Algorithmic Graph Theory on the Adriatic Coast

Koper (Slovenia)

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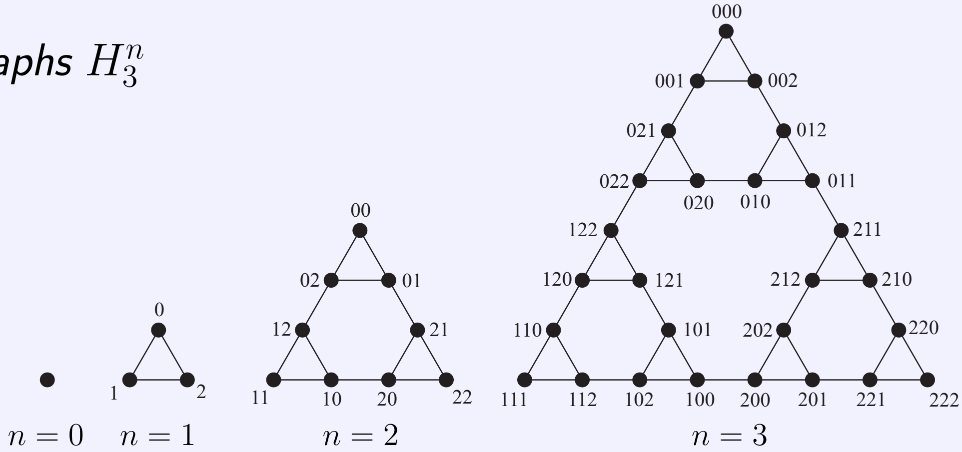
The Dudeney Algorithm[1]

Andreas M. Hinz
LMU Munich (Germany) & University of Maribor (Slovenia)

La Tour d'Hanoï (Édouard Lucas, 1883)

Hanoi graphs H_3^n



Metric properties: $\quad 2^n - 1 = \mathrm{d}_3(0^n, 1^n) = \varepsilon_3(0^n) = \mathrm{diam}(H_3^n)$.

$\mathrm{d}_3(s, j^n) = \sum_{d=1}^{n} (s_d \neq (s \vartriangle j)_d) \cdot 2^{d-1}$ unique solution

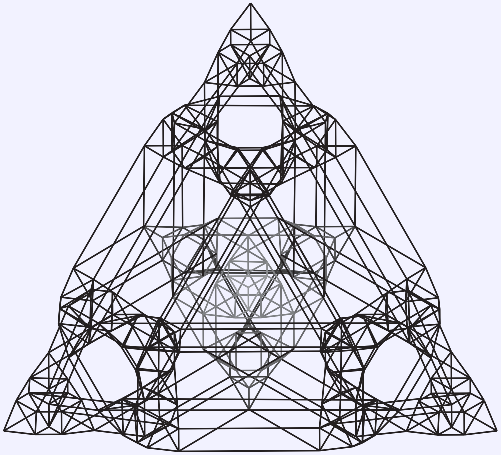
$\mathrm{d}_3(is, jt) = \mathrm{d}_3(s, (i \vartriangle j)^n) + 1 + \mathrm{d}_3(t, (i \vartriangle j)^n)$ unique solution, 1 LDM

or $= \mathrm{d}_3(s, j^n) + 1 + 2^n + \mathrm{d}_3(t, i^n)$ unique solution, 2 LDMs

or two solutions

Decision using automaton by Romik (2006)

Case $p = 4$ (*The Reve's Puzzle*, Dudeney 1902)



Hanoi graph H_4^4

Frame-Stewart numbers are defined as: $\forall\, n \in \mathbb{N}_0 :\ FS_3^n = 2^n - 1$;

$$FS_p^0 = 0,\ \forall\, n \in \mathbb{N} :\ FS_p^n = \min\left\{2FS_p^m + FS_{p-1}^{n-m} \ \middle|\ m \in [n]_0\right\};$$

$$\forall\, n \in \mathbb{N}_0 :\ \overline{FS}_p^n = \tfrac{1}{2}(FS_p^{n+1} - 1).$$

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10
2	0	1	3	6	10	15	21	28	36	45	55
3	0	1	4	10	20	35	56	84	120	165	220
4	0	1	5	15	35	70	126	210	330	495	715
5	0	1	6	21	56	126	252	462	792	1287	2002
6	0	1	7	28	84	210	462	924	1716	3003	5005
7	0	1	8	36	120	330	792	1716	3432	6435	11440
8	0	1	9	45	165	495	1287	3003	6435	12870	24310
9	0	1	10	55	220	715	2002	5005	11440	24310	48620

The hypertetrahedral array $\Delta_{q,\nu} = \binom{q+\nu-1}{q}$

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	-1	0	1	2	3	4	5	6	7	8	9
2	1	1	2	4	7	11	16	22	29	37	46
3	-1	0	2	6	13	24	40	62	91	128	174
4	1	1	3	9	22	46	86	148	239	367	541
5	-1	0	3	12	34	80	166	314	553	920	1461
6	1	1	4	16	50	130	296	610	1163	2083	3544
7	-1	0	4	20	70	200	496	1106	2269	4352	7896
8	1	1	5	25	95	295	791	1897	4166	8518	16414
9	-1	0	5	30	125	420	1211	3108	7274	15792	32206

The P-array $P_{q,\nu}$

$$2P_{q,\nu+1} = P_{q,\nu} + \Delta_{q,\nu+1}$$

$q \setminus \nu$	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	0	1	3	7	15	31	63	127	255
2	0	1	5	17	49	129	321	769	1793
3	0	1	7	31	111	351	1023	2815	7423
4	0	1	9	49	209	769	2561	7937	23297
5	0	1	11	71	351	1471	5503	18943	61183
6	0	1	13	97	545	2561	10625	40193	141569
7	0	1	15	127	799	4159	18943	78079	297727

Dudeney's array $a_{q,\nu}$

$$a_{q,0} = 0,\ a_{0,\nu} = \Delta_{0,\nu},\ a_{q+1,\nu+1} = 2a_{q+1,\nu} + a_{q,\nu+1}\,.$$

Dudeney's algorithm uses $\displaystyle \Delta_{h,\nu+1} = 1 + \sum_{q=1}^{h} \Delta_{q,\nu}\,.$

$$\Rightarrow\ \ \mathrm{d}_{2+h}(0^{\Delta_{h,\nu}}, 1^{\Delta_{h,\nu}}) \le a_{h,\nu} = P_{h-1,\nu} \cdot 2^{\nu} + (-1)^{h}$$

Fundamental relations

$$a_{h,\nu} = \sum_{k=0}^{\Delta_{h,\nu}-1} 2^{\nabla_{h,k}}\,,$$

where $\nabla_{h,k} = \max\{\mu \in \mathbb{N}_0 \mid \Delta_{h,\mu} \leq k\}$ is the *hypertetrahedral root* of k, for which

$$\forall\, x \in [\Delta_{h-1,\nu+1}]_0 :\ \nabla_{h,\Delta_{h,\nu}+x} = \nu\,.$$

$h \setminus n$	0	1	2	3	4	5	6	7	8	9
1	0	1	3	7	15	31	63	127	255	511
2	0	1	3	5	9	13	17	25	33	41
3	0	1	3	5	7	11	15	19	23	27
4	0	1	3	5	7	9	13	17	21	25

$$\forall\, x \leq \Delta_{h-1,\nu+1} :\ FS_{h+2}^{\Delta_{h,\nu}+x} = a_{h,\nu} + x \cdot 2^{\nu}$$

Altogether we know

$$FS^n_{2+h} = \sum_{k=0}^{n-1} 2^{\nabla_{h,k}} =: \Phi_{h-1}(n),\ \overline{FS}^n_{2+h} = \frac{1}{2}\sum_{k=1}^{n} 2^{\nabla_{h,k}} =: \overline{\Phi}_{h-1}(n)\,.$$

To prove the

Frame-Stewart Conjecture $\qquad \mathrm{d}_p(0^n, 1^n) = FS^n_p\,,$

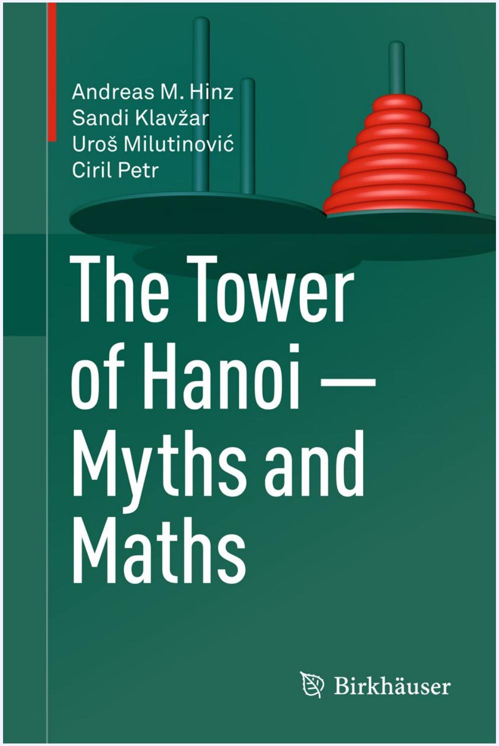
it suffices to show that

$$\forall\, t \in [h]^n:\ \mathrm{d}_{2+h}(0^n, t) \geq \overline{\Phi}_{h-1}(n)\,.$$

Note that for $p = 3$, i.e. $h = 1$, we have $t = 1^n$ and $\overline{\Phi}_0(n) = 2^n - 1$.

Thierry Bousch has published an attempt at $p = 4$, i.e. $h = 2$, in 2014.

Further reading:



Basel, 2013

http://www.tohbook.info

Bousch, T., La quatrième tour de Hanoï, Bull. Belg. Math. Soc. Simon Stevin 21(2014), 895–912.