Obstructions against 3-coloring graphs without long induced paths

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Graph coloring

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- a **k-coloring** is an assignment of numbers $\{1, 2, \ldots, k\}$ to the vertices such that any two adjacent vertices receive distinct numbers

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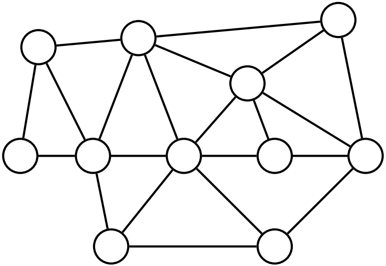

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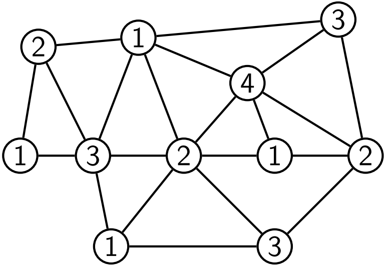

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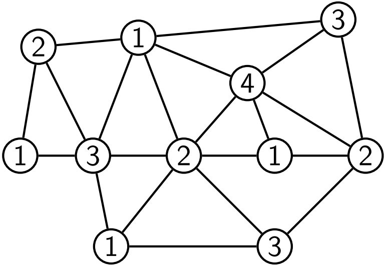

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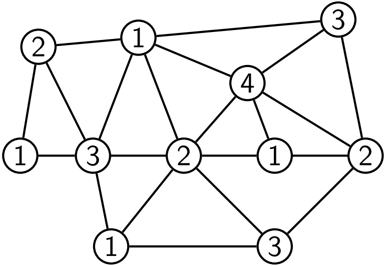

- the related decision problem is called **k-colorability**
- it is NP-complete for every $k \geq 3$

k-colorability in H-free graphs

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Let H be any graph that is not the disjoint union of paths. Then k-colorability is NP-complete in the class of H-free graphs, for all $k \geq 3$.

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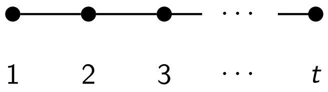
- leads to the study of P_t-free graphs



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Open Problem

Is there any t such that 3-colorability is NP-hard for P_t-free graphs?

Obstructions against 3-colorability

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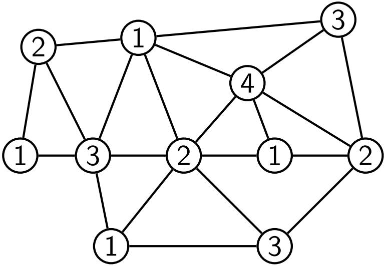

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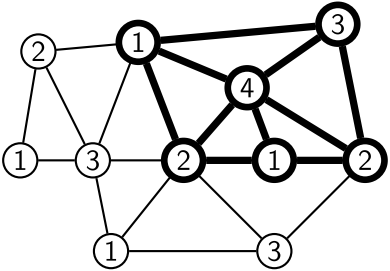

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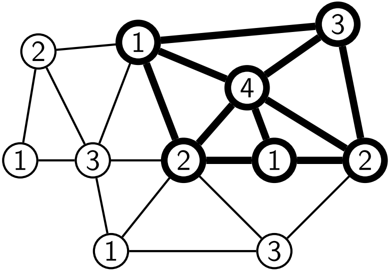

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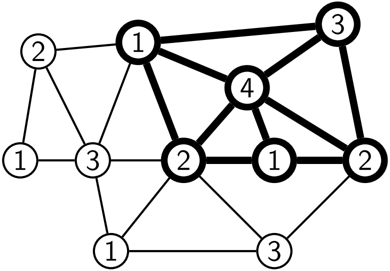

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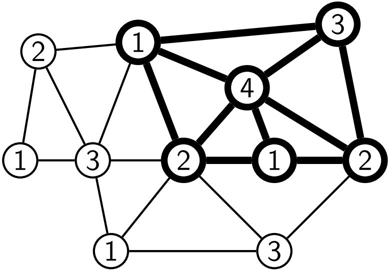

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- call such a graph an **obstruction** against 3-colorability
- useful in the design of certifying algorithms

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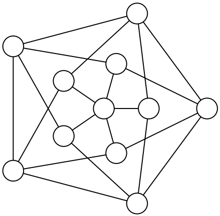

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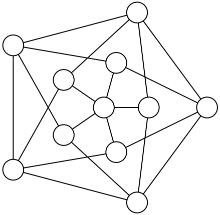

Theorem (Bruce, Hòang and Sawada 2009)

There are six obstructions in the class of P_5-free graphs.

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Theorem (Chudnovsky, Goedgebeur, S and Zhong 2015)

There are 24 obstructions in the class of P_6-free graphs.

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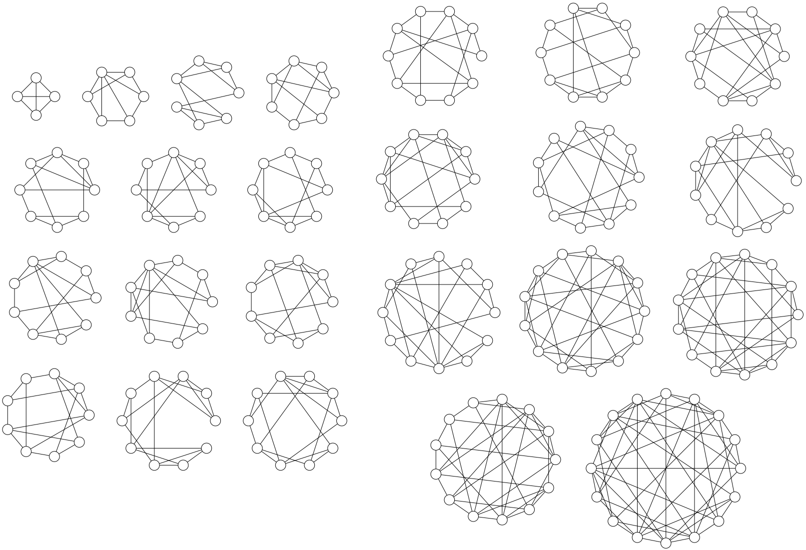
- **Seymour:** for which connected graphs H exist only finitely many obstructions in the class of H-free graphs?

Theorem (Chudnovsky, Goedgebeur, S and Zhong 2015)

There are 24 obstructions in the class of P_6-free graphs.

Moreover, if H is connected and not a subgraph of P_6, there are infinitely many obstructions in the class of H-free graphs.



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- Prove that our list is complete in the $(P_6, \text{diamond})$-free case
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- Prove that our list is complete up to 28 vertices
  - Use same enumeration algorithm
- Prove that our list is complete in the full case
  - Structural analysis by hand
  - Contraction/Decontraction of maximal tripods

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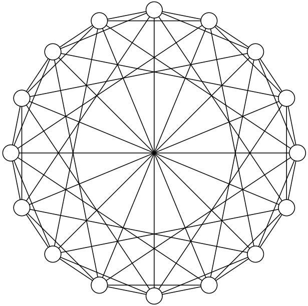

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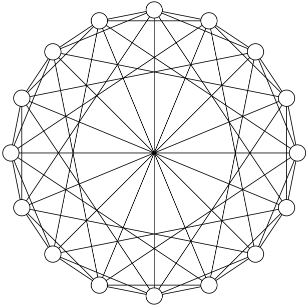

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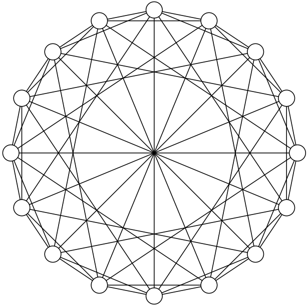

- Easy: infinte familes of claw-free obstructions, and obstructions of large girth
- this yields the desired dichotomy

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- Formulate a dichotomy theorem for general H
- Is 3-colorability solvable in polytime on P_t-free graphs?
- Is 4-colorability solvable in polytime on P_6-free graphs?
- Is k-colorability FPT in the class of P_5-free graphs?

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Thanks!