The 4-ordered Hamiltonicity and the spanning K_4^--linkedness in planar graphs

Kenta Ozeki

(National Institute of Informatics, Japan)

(JST, ERATO, Kawarabayashi Large Graph Project)

k-ordered (Hamiltonian)

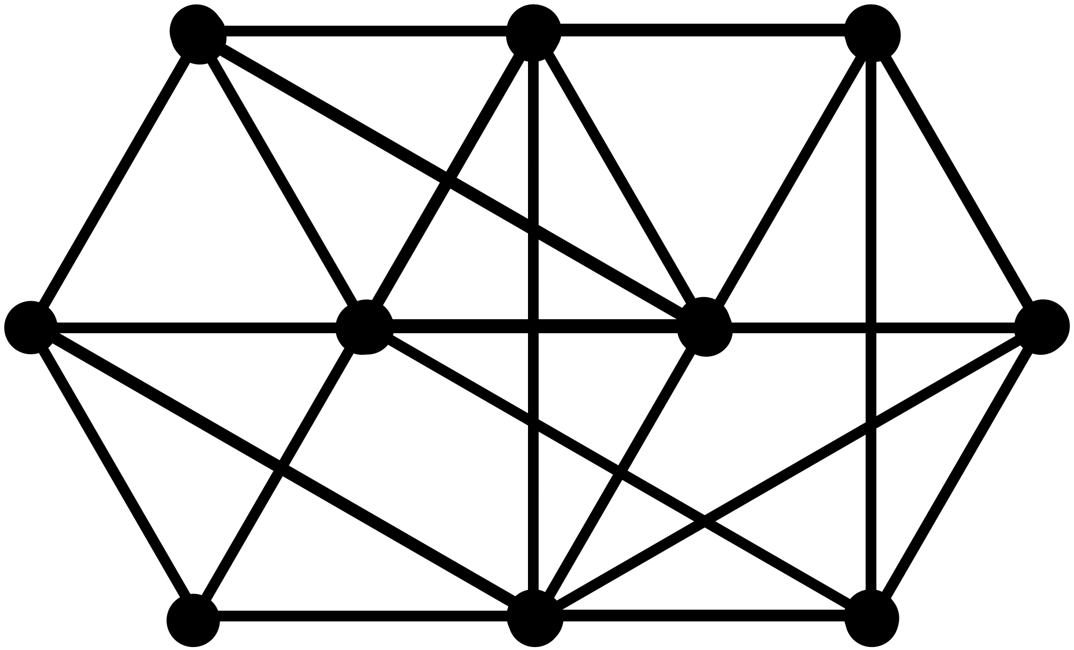
G : graph

- Hamiltonian cycle in G
  $\Leftrightarrow$ A cycle containing all of the vertices in G

- G : k-ordered (Hamiltonian)
  $\Leftrightarrow$ For $\forall v_1, v_2, \ldots, v_k \in V(G)$,
  $\exists$ (Hamiltonian) cycle
  containing all of them in this order

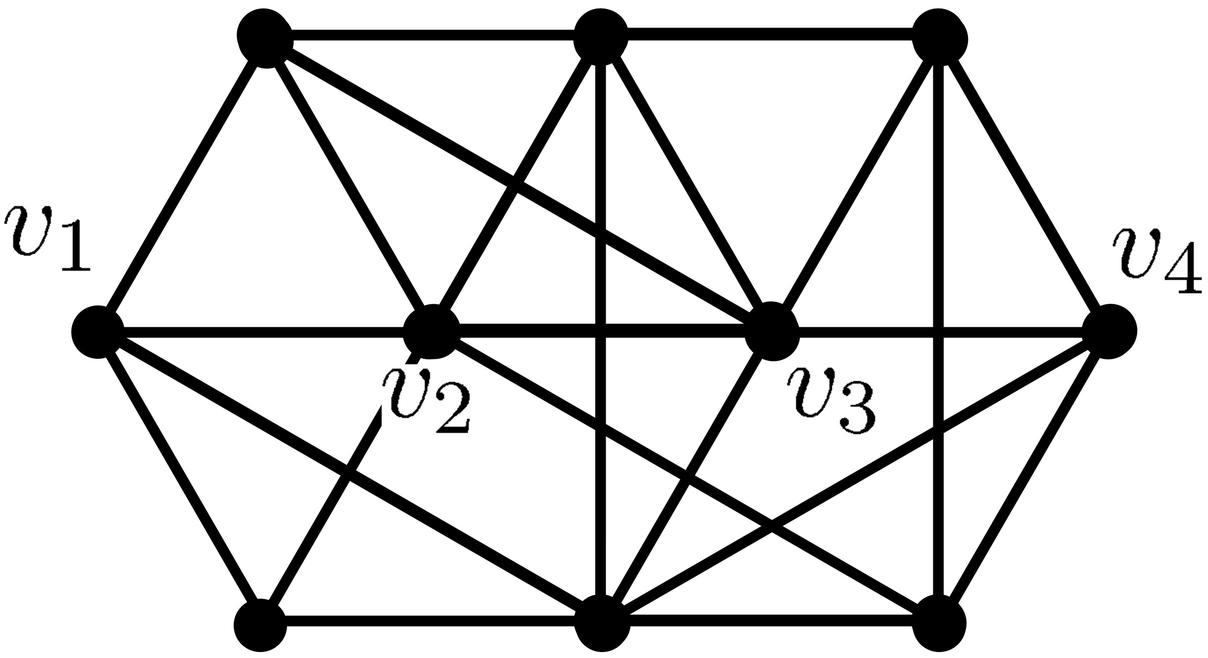
``$k$-ordered'' + ``Hamiltonian''

k-ordered (Hamiltonian)



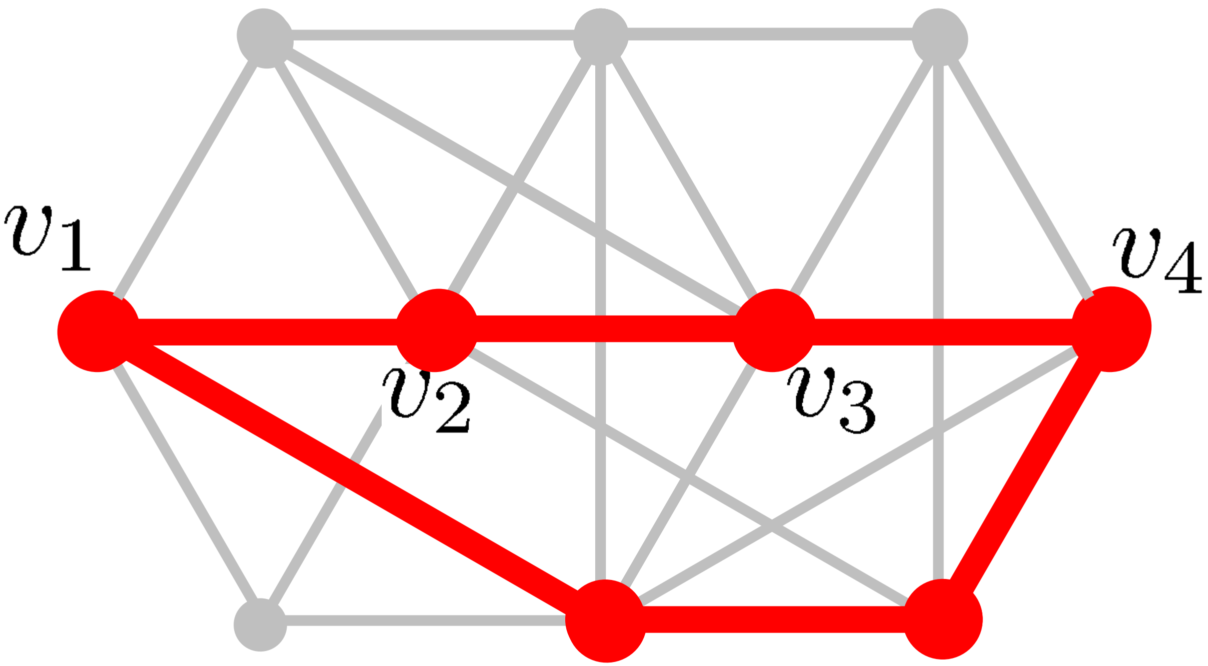
✓ Hamiltonian

k-ordered (Hamiltonian)



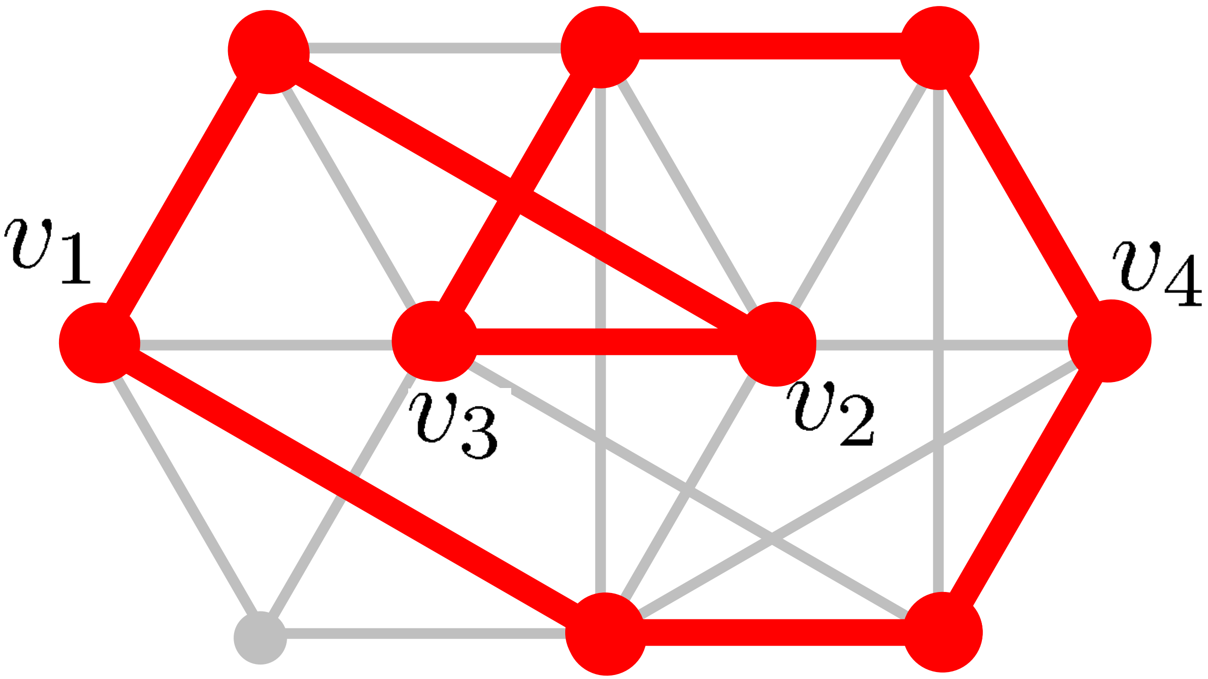
- ✓ Hamiltonian
- ✓ 4-ordered

k-ordered (Hamiltonian)



- ✓ Hamiltonian
- ✓ 4-ordered

k-ordered (Hamiltonian)

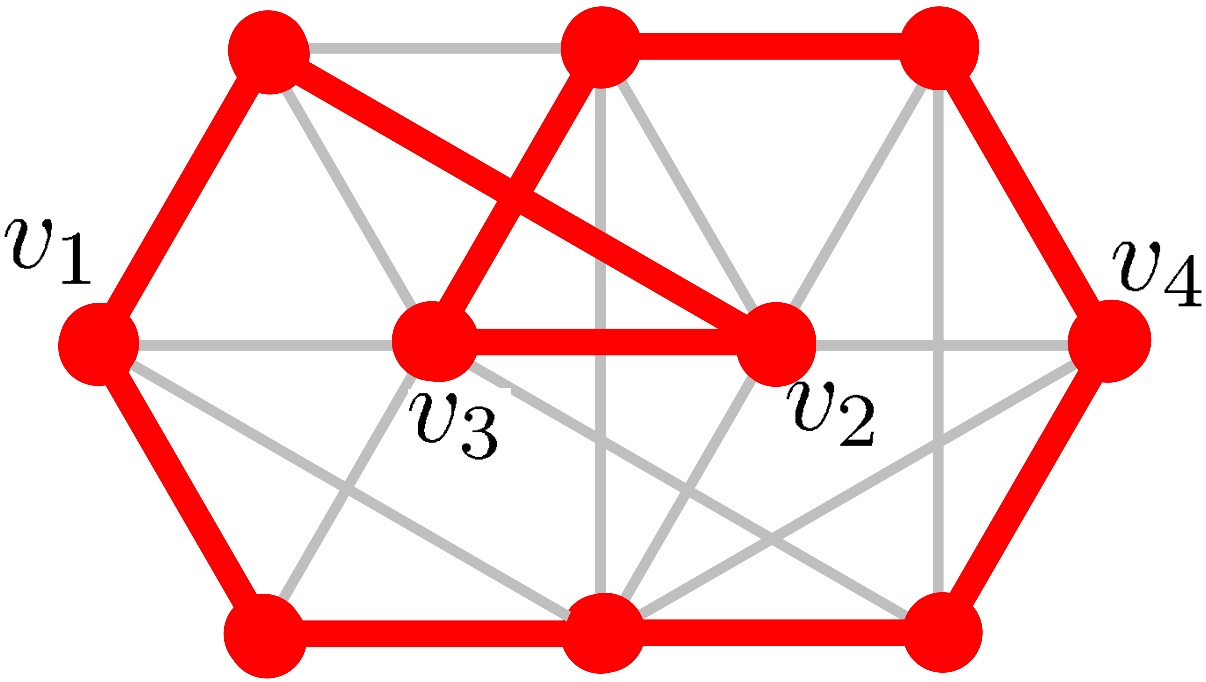


- ✓ Hamiltonian
- ✓ 4-ordered

  3 possibilities

  (1, 2, 3, 4)
  (1, 2, 4, 3)
  (1, 3, 2, 4)

k-ordered (Hamiltonian)



- ✓ Hamiltonian
- ✓ 4-ordered
  3 possibilities
  (1, 2, 3, 4)
  (1, 2, 4, 3)
  (1, 3, 2, 4)
- ✓ 4-ordered Hamiltonian

\`\`k-ordered'' + \`\`Hamiltonian''

3-ordered Hamiltonian

3-ordered = 3-cyclable

(any 3 vertices are contained in a cycle)

Such a graph is characterized by Dirac `52

3-ordered Hamiltonian

3-ordered = 3-cyclable

(any 3 vertices are contained in a cycle)

Such a graph is characterized by Dirac `52

Proposition

G : Hamiltonian $\Leftrightarrow$ G : 3-ordered Hamiltonian

3-ordered Hamiltonian

3-ordered = 3-cyclable

(any 3 vertices are contained in a cycle)

Such a graph is characterized by Dirac `52

Proposition

G : Hamiltonian $\Leftrightarrow$ G : 3-ordered Hamiltonian

4-ordered (Hamiltonian) is more interesting!

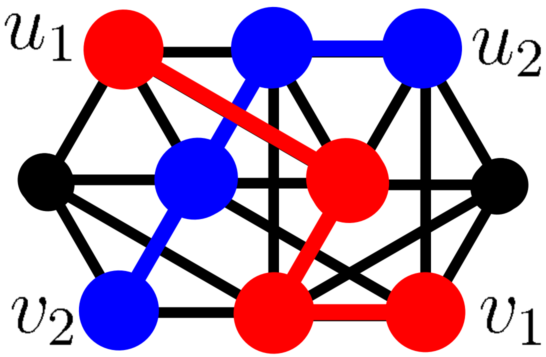
4-ordered and 2-linked

- 2-linked

  For $\forall u_1, u_2, v_1, v_2 \in V(G)$

  $\exists P_1, P_2$: disjoint paths

  s.t. P_i connects u_i and v_i



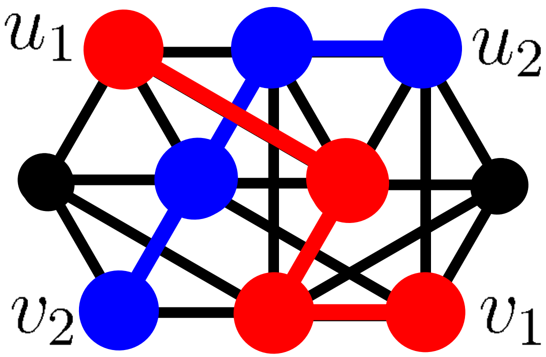
4-ordered and 2-linked

- 2-linked

  For $\forall u_1, u_2, v_1, v_2 \in V(G)$

  $\exists P_1, P_2$: disjoint paths

  s.t. P_i connects u_i and v_i



Proposition

G : 4-ordered $\Rightarrow$ G : 2-linked

4-ordered and 2-linked

Theorem (Seymour, Shiloach, Thomassen)

G : 4-connected, $u_1, u_2, v_1, v_2 \in V(G)$

$\Rightarrow$ $\exists P_1, P_2$: disjoint paths

s.t. P_i connects u_i and v_i

2-linked

or G can be embedded into the plane

so that u_1, u_2, v_1, v_2 appears

in a face boundary in this order.

4-ordered and 2-linked

Theorem (Seymour, Shiloach, Thomassen)

G : 4-connected, $u_1, u_2, v_1, v_2 \in V(G)$

$\Rightarrow$ $\exists P_1, P_2$: disjoint paths

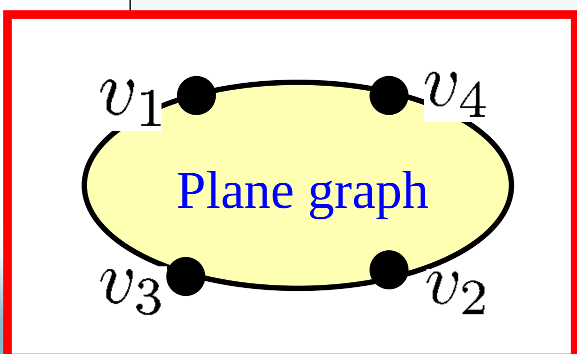
s.t. P_i connects u_i and v_i

2-linked

or G can be embedded into the plane

o that u_1, u_2, v_1, v_2 appears

n a face boundary in this order.



4-connected plane triangulation

Theorem

G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

4-connected plane triangulation

Theorem

G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

Sharpness of the assumptions

	4-connected	plane	triangulation
(I)	Best ∃ Some examples	Improved (mentioned later)	Improved to planar graphs (Tutte `56)
(II)	Best	Improved (mentioned later)	Best

4-connected plane triangulation

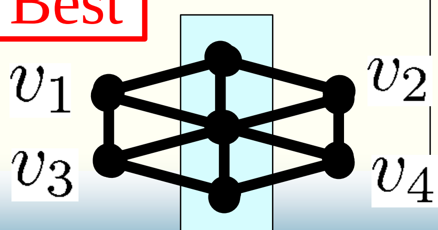
Theorem

G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

Sharpness of the assumptions

	4-connected	plane	triangulation
(I)	Best ∃ Some examples	Improved (mentioned later)	Improved to planar graphs (Tutte `56)
(II)	Best 	Improved (mentioned later)	Best

4-connected plane triangulation

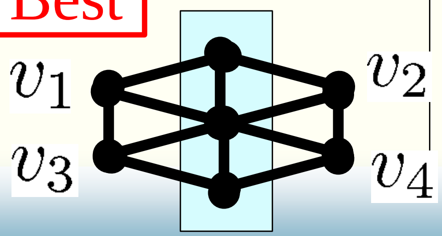
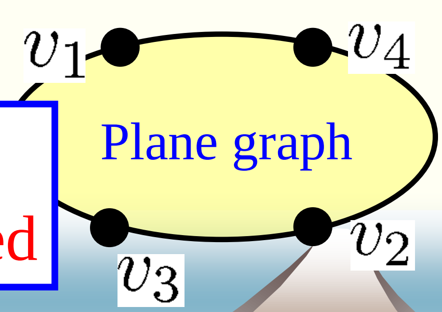
Theorem

G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

Sharpness of the assumptions

	4-connected	plane	triangulation
(I)	Best ∃ Some examples	Improved (mentioned later)	Improved to planar graphs (Tutte `56)
(II)	Best 	Improved (mentioned later)	Best NOT 2-linked 

4-connected plane triangulation

Theorem

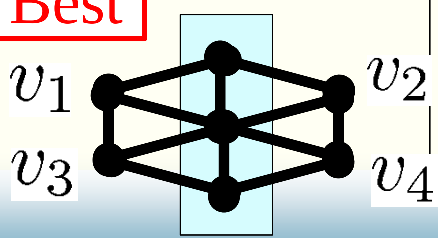
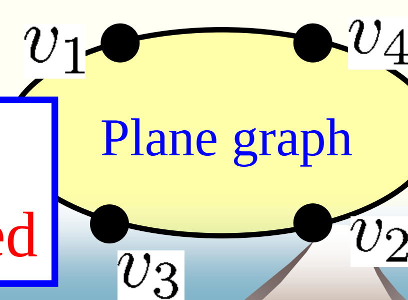
G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

Sharpness of the assumptions

``$k$-ordered'' + ``Hamiltonian''

	4-co…		…ulation
(I)	Best $\exists$ Some examples	Improved (mentioned later)	Improved to planar graphs (Tutte `56)
(II)	Best 	Improved (mentioned later)	Best NOT 2-linked 

4-connected plane triangulation

Theorem

G : 4-connected plane triangulation

$\Rightarrow$ (I) G : Hamiltonian (Whitney, `31)

(II) G : 4-ordered (Goddard, `02)

``$k$-ordered" + ``Hamiltonian"

Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

4-connected plane triangulation

Theorem

G : 5-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

``$k$-ordered'' + ``Hamiltonian''

Conjecture

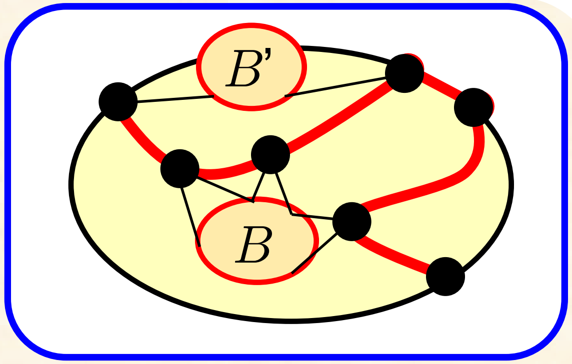
G : 4-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

Idea for the proof

T : D-Tutte path

$\Leftrightarrow$ For $\forall B$: component of $G - V(T)$,
B has ≤ 3 neighbors on T
and ≤ 2 neighbors if B contains a vertex in D



proof

G : 4-conn. and $|T| \geq 4$

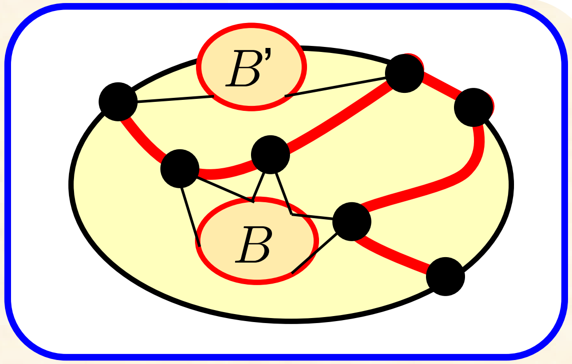
$\Rightarrow T$: H-path in G

T : D-Tutte path

$\Leftrightarrow$ For $\forall B$: component of $G - V(T)$,

B has ≤ 3 neighbors on T

and ≤ 2 neighbors if B contains a vertex in D



proof

G : 4-conn. and $|T| \geq 4$

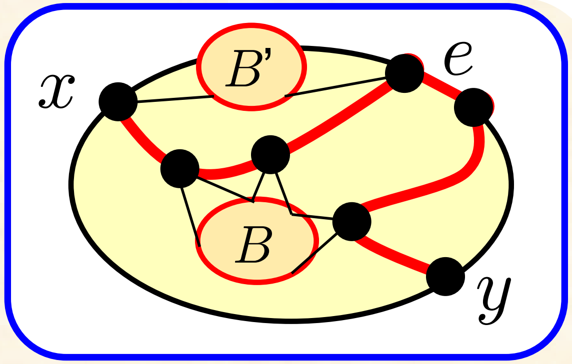
$\Rightarrow T$: H-path in G

T : D-Tutte path

$\Leftrightarrow$ For $\forall B$: component of $G - V(T)$,

B has ≤ 3 neighbors on T

and ≤ 2 neighbors if B contains a vertex in D



Theorem (Thomassen `83)

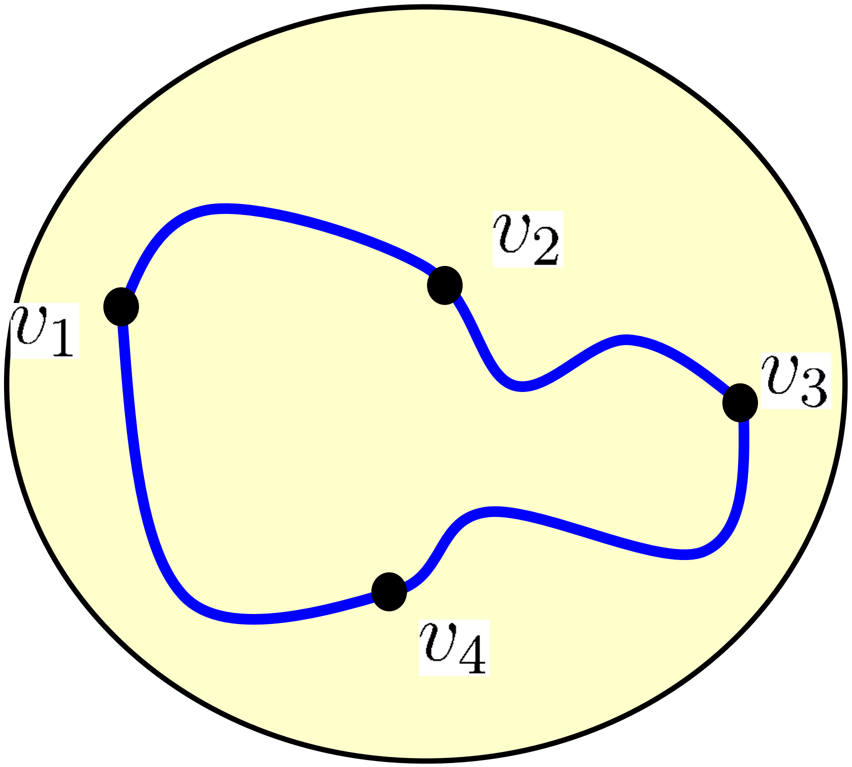
G : plane graph, D : outer cycle $x, y \in V(C)$ $e \in E(C)$

$\exists$ path from x to y through e

$O(n^2)$-algorithm

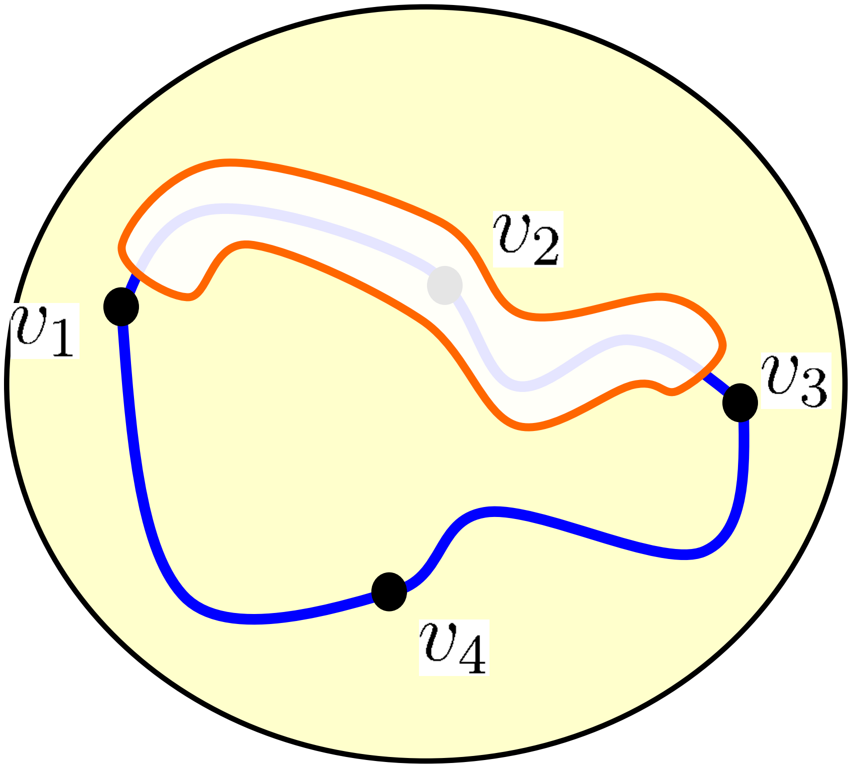
$\Rightarrow \exists$ D-Tutte path from x to y through e

Idea for the proof



C : 4-ordered cycle

Idea for the proof



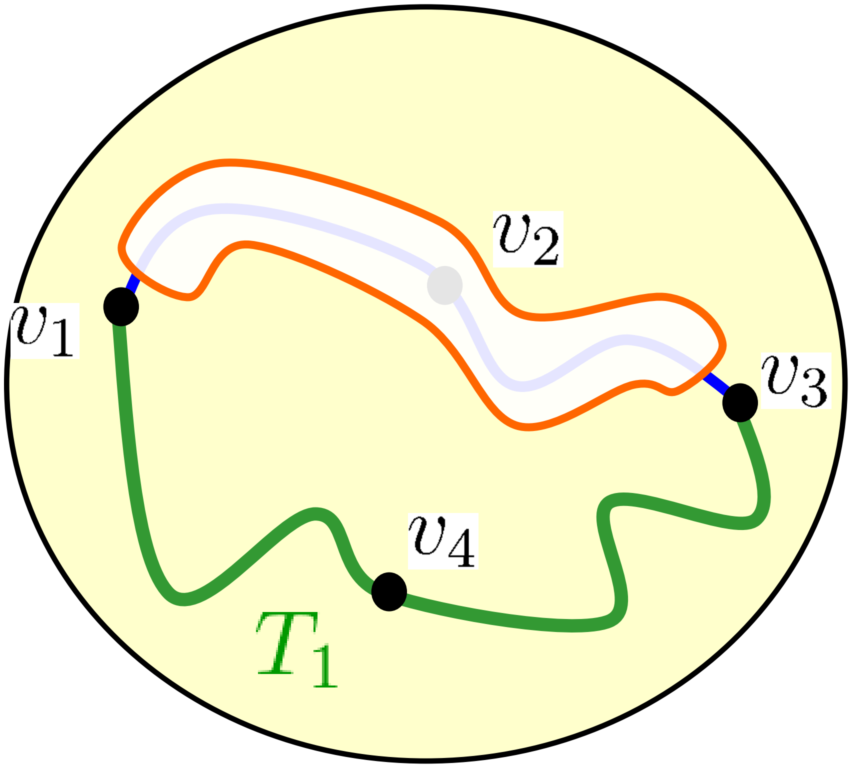
C : 4-ordered cycle

$G_1 = G - C(v_1, v_3)$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

Idea for the proof



C : 4-ordered cycle

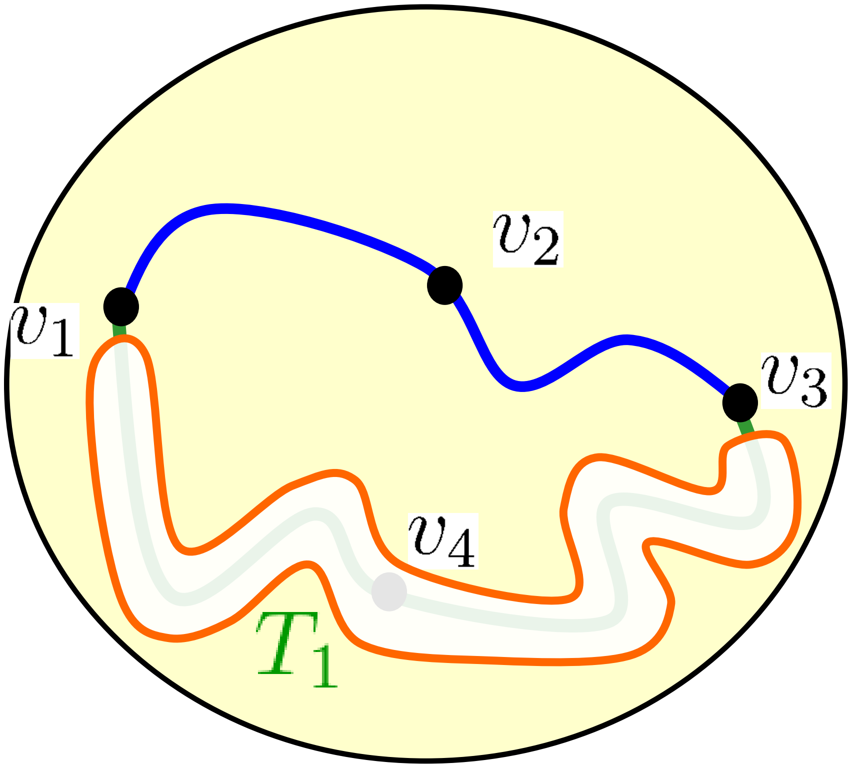
$G_1 = G - C(v_1, v_3)$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

Idea for the proof



C : 4-ordered cycle

$$G_1 = G - C(v_1, v_3)$$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

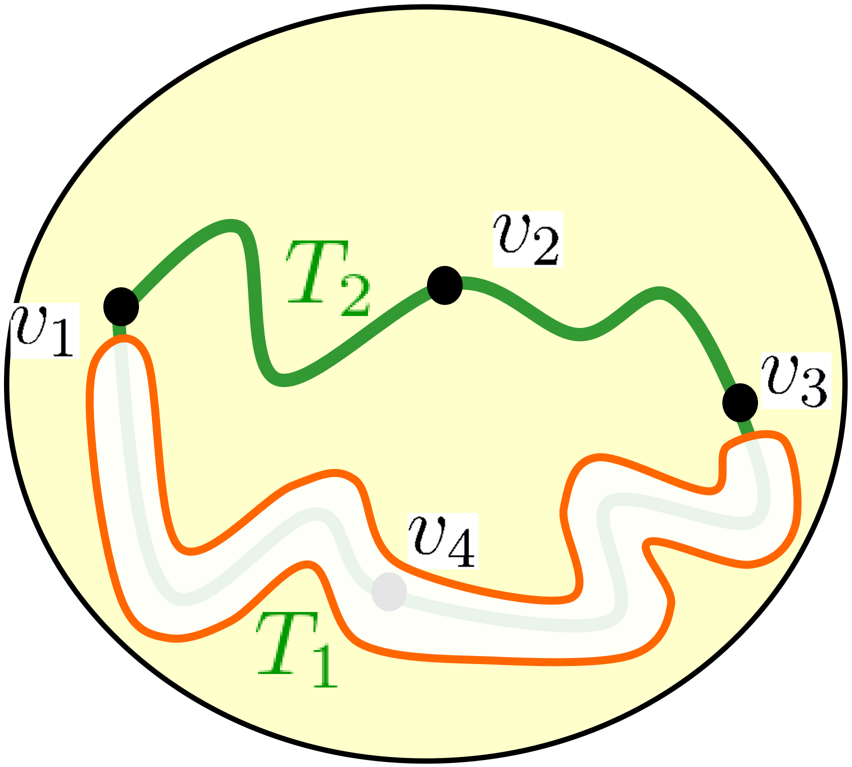
T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

$$G_2 = G - T_1(v_3, v_1)$$

D_2 : a ``hole'' on $T_1(v_3, v_1)$

e_2 : an edge ``close'' to v_2

Idea for the proof



C : 4-ordered cycle

$G_1 = G - C(v_1, v_3)$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

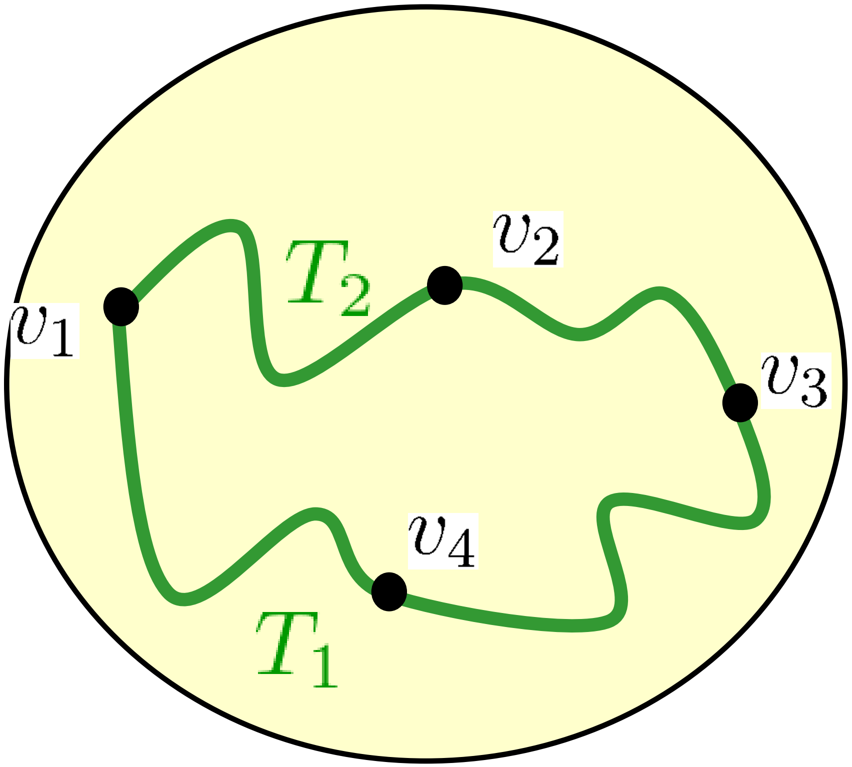
$G_2 = G - T_1(v_3, v_1)$

D_2 : a ``hole'' on $T_1(v_3, v_1)$

e_2 : an edge ``close'' to v_2

T_2 : D_2 -Tutte path $v_1 \xrightarrow{e_2} v_3$

Idea for the proof



C : 4-ordered cycle

$G_1 = G - C(v_1, v_3)$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

$G_2 = G - T_1(v_3, v_1)$

D_2 : a ``hole'' on $T_1(v_3, v_1)$

e_2 : an edge ``close'' to v_2

T_2 : D_2 -Tutte path $v_1 \xrightarrow{e_2} v_3$

$T_1 \cup T_2$ is a Hamiltonian cycle if G is 5-connected.

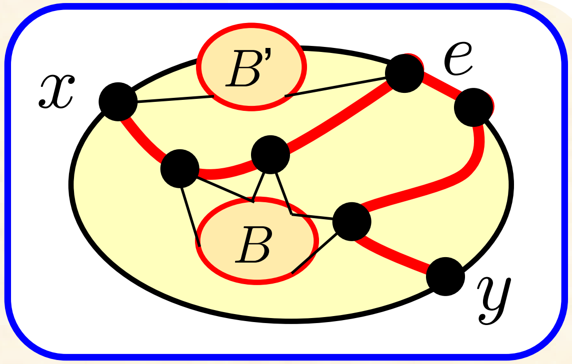
Idea for the proof

T : D-Tutte path

$\Leftrightarrow$ For $\forall B$: component of $G - V(T)$,

B has ≤ 3 neighbors on T

and ≤ 2 neighbors if B contains a vertex in D



Theorem (Thomassen `83)

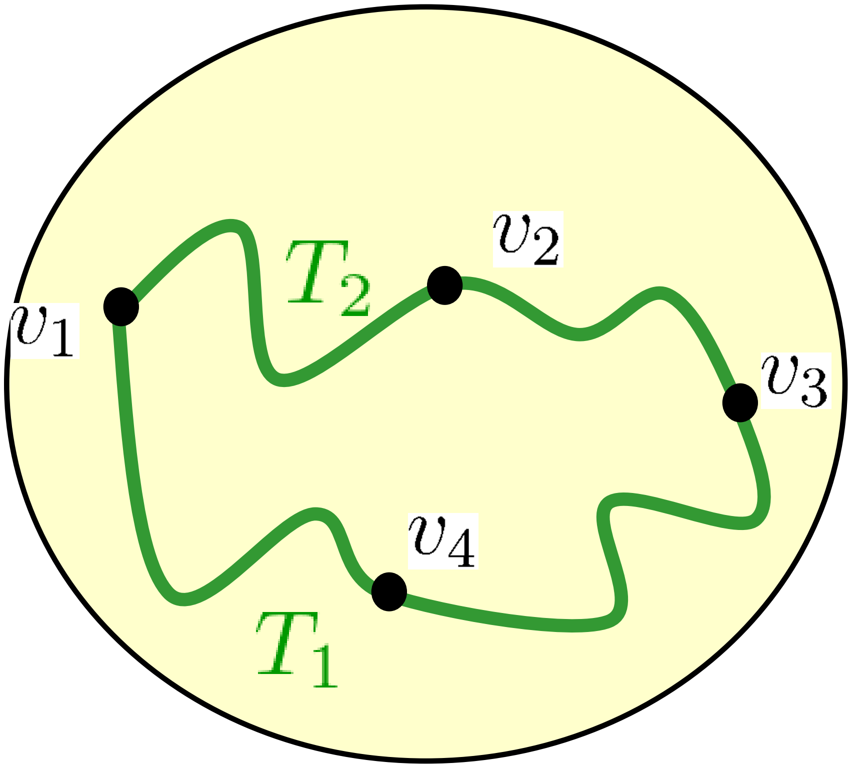
G : plane graph, D : outer cycle $x, y \in V(C)$ $e \in E(C)$

$\exists$ path from x to y through e

$\Rightarrow \exists$ D-Tutte path from x to y through e

$O(n^2)$-algorithm

Idea for the proof



C : 4-ordered cycle

$$G_1 = G - C(v_1, v_3)$$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

$$G_2 = G - T_1(v_3, v_1)$$

D_2 : a ``hole'' on $T_1(v_3, v_1)$

e_2 : an edge ``close'' to v_2

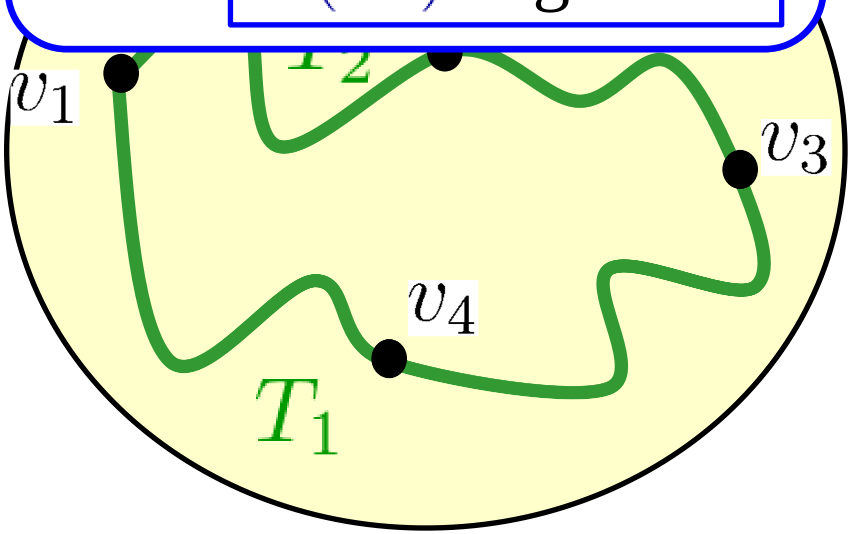
T_2 : D_2 -Tutte path $v_1 \xrightarrow{e_2} v_3$

$T_1 \cup T_2$ is a Hamiltonian cycle if G is 5-connected.

Idea for the proof

Use Thomassen's algorithm twice

→ $O(n^2)$-algorithm



$T_1 \cup T_2$ is a Hamiltonian cycle if G is 5-connected.

C : 4-ordered cycle

$$G_1 = G - C(v_1, v_3)$$

D_1 : a ``hole'' on $C(v_1, v_3)$

e_1 : an edge ``close'' to v_4

T_1 : D_1 -Tutte path $v_3 \xrightarrow{e_1} v_1$

$$G_2 = G - T_1(v_3, v_1)$$

D_2 : a ``hole'' on $T_1(v_3, v_1)$

e_2 : an edge ``close'' to v_2

T_2 : D_2 -Tutte path $v_1 \xrightarrow{e_2} v_3$

4-connected plane triangulation

Theorem

G : 5-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

``$k$-ordered'' + ``Hamiltonian''

Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

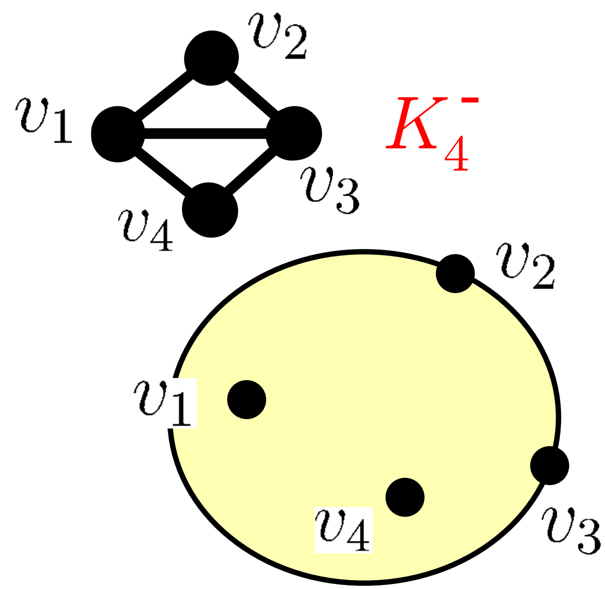
4-connected plane triangulation

Theorem

G : 4-connected plane triangulation

$\Rightarrow G$: K_4^--linked

(Ellingham, Plummer & Yu, `12)



G : K_4^--linked

$\Leftrightarrow \forall v_1, v_2, v_3, v_4 \in V(G)$

$\exists$ subdivision of K_4^- with ``base'' v_1, v_2, v_3, v_4

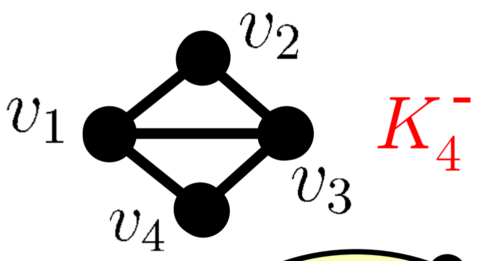
4-connected plane triangulation

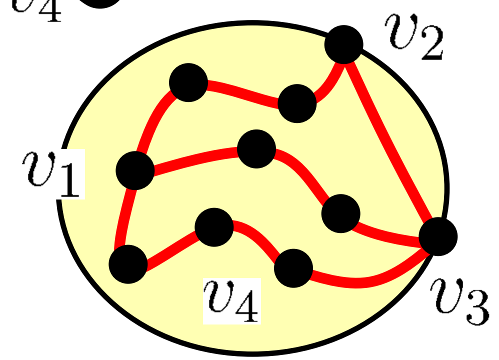
Theorem

G : 4-connected plane triangulation

$\Rightarrow$ G : K_4^--linked

(Ellingham, Plummer & Yu, `12)



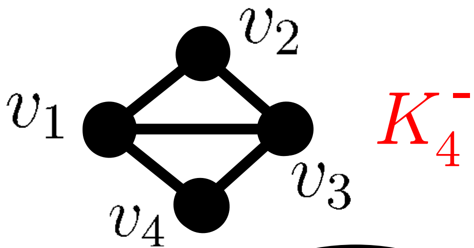


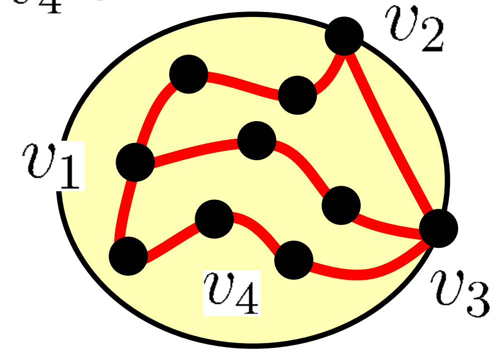
G : K_4^--linked

$\Leftrightarrow$ $\forall v_1, v_2, v_3, v_4 \in V(G)$

$\exists$ subdivision of K_4^- with ``base'' v_1, v_2, v_3, v_4

4-connected plane triangulation





Theorem

G : 4-connected plane triangulation

$\Rightarrow G$: K_4^--linked

(Ellingham, Plummer & Yu, `12)

G : K_4^--linked

$\Leftrightarrow \forall v_1, v_2, v_3, v_4 \in V(G)$

$\exists$ subdivision of K_4^- with ``base'' v_1, v_2, v_3, v_4

Proposition

G : K_4^--linked $\Rightarrow$ G : 4-ordered ($\Leftrightarrow$ C_4-linked)

$\Rightarrow$ G : 2-linked ($\Leftrightarrow$ $2K_2$-linked)

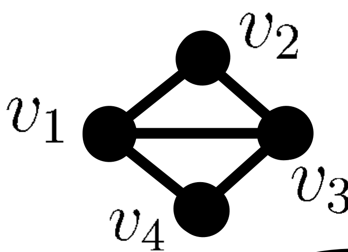
4-connected plane triangulation

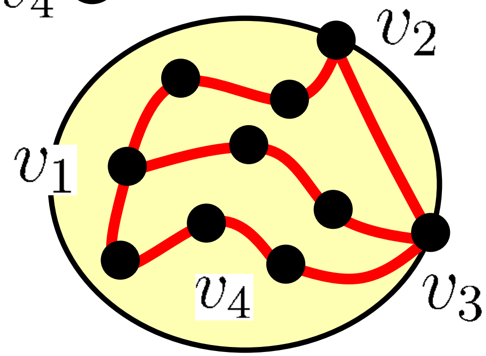
Theorem ``$k$-ordered'' + ``Hamiltonian''

G : 4-connected plane triangulation

$\Rightarrow G$: K_4^--linked

(Ellingham, Plummer & Yu, `12)





G : K_4^--linked

$\Leftrightarrow \forall v_1, v_2, v_3, v_4 \in V(G)$

$\exists$ subdivision of K_4^- with ``base'' v_1, v_2, v_3, v_4

Proposition

G : K_4^--linked $\Rightarrow$ G : 4-ordered ($\Leftrightarrow$ C_4-linked)

$\Rightarrow$ G : 2-linked ($\Leftrightarrow$ $2K_2$ -linked)

4-connected plane triangulation

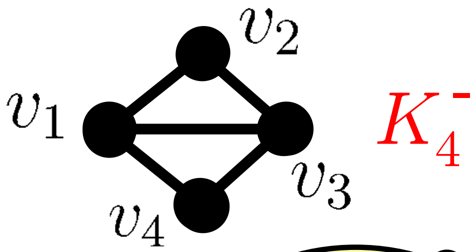
``$k$-ordered'' + ``Hamiltonian''

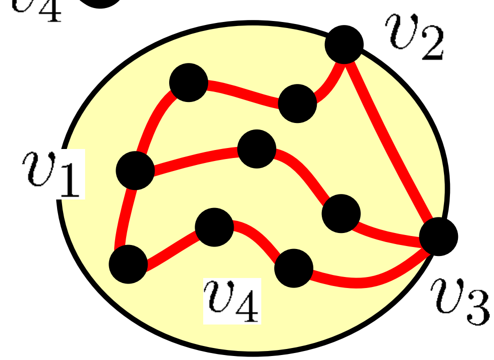
<u>Conjecture</u>

G : 4-connected plane triangulation

$\Rightarrow$ ✓ G : spanning K_4^- -linked

✓ G : spanning 2-linked



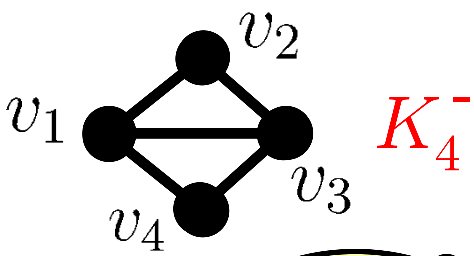


<u>Theorem</u>

This conjecture is true for 5-conn. plane triangulation

4-connected plane triangulation

``$k$-ordered'' + ``Hamiltonian''

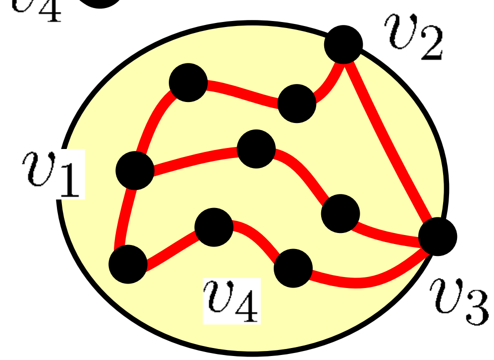


Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ ✓ G : spanning K_4^- -linked

✓ G : spanning 2-linked



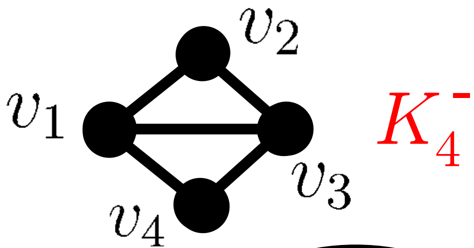
Theorem

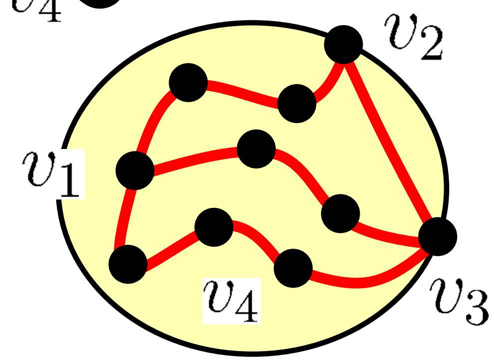
This conjecture is true for 5-conn. plane triangulation

(Yu `98) G : 5-conn. plane triangulation $\Rightarrow$ G : K_4 -linked

4-connected plane triangulation

"k-ordered" + "Hamiltonian"





Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ ✓ G : spanning K_4^- -linked

✓ G : spanning 2-linked

Theorem

This conjecture is true for 5-conn. plane triangulation

(Yu `98) G : 5-conn. plane triangulation $\Rightarrow$ G : K_4 -linked

Conjecture

G : 5-conn. plane triangulation $\Rightarrow$ G : spanning K_4 -linked

Results and problems for triangulations

		Plane		P.P., Torus, K-bottle		Other surfaces	
	-linked	4-conn.	5-conn.	4-conn.	5-conn.	4-conn.	5-conn.
	$2K_2$	○	○				
	C_4	○	○				
	K_4^-	○	○				
	K_4	✖	○				
spanning	$2K_2$	?	○				
spanning	C_4	?	○				
spanning	K_4^-	?	○				
spanning	K_4	✖	?				

4-connected plane triangulation

Theorem

G : 5-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

"k-ordered" + "Hamiltonian"

Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

The case of other surfaces

Theorem

G : 4-connected triangulation of the projective plane

$\Rightarrow$ (I) G : Hamiltonian (Thomas & Yu, `94)

(II) G : 4-ordered (Mukae & Oz., `10)

``$k$-ordered'' + ``Hamiltonian''

Conjecture

G : 4-connected plane triangulation

$\Rightarrow$ G : 4-ordered Hamiltonian

The case of other surfaces

Theorem

G : 4-connected triangulation of the projective plane

$\Rightarrow$ (I) G : Hamiltonian (Thomas & Yu, `94)

(II) G : 4-ordered (Mukae & Oz., `10)

``$k$-ordered'' + ``Hamiltonian''

Conjecture

G : 4-connected triangulation of the projective plane

$\Rightarrow$ G : 4-ordered Hamiltonian

The case of other surfaces

Theorem

G : 4-connected triangulation of the projective plane

$\Rightarrow$ (I) G : Hamiltonian (Thomas & Yu, `94)

(II) G : 4-ordered (Mukae & Oz., `10)

The case of the tours

(I) Hamiltonian (Kawarabayashi & Oz. ??)

(II) 4-ordered (Mukae & Oz., `10)

The case of the surface of genus ≥ 2

(I) ``Hamiltonian'' is false

The case of other surfaces

Conjecture

G : 4-connected triangulation of the projective plane

$\Rightarrow$ ✓ G : K_4^- -linked

✓ G : spanning K_4^- -linked

``$k$-ordered'' + ``Hamiltonian''

Conjecture

G : 4-connected triangulation of the projective plane

$\Rightarrow$ G : 4-ordered Hamiltonian

Results and problems for triangulations

-linked	Plane 4-conn.	Plane 5-conn.	P.P., Torus, K-bottle 4-conn.	P.P., Torus, K-bottle 5-conn.	Other surfaces (locally planar) 4-conn.	Other surfaces (locally planar) 5-conn.
$2K_2$	○	○	○	○	○	○
C_4	○	○	○	○	○	○
K_4^-	○	○	?	?	?	?
K_4	✖	○	?	?	?	?
spanning $2K_2$	?	○	?	?	✖	✖ (?)
spanning C_4	?	○	?	?	✖	✖ (?)
spanning K_4^-	?	○	?	?	✖	✖ (?)
spanning K_4	✖	?	?	?	✖	✖ (?)

Results and problems for triangulations

		Plane		P.P., Torus, K-bottle		Other surfaces (locally planar)	
-linked		4-conn.	5-conn.	4-conn.	5-conn.	4-conn.	5-conn.
	$2K_2$	○	○	○	○	○	○
	C_4	○	○	○	○	○	○
	K_4^-	○	○	?	?	?	?
	K_4	✖	○	?	?	?	?
spanning	$2K_2$	?	○	?	?	✖	✖ (?)
spanning	C_4	?	○	?	?	✖	✖ (?)
spanning	K_4^-	?	○	?	?	✖	✖ (?)
spanning	K_4	✖	?	?	?	✖	✖ (?)

Other problems

- What about the (non-planar) case of non-triangulation?
- What about other linkages??

  ✓ $K_2 \cup P_3,\ C_5,\ K_2 \cup K_3\ \cdots$

  (Those are impossible for planar case,

  but might be possible for other surfaces)

Problem

G : 5-connected non-planar triangulation

$\Rightarrow$ G : 5-ordered (C_5 -linked)

Thank you for your attention