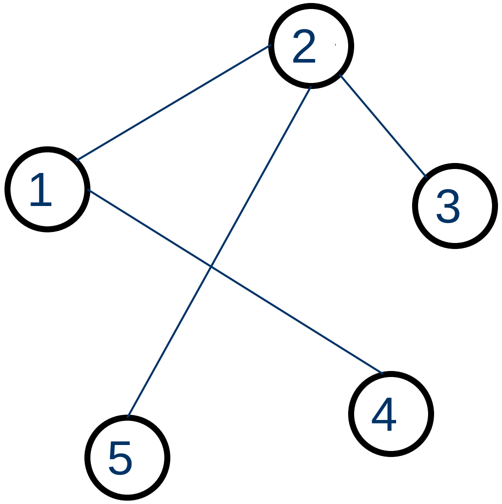




In this talk “intersection” means “edge intersection”

The EPG Graph $\mathrm{EPG}(\mathcal{P})$





- A graph is B_k-EPG if it has a representation with paths of at most k bends.
  (This is a B_3-EPG graph)

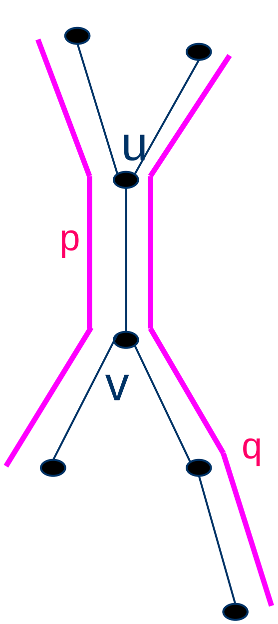
Results

- [2] Every graph is EPG
- [3] $B_1 - EPG \not\subset B_2 - EPG \not\subset \ldots$

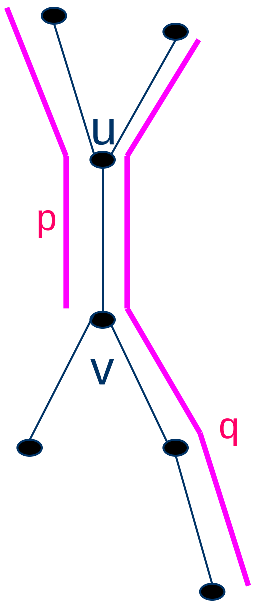
ENPT Graphs

[4] Boyacı, A.; Ekim, T.; Shalom, M. & Zaks, S., Graphs of Edge-Intersecting Non-Splitting Paths in a Tree: Towards Hole Representations, (WG2013)

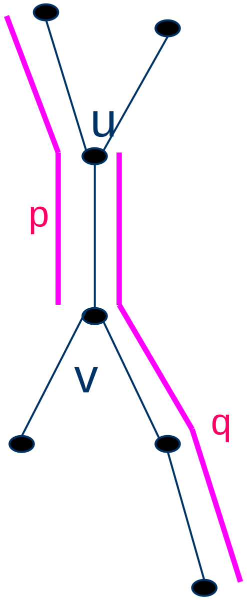
(Edge) Intersecting Paths (on a tree)



$split(p,q)=\{u,v\}$

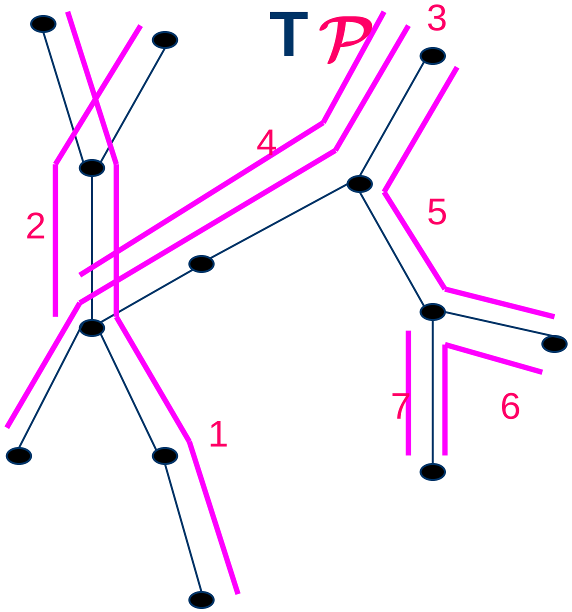


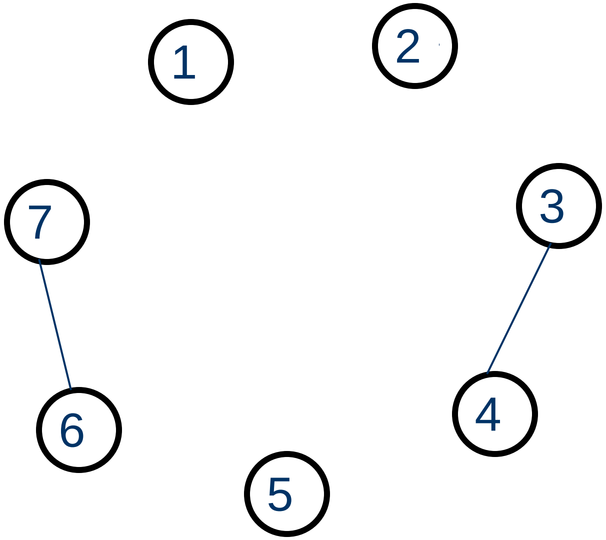
$split(p,q)=\{u\}$



$split(p,q)=\emptyset$

The ENPT Graph $\mathrm{ENPT}(\mathcal{P})$





$$V(ENPT(\mathcal{P})) = V(EPT(\mathcal{P})) = \mathcal{P}$$

$$E(ENPT(\mathcal{P})) \subseteq E(EPT(\mathcal{P}))$$

ENP/ENPG Graphs

Graphs of Edge-Intersecting and Non-Splitting Paths / in a Grid

Our Results

- ENP=ENPG

- Not every graph is ENPG

$$B_0 - ENPG \not\subset B_1 - ENPG \not\subset B_2 - ENPG \not\subset$$

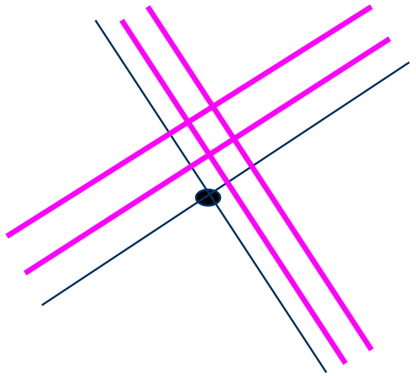
- $B_{k_1} - ENPG \not\subset B_{k_2} - ENPG \ldots$

$$\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = \sqrt{48}$$

ENP = ENPG

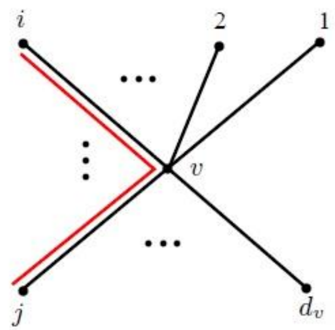
ENP = ENPG

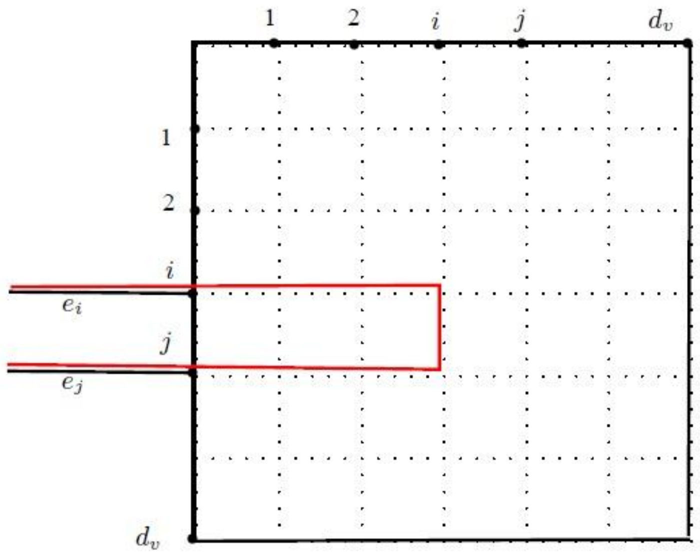
1) Every representation on any host graph H can be embedded in a plane in general position:
   - Edges are embedded to straight line segments
   - At most two edges intersect at any given point
2) H’ is planar.



ENP = ENPG

3) H’’ is planar with maximum degree at most 4.





4) Yanpei et al. (1991) → H’’’ is a Grid.

$$\mathrm{CO\text{-}BIPARTITE} \not\subset \mathit{ENPG}$$

Representation of a Clique

- The union of the paths representing a clique is a trail.
- If the trail is open there is an edge that intersects every path.
- If the trail is closed there is a set of at most two edges that intersects every path.

$\text{CO-BIPARTITE} \not\subset ENPG$

- Consider a co-bipartite graph C(K,K',E) with |K|=|K'|=n.
- There are 2^{n^2} such graphs. We now show that the number of possible ENPG representations is at most $(26n)!^3$
- The union of the paths representing a clique is a trail. Moreover, there is a set of at most two edges that intersects every path.
- The intersection of the two trails can be uniquely divided into a set of segments.
- Let S be the set of segments induced by the representations of the cliques K, K'.
- The paths representing two adjacent edges v of K and v' of K' can intersect only in edges of S.

$\text{CO-BIPARTITE} \not\subset ENPG$

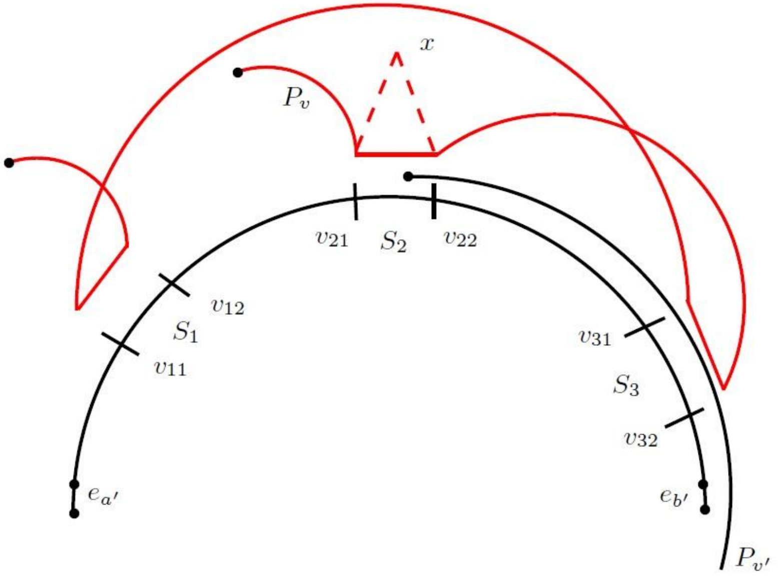
- The graph depends only on the order of the 2 |S| segments endpoints and 4n path endpoints on each trail.
- Lemma: The number of different orderings is at most

  $(4n)!\left(2n+2|S|\right)!^{2}$

- It remains to bound |S|.
- We show that for every representation, there is an equivalent one with |S| <= 12 n.

$\text{CO-BIPARTITE} \not\subset ENPG$

- A segment is *quiet* if it does not contain any path endpoints.
- The number of non-quiet segments is at most 4n.
- We now show that there are at most 4 quiet segments between two consecutive endpoints.



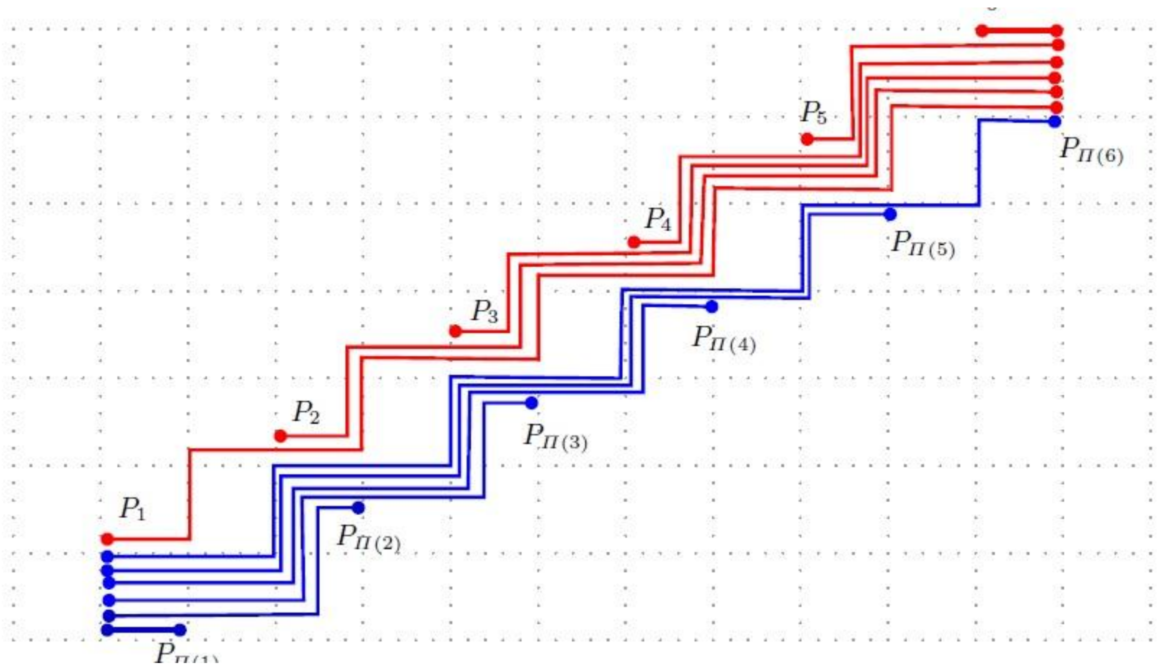
$$B_0 - ENPG \not\subset B_1 - ENPG \not\subset$$

$$B_{k_1} - ENPG \not\subset B_{k_2} - ENPG \ldots$$

$$\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = \sqrt{48}$$

Bend number of a "perfect matching"

Consider the co-bipartite graph PM_n=(K,K',E) where E is a perfect matching. $PM_n \in B_{2(n-1)} - ENPG$



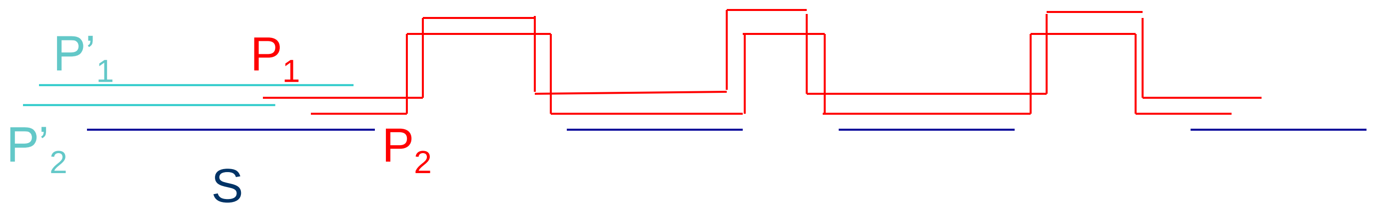
Bend number of a "perfect matching"

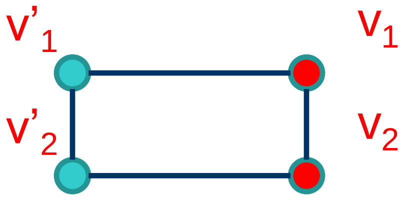
We show that for every k and for sufficiently big n

$$PM_n \notin B_k - ENPG$$

- We first observe that |S| <= 3k. (There are at most three paths covering the trail).
- Every edge of the perfect matching is realized in at least one segment.
- For sufficiently big n, there is at least one segment realizing at least 2|S| edges.
- The paths representing the corresponding vertices are either within the segment or going out from different parts of the segments.
- Therefore there are the least two paths from one side that their both endpoints are in the same segments, i.e. "equivalent".

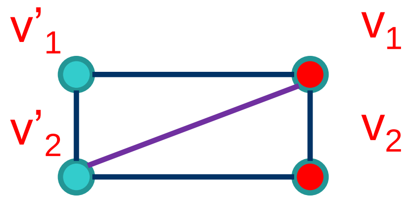
Bend number of a "perfect matching"





- Consider the vertices corresponding to these paths and their two neighbors in the matching.
- They contain a (not necessarily induced) C_4 (v_1,v_2,v'_2,v'_1).
- This C_4 is part of the corresponding EPG graph.
- We observe that the intersecting paths intersect also when restricted to the segment under consideration.

Bend number of a "perfect matching"

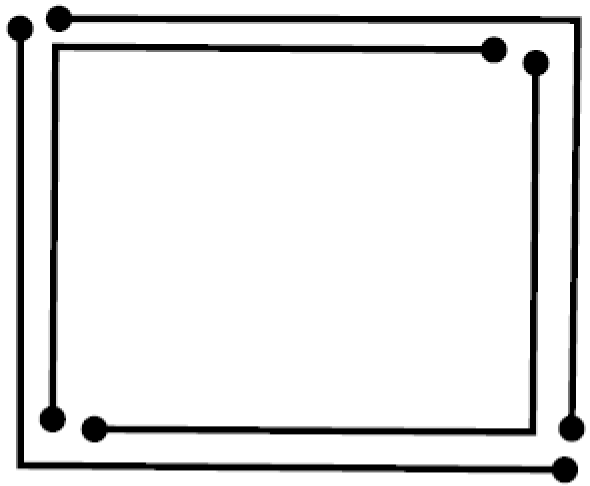


- Then this C_4 is part of some interval graph. Therefore it has a chord, w.l.o.g. (v_1,v'_2)
- This chord is not in the perfect matching.
- Therefore, the corresponding paths (P_1,P'_2) split from each other.
- On the other hand P_2 and P'_2 do not split.
- A contradiction to the "equivalence" of the two paths P_1,P_2

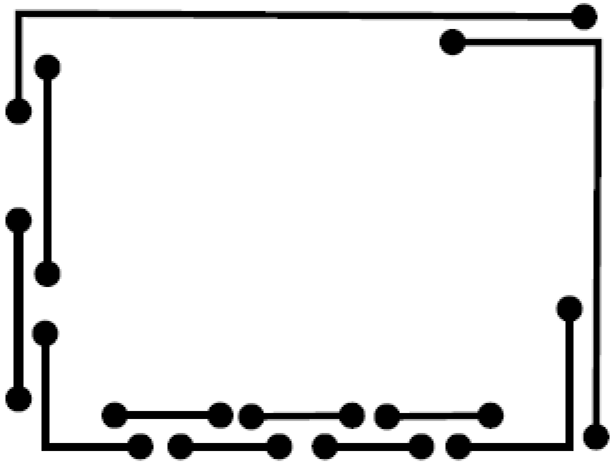
B_1-ENPG

- Trees and cycles are B_1-ENPG.
- If a twin free Split Graph is B_1-ENPG then $\sqrt{|K|} \le |S| < |K|^2$
- $B_1 - ENPG \not\subset B_2 - ENPG$
- The Recognition of B_1-ENPG is NP-C even for Split graphs.
- B_1-ENPG Co-bipartite graphs can be recognized in linear time.
- “at most k bends” is more powerful than “exactly k bends”.

Cycles are B_1-ENPG

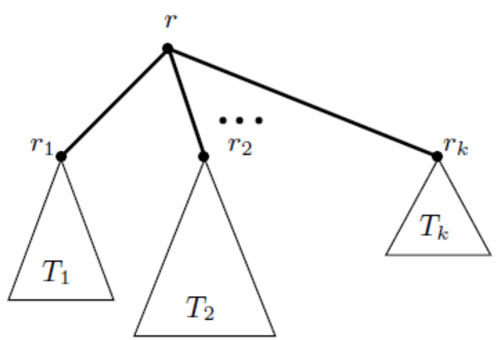


A C_4

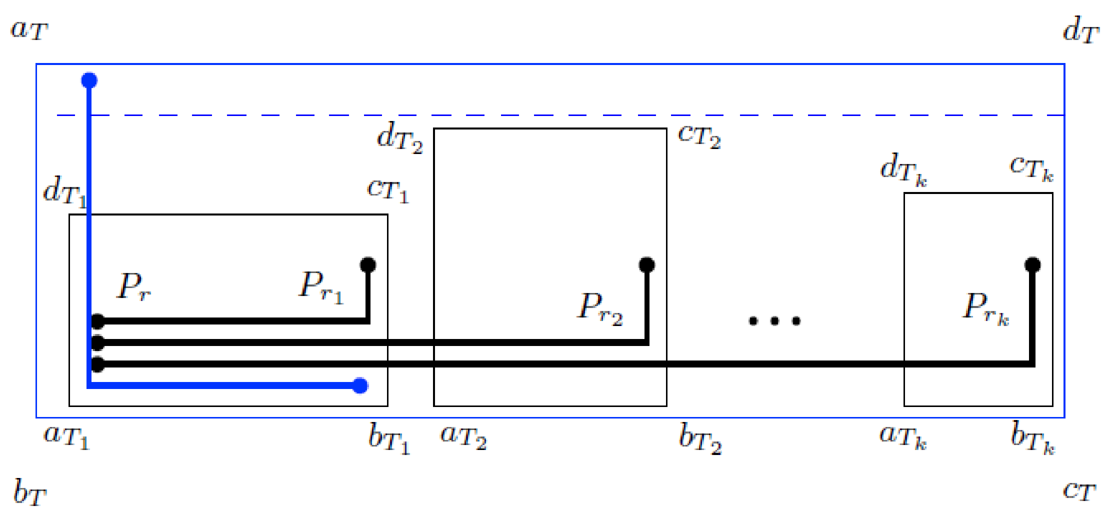


A C_{11}

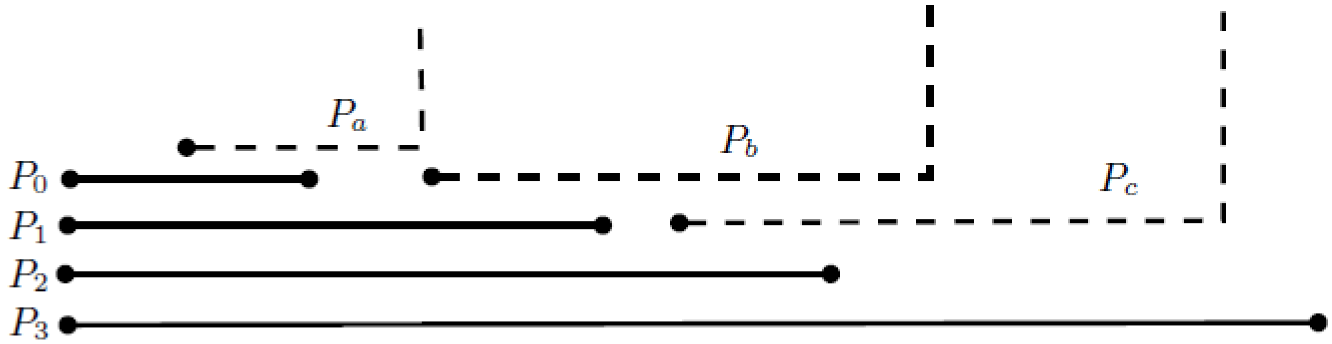
Trees are B_1-ENPG



- Every path has exactly one bend
- b_T is a bend of P_r
- a_T is an endpoint of P_r
- a_T is used only in P_r



Split Graph Representation

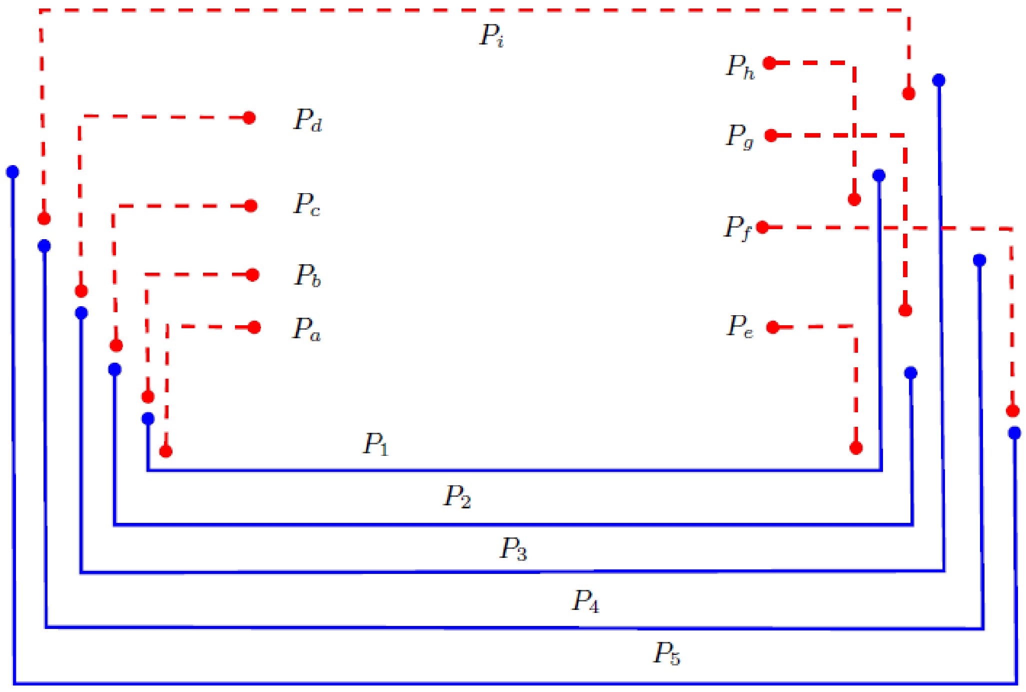


- If a twin free Split Graph is B_1-ENPG then

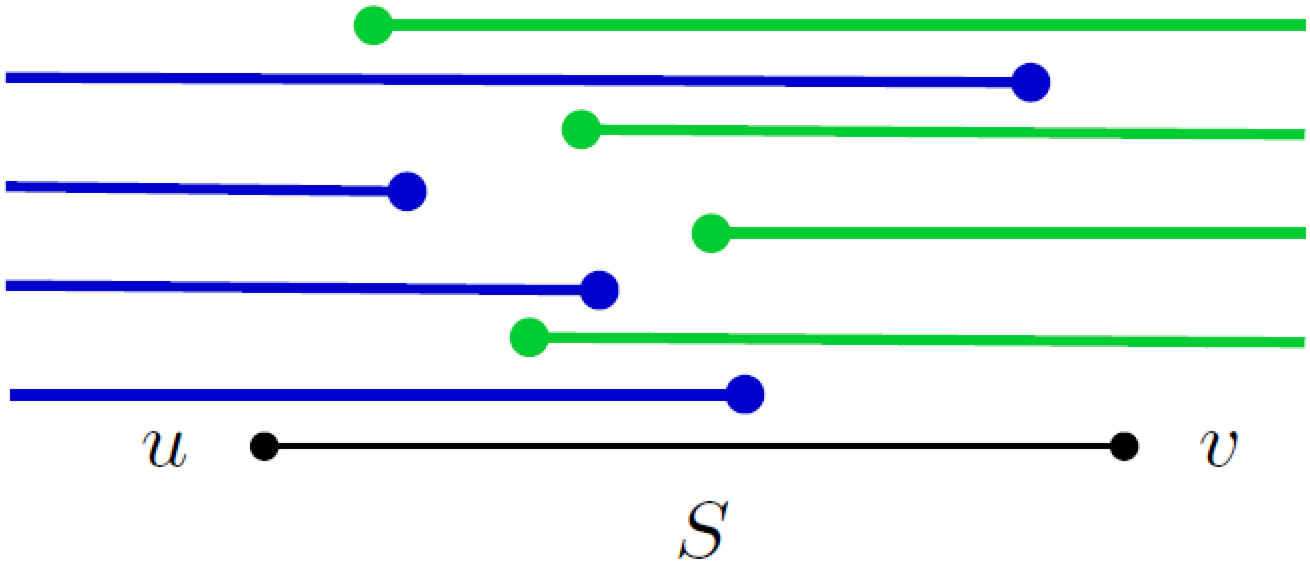
$$|S_d| \le 2(|K| + 1 - d)$$

$$\sqrt{|K|} \le |S| < |K|^2$$

B_1-ENPG $\not\subset$ B_2-ENPG

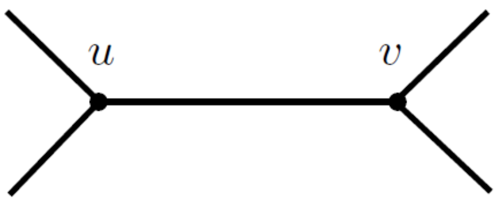


Co-bipartite Graph Representation

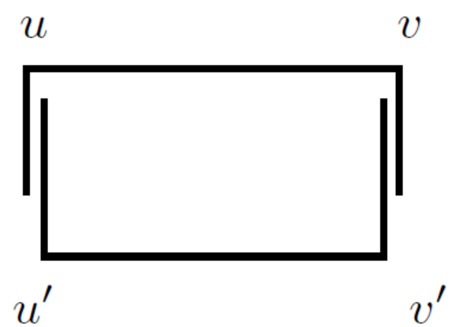


The corresponding intersection graph is a Difference Graph.
Difference graphs can be recognized in linear time

Co-bipartite Graph Representation



Type I B_1-ENPG



Type II B_1-ENPG

Consists of:
At most 4 special vertices+
A difference graph

Consists of:
“isolated” vertices +
Two difference graphs


Thanks


Teşekkürler


תודה