Parameterized Complexity of Secluded Connectivity Problems

Petr A. Golovach[1]

Fedor V. Fomin[1] Nikolay Karpov[2] Alexander S. Kulikov[2]

[1]University of Bergen

[2]Steklov Institute of Mathematics at St.Petersburg

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Outline

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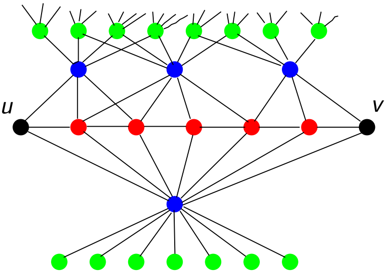
A W[1]-hard problem cannot be solved in FPT-time unless FPT=W[1].

Secluded paths and trees



Secluded paths and trees

SECLUDED STEINER TREE

Input: A graph G with a cost function $w\colon V(G) \to \mathbb{N}$, a set $S = \{s_1, \dots, s_p\} \subseteq V(G)$ of terminals, and non-negative integers k and C.

Question: Is there a connected subgraph T of G with $S \subseteq V(T)$ such that $|N_G[V(T)]| \leq k$ and $w(N_G[V(T)]) \leq C$?

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If $w(v) = 1$ for $v \in V(G)$ and $C = k$, then we have an instance of SECLUDED STEINER TREE without costs.

Previous work

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- Secluded Path without costs is NP-complete.
- Secluded Path without costs is solvable in time $\Delta^{\Delta} \cdot n^{O(1)}$, where Δ is the maximum vertex degree and and thus is FPT being parameterized by Δ.
- Secluded Steiner Tree problem is solvable in time $2^{O(t \log t)} \cdot n^{O(1)} \cdot \log W$ for graphs of treewidth at most t, where W is the maximum value of w.

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- We investigate Secluded Steiner Tree from kernelization perspective and provide several lower and upper bounds when parameters are the treewidth, the size of a vertex cover, maximum vertex degree and the solution size.

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- We investigate SECLUDED STEINER TREE from kernelization perspective and provide several lower and upper bounds when parameters are the treewidth, the size of a vertex cover, maximum vertex degree and the solution size.
- We refine the algorithmic result of Chechik et al. for SECLUDED STEINER TREE on graphs of bounded treewidth by improving the exponential dependence from the treewidth of the input graph.

Parameterization by the solution size

Theorem

SECLUDED PATH is solvable in time $O(3^{k/3} \cdot n \log W)$ and SECLUDED STEINER TREE can be solved in time $O(2^k \cdot (n+m)k^2 \log W)$, where W is the maximum value of w on the input graph G.

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Corollary

SECLUDED PATH is solvable in time $O(1.3896^n \cdot \log W)$, and SECLUDED STEINER TREE is solvable in time $O(1.7088^n \cdot \log W)$, where W is the maximum value of w on the input graph G.

Sketch of the proof

We use the color coding/random separation techniques introduced by Alon, Yuster and Zwick (1995) and Cai, Chan and Chan (2006).

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We say that a solution for the considered instance, i.e., a connected subgraph T of G with $S \subseteq V(T)$ such that $|N_G[V(T)]| \leq k$ and $w(N_G[V(T)]) \leq C$, is colored correctly if the vertices of T are red and the vertices of $N_G(V(T))$ are blue.

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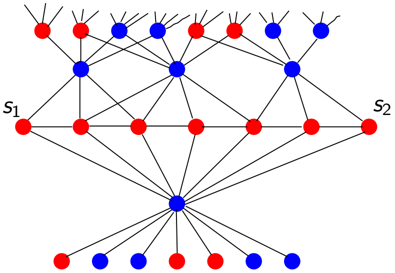

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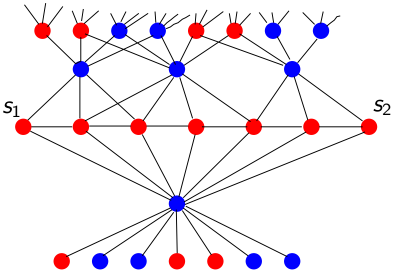

If there is a correctly colored solution T, then it can be found in a straightforward way: T is the component of the subgraph of G induced by the red vertices that contains all the terminals.

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The probability that for 2^k random colorings, the solution is colored incorrectly for all of them, is at most $(1 - \frac{1}{2^k})^{2^k} \leq \frac{1}{e}$.

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Suppose that F is a connected induced subgraph of G of minimum size such that $S \subseteq V(F)$, i.e., F is a minimum vertex Steiner tree. Then $\ell = |V(F)| \leq |V(T)| \leq |N_G[V(T)]|$ for any solution T for the considered instance.

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We ask whether there is a solution for $k = \ell + r$, where r is the parameter.

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The currently best FPT-algorithms for STEINER TREE running in time $O^*(2^p)$ are given by Björklund et al. (2007) and Nederlof (2013) (the first algorithm demands exponential in p space and the latter uses polynomial space).

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Auxiliary lemmas

Lemma

Let G be a connected graph and $S \subseteq V(G)$, $p = |S|$. Let F be an inclusion minimal induced subgraph of G such that $S \subseteq V(F)$ and $X = \{v \in V(F) | d_F(v) \geq 3\} \cup S$. Then

- **i)** $|X| \leq 4p - 6$, *and*
- **ii)** $|N_F(X)| \leq 4p - 6$.

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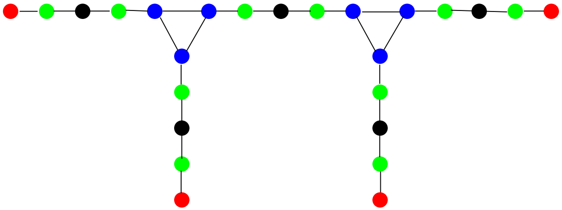

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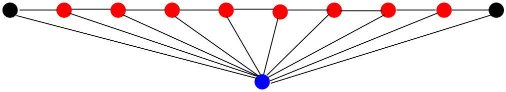

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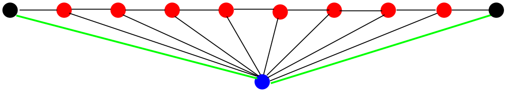

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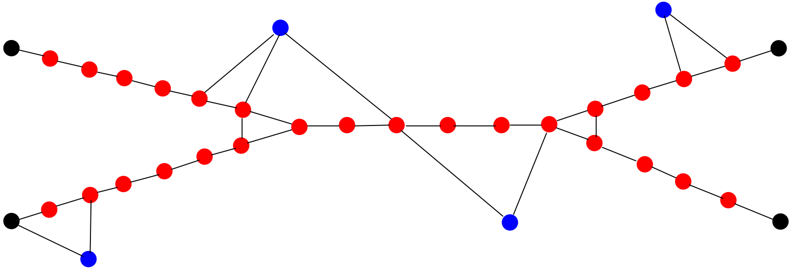

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- the vertices of $Y = N_G(V(T))$ are blue,
- the vertices of $X = \{v \in V(T) \mid d_F(v) \geq 3\} \cup S$ are red,
- the vertices of $N_T(X)$ are red,
- the vertices of $Z = N_G(Y) \cap V(T)$ are red,
- for any $z \in Z \setminus S$, at least two neighbors of z in T are red.

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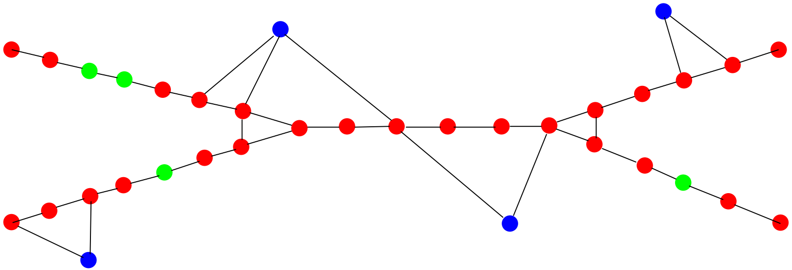

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Theorem

SECLUDED STEINER TREE *without costs is co-W[1]-hard when parameterized by* r*, where* $r = k - \ell$ *and* ℓ *is the size of a Steiner tree for* S*.*

Kernelization

A kernelization for a parameterized problem is a polynomial algorithm that maps each instance (x, k) with the input x and the parameter k to an instance (x', k') such that

i) (x, k) is a YES-instance if and only if (x', k') is a YES-instance of the problem,

ii) the size of x' is bounded by $f(k)$ for a computable function f, and

iii) k' is bounded by some $g(k)$.

The output (x', k') is called a kernel. The function f is said to be a size of a kernel. Respectively, a kernel is polynomial if f is polynomial.

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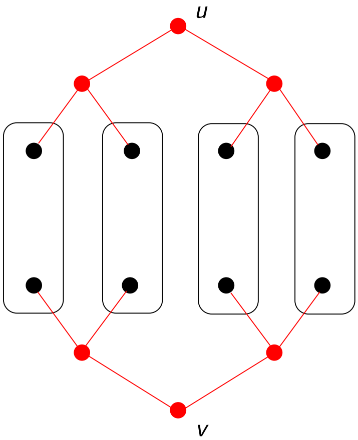
It is widely believed that not all FPT problems have polynomial kernels. In particular, Bodlaender et al. (2009) and Bodlaender, Jansen and Kratsch (2014) introduced techniques that allow to show that a parameterized problem has no polynomial kernel unless NP$\subseteq$co-NP/poly.

Kernelization lower bounds

Theorem

SECLUDED PATH without costs on graphs of treewidth at most t and maximum degree at most Δ admits no polynomial kernel unless $\mathrm{NP} \subseteq \mathrm{co\text{-}NP/poly}$ *when parameterized by* $k + t + \Delta$ *or* $(k - \ell) + t + \Delta$*, where* ℓ *is the length of the shortest path between terminals.*

Idea of the proof



Kernelization for graphs with bounded vertex cover number

A set of vertices U is a vertex cover of a graph G if for any $uv \in E(G)$, $u \in U$ or $v \in U$. The vertex cover number is the size of a minimum vertex cover.

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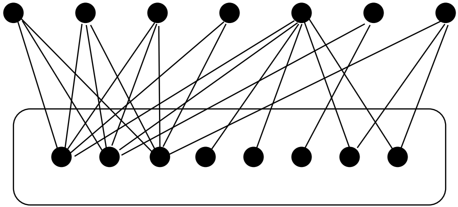
SECLUDED PATH *without costs on graphs with the vertex cover number at most* t *admits no polynomial kernel unless* $\mathrm{NP} \subseteq \mathrm{co\text{-}NP/poly}$ *when parameterized by* t.

Sketch of the proof



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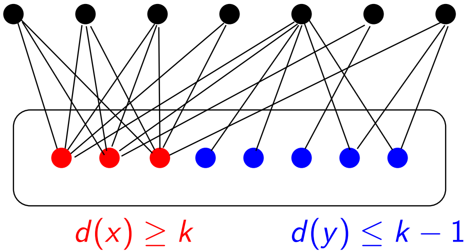

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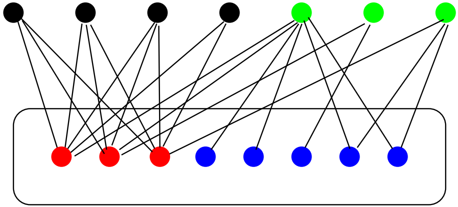

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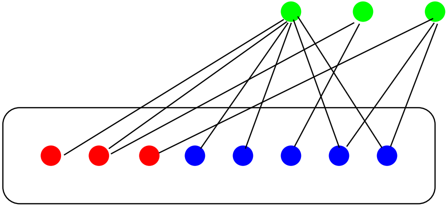

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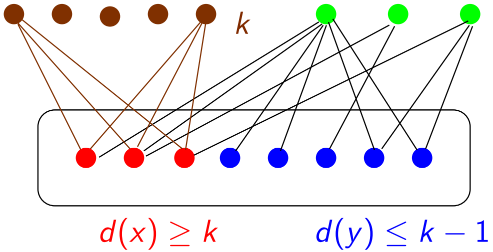

Secluded Steiner Tree for graphs of bounded treewidth

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Theorem

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It is possible to solve SECLUDED STEINER TREE deterministically in time $O((2 + 2^{\omega+1})^t \cdot (n + \log W)^{O(1)})$ (here ω is the matrix multiplication constant).

Open problems

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- What can be said about Secluded Path/Secluded Steiner Tree for planar graphs?

Thank You!