High order parametric polynomial approximation of conic sections

Gašper Jaklič [a,b,c], Jernej Kozak [a,b], Marjeta Krajnc [a,b], Vito Vitrih [c], Emil Žagar [a,b,*]

[a] *FMF, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia*

[b] *IMFM, Jadranska 19, Ljubljana, Slovenia*

[c] *PINT, University of Primorska, Muzejski trg 2, Koper, Slovenia*

Abstract

In this paper, a particular shape preserving parametric polynomial approximation of conic sections is studied. The approach is based upon a general strategy to the parametric approximation of implicitly defined planar curves. Polynomial approximants derived are given in a closed form and provide the highest possible approximation order. Although they are primarily studied to be of practical use, their theoretical background is not to be underestimated: they confirm the Höllig-Koch's conjecture for the Lagrange interpolation of conic sections too.

Key words: conic section, parametric curve, implicit curve, approximation order, Lagrange interpolation, approximation

1 Introduction

Conic sections are standard objects in CAGD (computer aided geometric design) and many computer graphics systems include them by default. An ellipse and a hyperbola can be represented in a parametric form using e.g. trigonometric and hyperbolic functions. In contrast to a parabola, they do not have a parametric polynomial parameterization, but they can be written as quadratic rational Bézier curves. In many applications a parametric polynomial approximation of conic sections is needed and it is important to derive accurate polynomial approximants.

* Corresponding author.
Email address: emil.zagar@fmf.uni-lj.si (Emil Žagar).

General results on Hermite type approximation of conic sections by parametric polynomial curves of odd degree are given in [1] and [2]. However, the results hold true only asymptotically, i.e., for small segments of a particular conic section. Among conic sections circular arcs are the most important geometric objects in practice. A lot of papers consider good approximation of circular segments with the radial error as the parametric distance. In [3], the authors study cubic Bézier Hermite type interpolants which are sixth-order accurate, and in [4] a similar problem with various boundary conditions is presented. Quadratic Bézier approximants are considered in [5], and some new special types of Hermite interpolation schemes are derived in [6] and [7].

An interesting closed form solution of the Taylor type interpolation of a circular arc by parametric polynomial curves of odd degree goes back to [8]. In that paper the authors constructed an explicit formula for parametric polynomial approximants. The results have been later extended to even degree curves in [9] and to a more general class of rational parametric curves in [10].

As a motivation to improve the results obtained in [8] and [9], consider the following example. Take a particular parametric quintic polynomial approximant of the unit circle, given in [8,9] as

$$\begin{pmatrix} 1 - 2\,t^2 + 2\,t^4 \\ 2\,t - 2\,t^3 + t^5 \end{pmatrix}. \tag{1}$$

It is shown in Fig. 1 together with a new quintic approximant

$$\begin{pmatrix} 1 - (3+\sqrt{5})t^2 + (1+\sqrt{5})t^4 \\ (1+\sqrt{5})t - (3+\sqrt{5})t^3 + t^5 \end{pmatrix}. \tag{2}$$

Quite clearly, the latter has much better approximation properties. One of the aims of this paper is to establish a general framework for a construction of parametric polynomial approximants of conic sections such as (2).

The main problem considered turns out to be, how to find two nonconstant polynomials x_n, $y_n \in \mathbb{R}[t]$ of degree $\leq n$, such that

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n} \tag{3}$$

for the elliptic case, and

$$x_n^2(t) - y_n^2(t) = 1 \pm t^{2n} \tag{4}$$

for the hyperbola. The implicit form of the unit circle or the unit hyperbola is

$$x^2 \pm y^2 = 1. \tag{5}$$

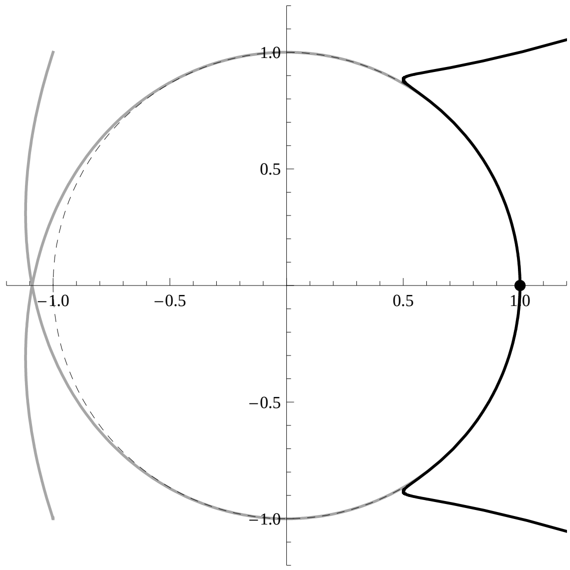

Fig. 1. The unit circle (dashed), quintic parametric polynomial approximant given by (1) (black) and a new parametric approximant of the same degree given by (2) (grey).

This clearly indicates that the considered problem is equivalent to finding a good parametric polynomial approximation of the implicit representation (5).

The importance of the equation (3) has already been noted in [8] and the existence of a solution has been established for odd n. However, in [9] it has been shown that the equation (3) has at least one real solution for all $n \in \mathbb{N}$. It is based upon a particular rational parameterization of the circle, and the coefficients of polynomials x_n and y_n can be elegantly expressed in terms of Chebyshev polynomials of the first and the second kind.

However, such a solution is far from optimal (see Fig. 1). In this paper, all solutions of (3) and (4) are constructed in a closed form, and the best ones with respect to the approximation error are studied in detail. It turns out that such an approximation is excellent since the error decays exponentially with the degree n. On the other hand, the existence of the approximants has a surprisingly deep theoretical impact too. Namely, it confirms a very well known Höllig-Koch's conjecture ([11]) on geometric Lagrange interpolation of conic sections.

The paper is organized as follows. In Section 2 a general approach to a parametric approximation of implicitly defined planar curves is outlined. The normal distance is studied as an upper bound for the Hausdorff as well as for a parametric distance. In Section 3 conic sections as a special class of implicit curves are studied. The approximation problem is precisely defined. In the following section a construction of all appropriate solutions is outlined. In Section 5 the best solution is studied in detail. Section 6 deals with the error analysis of the best solution. In Section 7 the most important theoretical result of the paper is presented, the Höllig-Koch's conjecture for conic sections is confirmed. The paper is concluded by some numerical examples in Section 8.

2 Parametric approximation of implicit functions

Suppose that the equation

$$f(x,y)=0, \quad (x,y)\in\mathcal{D}\subset\mathbb{R}^2, \tag{6}$$

defines a segment $\boldsymbol{f}$ of a regular smooth planar curve. Further, let

$$\boldsymbol{r}:[a,b]\to\mathbb{R}^2, \quad t\mapsto\begin{pmatrix}x_r(t)\\ y_r(t)\end{pmatrix}, \tag{7}$$

denote a parametric approximation of the curve segment $\boldsymbol{f}$ that satisfies the implicit equation (6) approximately,

$$f(x_r(t),y_r(t))=:\varepsilon(t), \quad t\in[a,b]. \tag{8}$$

What can be concluded about approximation properties from the approximated implicit equation if ε is small enough? Let $(x,y)\in\mathcal{D}$ be fixed. The first order expansion of the equation (8) reveals

$$f_x(x,y)\ (x_r(t)-x)+f_y(x,y)\ (y_r(t)-y)=\varepsilon(t)+\delta(t), \tag{9}$$

where f_x and f_y are partial derivatives and $\delta(t)$ denotes higher order terms in differences $x_r(t)-x$ and $y_r(t)-y$, i.e.,

$$\delta(t):=\mathcal{O}\left((x_r(t)-x)^2\right)+\mathcal{O}\left((x_r(t)-x)\,(y_r(t)-y)\right)+\mathcal{O}\left((y_r(t)-y)^2\right). \tag{10}$$

Suppose now that the curve $\boldsymbol{r}$ can be regularly reparameterized by a normal to the curve (6) (see [12]). More precisely, take a normal on $\boldsymbol{f}$ at a particular point (x,y) and find its nearest intersection with $\boldsymbol{r}$ (see Fig. 2). The corresponding parameter $t=t(x,y)$ is then determined by the equation

$$f_y(x,y)\ (x_r(t)-x)-f_x(x,y)\ (y_r(t)-y)=0. \tag{11}$$

Since $\boldsymbol{f}$ is regular, $f_x^2(x,y)+f_y^2(x,y)\neq 0$, and the equations (9) and (11) imply

$$\begin{aligned} x_r(t)&=x+\frac{f_x(x,y)}{f_x^2(x,y)+f_y^2(x,y)}(\varepsilon(t)+\delta(t)),\\ y_r(t)&=y+\frac{f_y(x,y)}{f_x^2(x,y)+f_y^2(x,y)}(\varepsilon(t)+\delta(t)). \end{aligned} \tag{12}$$

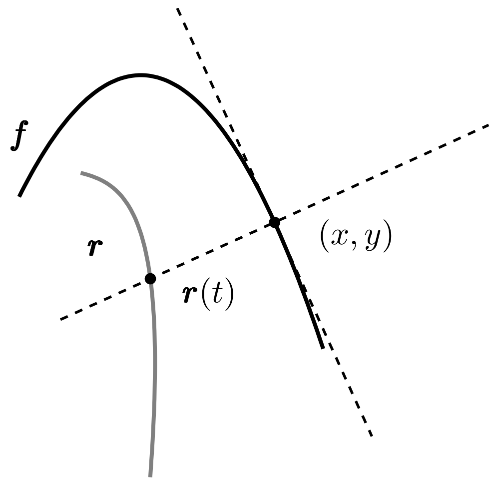

Fig. 2. The normal distance between a curve segment $\boldsymbol{f}$, satisfying (6), and a parametric curve $\boldsymbol{r}$ at a point (x, y).

Recall the behaviour of (10). If $\varepsilon(t)$ at a particular $t = t(x, y)$ is small enough, one may apply the Banach contraction principle on (12) to conclude that $\delta(t) = \mathcal{O}\left(\varepsilon^2(t)\right)$. The normal distance at this point is

$$\rho(x,y) := \sqrt{\left(x_r(t)-x\right)^2 + \left(y_r(t)-y\right)^2} = \frac{|\varepsilon(t)+\delta(t)|}{\sqrt{f_x^2(x,y)+f_y^2(x,y)}} \tag{13}$$
$$= \frac{|\varepsilon(t)|}{\sqrt{f_x^2(x,y)+f_y^2(x,y)}} + \mathcal{O}\left(\varepsilon^2(t)\right).$$

This provides a basis to obtain an upper bound on parametric and Hausdorff distance between curves (see [8]), and quite clearly indicates the importance of ε in (8) being as small as possible. Let us summarize the preceding discussion.

Theorem 1 *Let a parametric curve* $\boldsymbol{r}$*, defined by* (7)*, approximate a smooth curve segment* $\boldsymbol{f}$*, given by the implicit equation* (6)*. Suppose that*

$$f(x_r(t), y_r(t)) = \varepsilon(t), \quad t \in [a,b].$$

If the curve $\boldsymbol{r}$ *can be regularly reparameterized by the normal to* $\boldsymbol{f}$ *and* ε *is small enough, the normal distance between curves is bounded by*

$$\max_{(x,y)\in\mathcal{D}} \frac{|\varepsilon(t(x,y))|}{\sqrt{f_x^2(x,y)+f_y^2(x,y)}} + \mathcal{O}\left(\varepsilon^2\left(t\left(x,y\right)\right)\right). \tag{14}$$

3 Conic sections

In this section parametric polynomial approximation of implicitly defined conic sections will be considered. For a chosen error term ε, defined in (8), an appropriate parametric polynomial approximant $\boldsymbol{p}_n$ of degree n,

$$\boldsymbol{r}(t) = \boldsymbol{p}_n(t) := \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix},$$

which fulfills (8), will be determined. Since a parabola has a polynomial parameterization, it is interesting to study ellipse and hyperbola only. By choosing an appropriate coordinate system (see Section 7 for details), they are given as

$$\left(\frac{x - x_0}{a}\right)^2 \pm \left(\frac{y - y_0}{b}\right)^2 = 1.$$

Further, by a translation and scaling of the coordinate system, the above equation can be rewritten to (5). The main problem considered is to find two nonconstant polynomials x_n and y_n of degree at most n, such that

$$x_n^2(t) \pm y_n^2(t) = 1 + \varepsilon(t). \tag{15}$$

A residual polynomial ε is of degree at most $2n$. Since at least one point on the conic should be interpolated, let us choose $\varepsilon(0) = 0$. In order for the approximation error to be as small as possible, ε should not involve low degree terms, and should be spanned by t^{2n} only. Moreover, without loss of generality we can assume that

$$x_n(0) = 1, \quad x_n'(0) = 0, \quad y_n(0) = 0, \quad y_n'(0) = 1.$$

The unknown polynomials can thus be written as

$$x_n(t) := 1 + \sum_{\ell=2}^{n} a_\ell\, t^\ell, \quad y_n(t) := t + \sum_{\ell=2}^{n} b_\ell\, t^\ell,$$

which transforms (15) to

$$\left(1 + \sum_{\ell=2}^{n} a_\ell\, t^\ell\right)^2 \pm \left(t + \sum_{\ell=2}^{n} b_\ell\, t^\ell\right)^2 = 1 + \left(a_n^2 \pm b_n^2\right) t^{2n}. \tag{16}$$

The equation (16) is actually a system of $2n - 2$ nonlinear equations for $2n - 2$ unknowns $(a_\ell)_{\ell=2}^n$ and $(b_\ell)_{\ell=2}^n$. It can further be simplified by a suitable reparameterization. Let

$$A := \frac{1}{\sqrt[2n]{|a_n^2 \pm b_n^2|}}.$$

A linear parameter scaling $t \mapsto t/A$ and new variables

$$\alpha_\ell := a_\ell\, A^\ell, \quad \beta_\ell := b_\ell\, A^\ell, \quad \ell = 1, 2, \ldots, n, \tag{17}$$

where $a_1 := 0, b_1 := 1$, transform the problem into the problem of finding two nonconstant polynomials

$$x_n(t) := 1 + \sum_{\ell=2}^{n} \alpha_\ell \, t^\ell, \quad y_n(t) := \sum_{\ell=1}^{n} \beta_\ell \, t^\ell, \quad \beta_1 > 0, \tag{18}$$

such that

$$x_n^2(t) \pm y_n^2(t) = 1 + \operatorname{sign}(a_n^2 \pm b_n^2)\, t^{2n}. \tag{19}$$

The hyperbolic case involves two possibilities, since $a_n^2 = b_n^2$ can not happen, while the elliptic case implies only one.

Note that a similar problem is related to Pell's equation and a solution via Chebyshev polynomials ([13]).

4 Solutions

Solving the equation (19) is equivalent to solving

$$x_n^2(t) \pm y_n^2(t) = 1$$

in the factorial ring $\mathbb{R}[t]/t^{2n}$. But since there are additional restrictions (18), the problem can not be tackled by classical algebraic tools.

Fortunately, there is an another way - the special form of the equation (19) enables an approach that straightforwardly yields all the solutions, satisfying particular requirements. Let us consider each case separately.

For the elliptic case, the equation (19) can be rewritten as

$$\left(x_n(t) + \mathrm{i}\, y_n(t)\right)\left(x_n(t) - \mathrm{i}\, y_n(t)\right) = \prod_{k=0}^{2n-1} \left(t - e^{\mathrm{i}\, \frac{2k+1}{2n}\pi}\right), \tag{20}$$

where the right-hand side is the factorization of $1 + t^{2n}$ over $\mathbb{C}$. From the uniqueness of the polynomial factorization over $\mathbb{C}$ up to a constant factor, and from the fact that the factors in (20) appear in conjugate pairs, it follows

$$x_n(t) + \mathrm{i}\, y_n(t) = \gamma \prod_{k=0}^{n-1} \left(t - e^{\mathrm{i}\, \sigma_k \frac{2k+1}{2n}\pi}\right), \quad \gamma \in \mathbb{C}, \quad |\gamma| = 1,$$

where $\sigma_k = \pm 1$. In order to interpolate the point $(1, 0)$, γ must be chosen as

$$\gamma := (-1)^n \prod_{k=0}^{n-1} e^{-\mathrm{i}\, \sigma_k \frac{2k+1}{2n}\pi},$$

which implies

$$x_n(t) + \mathrm{i}\, y_n(t) = (-1)^n \prod_{k=0}^{n-1} \left(t\, e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi} - 1 \right) =: p_e(t;\boldsymbol{\sigma}), \tag{21}$$

with $\boldsymbol{\sigma} = (\sigma_k)_{k=0}^{n-1} \in \{-1,1\}^n$. This leads to 2^n solutions, but those with $\beta_1 = 0$ must be excluded. Since the remaining ones appear in pairs $(x_n, \pm y_n)$, precisely half of them fulfill the requirement $\beta_1 > 0$.

Let us now consider the hyperbolic case. Similarly as in (20) the expression (19) can be rewritten as

$$(x_n(t) + y_n(t))\,(x_n(t) - y_n(t)) = 1 - t^{2n} = (1 - t^2) \prod_{k=1}^{n-1} \left(t^2 - 2\cos\left(\frac{k\pi}{n}\right) t + 1 \right) \tag{22}$$

for the case $a_n^2 < b_n^2$, or as

$$(x_n(t) + y_n(t))\,(x_n(t) - y_n(t)) = 1 + t^{2n} = \prod_{k=0}^{n-1} \left(t^2 - 2\cos\left(\frac{2k+1}{2n}\pi\right) t + 1 \right) \tag{23}$$

for $a_n^2 > b_n^2$. The right-hand side is obtained by the factorization of $1 \pm t^{2n}$ using roots of unity and by joining conjugate complex factors into quadratic real ones. The idea now is to write the right-hand side of (22) and (23) as a product of two polynomials p_h and q_h and to define x_n and y_n as

$$x_n(t) = \frac{1}{2}(p_h(t) + q_h(t)), \quad y_n(t) = \pm\frac{1}{2}(p_h(t) - q_h(t)). \tag{24}$$

Since x_n and y_n have to be of degree $\leq n$, polynomials p_h and q_h must both be of degree n, otherwise the degree of x_n or y_n would be too high. Therefore

$$p_h(t) := p_h\left(t;\mathcal{I}_n\right) := (1+t)^{\frac{1-(-1)^n}{2}} \prod_{\substack{k \in \mathcal{I}_n \subseteq \{1,2,\ldots,n-1\} \\ |\mathcal{I}_n| = \left\lfloor \frac{n}{2} \right\rfloor}} \left(t^2 - 2\cos\left(\frac{k\pi}{n}\right) t + 1 \right) \tag{25}$$

for $a_n^2 < b_n^2$. In the factorization (23) only even degree factors are available. A solution thus exists only for even n and

$$p_h(t) := p_h\left(t;\mathcal{I}_n\right) := \prod_{\substack{k \in \mathcal{I}_n \subseteq \{1,2,\ldots,n-1\} \\ |\mathcal{I}_n| = \frac{n}{2}}} \left(t^2 - 2\cos\left(\frac{2k+1}{2n}\pi\right) t + 1 \right). \tag{26}$$

As in the elliptic case, the solutions with $\beta_1 = 0$ must be excluded and from the remaining pairs $(x_n, \pm y_n)$ those with $\beta_1 > 0$ are kept.

The number of admissible solutions is growing exponentially with n as can be seen from Tab. 1. The choice of a particular solution with minimal approximation error and its explicit formula will be given in the next section.

Table 1
The number of admissible solutions in all three cases for $n = 2, 3, \ldots, 10$.

n	2	3	4	5	6	7	8	9	10
elliptic case	1	3	6	15	27	64	120	254	495
hyperbolic case $a_n^2 < b_n^2$	0	1	2	5	8	20	32	70	120
hyperbolic case $a_n^2 > b_n^2$	1	0	2	0	9	0	32	0	125

5 Best solution

For both, elliptic and hyperbolic case, the best solution is the one with the maximal possible $\beta_1 > 0$. This can clearly be seen from (15) and (17), since the error term in the given parameterization will be the smallest for A as large as possible.

Theorem 2 *The best solution for the elliptic case is*

$$x_n(t) = \mathrm{Re}\left(p_e\left(t; \boldsymbol{\sigma}^*\right)\right), \quad y_n(t) = \mathrm{Im}\left(p_e\left(t; \boldsymbol{\sigma}^*\right)\right), \quad \boldsymbol{\sigma}^* = (1)_{k=0}^{n-1}, \tag{27}$$

and the best solution for the hyperbolic case is

$$x_n(t) = \frac{1}{2}\left(p_h\left(t; \mathcal{I}_n^*\right) + p_h\left(-t; \mathcal{I}_n^*\right)\right), \quad y_n(t) = \frac{1}{2}\left(p_h\left(t; \mathcal{I}_n^*\right) - p_h\left(-t; \mathcal{I}_n^*\right)\right), \tag{28}$$

where p_h is defined by (25) for odd n and by (26) for even n, and

$$\mathcal{I}_n^* = \left\{\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n-1\right\}.$$

In all cases

$$\beta_1 = \frac{1}{\omega_n}, \quad \omega_n := \sin\frac{\pi}{2n}.$$

PROOF. Consider first the elliptic case. From (21) it is straightforward to obtain

$$\alpha_1 + \mathrm{i}\,\beta_1 = -\sum_{k=0}^{n-1} e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi}, \tag{29}$$

which leads to

$$\beta_1 = \sum_{k=0}^{n-1} \sigma_k \sin\left(\frac{2k+1}{2n}\pi\right).$$

Since all considered sines are positive, the largest β_1 is obtained for the choice $\sigma_k = 1$, $k = 0, 1, \ldots, n-1$. By (29),

$$\beta_1 = \operatorname{Im}\left(-e^{-\mathrm{i}\frac{\pi}{2n}} \sum_{k=0}^{n-1} \left(e^{-\mathrm{i}\frac{\pi}{n}}\right)^k\right) = 2\,\frac{\sin\frac{\pi}{2n}}{1-\cos\frac{\pi}{n}} = \frac{1}{\omega_n}. \tag{30}$$

In the hyperbolic case, let us first observe (22). From (24) and (25) it is straightforward to derive

$$\beta_1 = \pm\left(1 + \sum_{k\in\mathcal{I}_n^c} \cos\frac{k\pi}{n} - \sum_{k\in\mathcal{I}_n} \cos\frac{k\pi}{n}\right),\ \ \mathcal{I}_n \subseteq \{1,2,\ldots,n-1\},\ |\mathcal{I}_n| = \left\lfloor\frac{n}{2}\right\rfloor,$$

where $\mathcal{I}_n^c$ denotes the complement of $\mathcal{I}_n$ in $\{1,2,\ldots,n-1\}$. The largest possible β_1, which implies the best solution, is obtained for $\mathcal{I}_n^*$ and is equal to

$$\beta_1 = 1 + 2\sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \cos\frac{k\pi}{n} = \begin{cases} \dfrac{1}{\omega_n}, & n \text{ is odd},\\ \sqrt{\dfrac{1}{\omega_n^2} - 1}, & n \text{ is even},\end{cases}$$

which can be derived as in (30).

But for even n the numerical experiences show that the resulting curve is not symmetric. Furthermore, the case $1+t^{2n}$ gives a larger β_1. Namely, from (26) it follows

$$\beta_1 = \pm\left(\sum_{k\in\mathcal{I}_n^c} \cos\left(\frac{2k+1}{2n}\pi\right) + \cos\frac{\pi}{2n} - \sum_{k\in\mathcal{I}_n} \cos\left(\frac{2k+1}{2n}\pi\right)\right),$$

$\mathcal{I}_n \subseteq \{1,2,\ldots,n-1\}$, $|\mathcal{I}_n| = \frac{n}{2}$, and the optimal β_1 is achieved for $\mathcal{I}_n^*$. It simplifies to

$$\beta_1 = 2\sum_{k=0}^{\frac{n}{2}-1} \cos\left(\frac{2k+1}{2n}\pi\right) = \frac{1}{\omega_n}.$$

The proof is completed.

Any solution for the elliptic case, for which x_n is an even and y_n is an odd function, can be transformed into a solution for the hyperbolic case by using the map

$$\begin{aligned} x_n(t) &\mapsto x_n(\mathrm{i}\,t),\\ y_n(t) &\mapsto -\mathrm{i}\,y_n(\mathrm{i}\,t). \end{aligned} \tag{31}$$

The coefficient β_1 for the best solution in the elliptic and in the hyperbolic case is the same and it is preserved by the map (31). Therefore (31) maps the best solution for the elliptic case into the best solution for the hyperbolic case provided that x_n is even and y_n is odd. This follows from the next theorem.

Theorem 3 *Coefficients of the best solution for the elliptic case are obtained as*

$$\alpha_k + \mathrm{i}\,\beta_k = (-1)^k S_k, \quad S_k := \sum_{\substack{i_1<i_2<\cdots<i_k \\ i_j\in\{1,2,\ldots,n\}}} e^{-\frac{\mathrm{i}\pi}{2n}(2(i_1+i_2+\ldots+i_k)-k)}. \tag{32}$$

Moreover,

$$\alpha_k = \begin{cases} \displaystyle\sum_{j=0}^{k(n-k)} P(j,k,n-k)\cos\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is even,} \\ 0, & k \text{ is odd,} \end{cases}$$

$$\beta_k = \begin{cases} 0, & k \text{ is even,} \\ \displaystyle\sum_{j=0}^{k(n-k)} P(j,k,n-k)\sin\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is odd,} \end{cases}$$

where $P(j,k,r)$ denotes the number of integer partitions of $j \in \mathbb{N}$ with $\leq k$ parts, all between 1 and r, where $k, r \in \mathbb{N}$, and $P(0,k,r) := 1$.

PROOF. Let us define $z_k := e^{\mathrm{i}\,\frac{2k-1}{2n}\pi}$. From (21) and (27) we obtain

$$\alpha_k + \mathrm{i}\,\beta_k = (-1)^k \sum_{\substack{i_1<i_2<\cdots<i_k \\ i_j\in\{1,2,\ldots,n\}}} \overline{z}_{i_1}\overline{z}_{i_2}\cdots\overline{z}_{i_k}. \tag{33}$$

The equation (32) now follows straightforwardly from (33). Let us introduce

$$\mathcal{L}_k := \{\boldsymbol{\ell} = (\ell_j)_{j=1}^k := (2i_1-1, 2i_2-1, \ldots, 2i_k-1),\ i_1 < \cdots < i_k,\ i_j \in \{1,\ldots,n\}\},$$

and $|\boldsymbol{\ell}| := \sum_{j=1}^k \ell_j$. Moreover, let

$$c : \mathbb{N}^k \to \mathbb{N}^k, \quad c(\boldsymbol{\ell}) := (2n-\ell_k, 2n-\ell_{k-1}, \ldots, 2n-\ell_1).$$

If $\boldsymbol{\ell} \in \mathcal{L}_k$ and $|\boldsymbol{\ell}| \neq kn$, then $c(\boldsymbol{\ell}) \in \mathcal{L}_k$. Since

$$\frac{|c(\boldsymbol{\ell})|}{2n}\pi = \frac{2nk - |\boldsymbol{\ell}|}{2n}\pi = k\pi - \frac{|\boldsymbol{\ell}|}{2n}\pi,$$

it follows

$$\sin\left(\frac{|\boldsymbol{\ell}|}{2n}\pi\right) = (-1)^{k+1}\sin\left(\frac{|c(\boldsymbol{\ell})|}{2n}\pi\right), \quad \cos\left(\frac{|\boldsymbol{\ell}|}{2n}\pi\right) = (-1)^k\cos\left(\frac{|c(\boldsymbol{\ell})|}{2n}\pi\right). \tag{34}$$

By (32), we have to show that S_k is a real number for even k and a pure imaginary number for odd k. By (34), only vectors $\boldsymbol{\ell} \in \mathcal{L}_k$ with $|\boldsymbol{\ell}| = nk$ have to be considered. Since

$$\sin\frac{k\pi}{2} = \begin{cases} 0, & k \text{ is even}, \\ (-1)^{\frac{k-1}{2}}, & k \text{ is odd}, \end{cases} \qquad \cos\frac{k\pi}{2} = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{\frac{k}{2}}, & k \text{ is even}, \end{cases}$$

we obtain

$$S_k = \begin{cases} \sum\limits_{\boldsymbol{\ell}\in\mathcal{L}_k} \cos\left(\frac{|\boldsymbol{\ell}|}{2n}\pi\right), & k \text{ is even}, \\ -\,\mathrm{i} \sum\limits_{\boldsymbol{\ell}\in\mathcal{L}_k} \sin\left(\frac{|\boldsymbol{\ell}|}{2n}\pi\right), & k \text{ is odd}. \end{cases}$$

For each vector $\boldsymbol{\ell} \in \mathcal{L}_k$, it holds $|\boldsymbol{\ell}| \in \{k^2 + 2j,\ j = 0, 1, \ldots, k(n-k)\}$, and there are exactly $P(j, k, n-k)$ vectors with $|\boldsymbol{\ell}| = k^2 + 2j$. Thus

$$S_k = \begin{cases} \sum\limits_{j=0}^{k(n-k)} P(j,k,n-k)\cos\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is even}, \\ -\,\mathrm{i}\sum\limits_{j=0}^{k(n-k)} P(j,k,n-k)\sin\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is odd}. \end{cases}$$

Using (32), the proof is completed.

Corollary 4 *For the best solution in both (elliptic and hyperbolic) cases, the polynomial x_n is an even function and y_n is an odd one.*

Corollary 5 *In the elliptic case, the coefficients possess a particular symmetry,*

$$\alpha_{n-k} + \mathrm{i}\,\beta_{n-k} = \mathrm{i}^n(\alpha_k - \mathrm{i}\,\beta_k), \quad k = 0, 1, \ldots, \lfloor n/2 \rfloor. \tag{35}$$

Furthermore, for $k = 0, 1, \ldots, n$,

$$\begin{aligned} &\alpha_k = \pm\beta_{n-k}, \quad \text{for } n = 4\ell \pm 1, \\ &\alpha_{n-k} = \mp\alpha_k, \quad \beta_{n-k} = \pm\beta_k, \quad \text{for } n = 4\ell + 1 \pm 1. \end{aligned}$$

PROOF. Since

$$(-1)^n \prod_{k=1}^{n} e^{-\mathrm{i}\,\frac{2k-1}{2n}\pi} = \mathrm{i}^n,$$

by (33) it follows

$$\alpha_{n-k} + \mathrm{i}\,\beta_{n-k} = (-1)^k\,\mathrm{i}^n \sum_{\substack{i_1<i_2<\cdots<i_k \\ i_j\in\{1,2,\ldots,n\}}} z_{i_1} z_{i_2} \cdots z_{i_k} = \mathrm{i}^n(\alpha_k - \mathrm{i}\,\beta_k).$$

The second part can be derived directly from (35) and the proof is completed.

Note that by using (31), a similar result can be obtained for the hyperbolic case.

In Tab. 2, the best approximants of degrees $n = 2, 3, \ldots, 6$, defined in Thm. 2, are presented and in Fig. 3 the elliptic ones are shown for $n = 4, 5, 6$.

Table 2
The best approximants, defined in Thm. 2. The upper sign in $\mp$ stands for the elliptic case and the lower sign is for the hyperbolic one.

n	$x_n(t), \quad y_n(t)$
2	$x_2(t) = 1 \mp t^2, \quad y_2(t) = \sqrt{2}\, t$
3	$x_3(t) = 1 \mp 2\, t^2, \quad y_3(t) = 2\, t \mp t^3$
4	$x_4(t) = 1 \mp (2+\sqrt{2})t^2 + t^4$ $y_4(t) = \sqrt{4+2\sqrt{2}}(t \mp t^3)$
5	$x_5(t) = 1 \mp (3+\sqrt{5})t^2 + (1+\sqrt{5})t^4$ $y_5(t) = (1+\sqrt{5})t \mp (3+\sqrt{5})t^3 + t^5$
6	$x_6(t) = 1 \mp 2(2+\sqrt{3})t^2 + 2(2+\sqrt{3})t^4 \mp t^6$ $y_6(t) = (\sqrt{2}+\sqrt{6})t \mp \sqrt{2}\,(3+2\sqrt{3})t^3 + (\sqrt{2}+\sqrt{6})t^5$

Note that the coefficients in the best solution (28) for the hyperbolic case are nonnegative. This follows from the positiveness of the coefficients in $p_h\left(t; \mathcal{I}_n^*\right)$.

6 Error analysis

In this section, the analysis of the normal distance, introduced in Section 2, between a conic section, defined by the implicit equation (5), and the best polynomial approximant $(x_n, y_n)^T$, that satisfies (15), is outlined. The normal reparameterization $(x, y) \mapsto t = t(x, y)$, determined by the equation (11), here simplifies to

$$\pm\, y\, x_n(t) - x\, y_n(t) = x\, y\, (\pm 1 - 1)\,, \tag{36}$$

where the upper sign stands for the elliptic case and the lower sign for the hyperbolic one. The particularly simple form of the equation (36) helps us to establish the normal distance (13) precisely.

Lemma 6 *Suppose that* $(x_n(t), y_n(t))^T$ *satisfies* (15). *The normal reparame-*

terization $(x,y) \mapsto t = t(x,y)$ of a conic section (5) *at a point (x,y) satisfies*

$$x_n(t) = x + \frac{\varepsilon(t)\, x}{x^2 + y^2 + \sqrt{(x^2+y^2)^2 + \varepsilon(t)}}, \tag{37}$$

$$y_n(t) = y \pm \frac{\varepsilon(t)\, y}{x^2 + y^2 + \sqrt{(x^2+y^2)^2 + \varepsilon(t)}}. \tag{38}$$

Furthermore, the normal distance (13) *is*

$$\rho(x,y) = \frac{|\varepsilon(t)|\, \sqrt{x^2+y^2}}{x^2 + y^2 + \sqrt{(x^2+y^2)^2 + \varepsilon(t)}}\,. \tag{39}$$

PROOF. From the equation (36) one obtains

$$y_n(t) - y = \pm \frac{y}{x} \left(x_n(t) - x\right), \tag{40}$$

which simplifies (15) to a quadratic equation for the difference $x_n(t) - x$,

$$\left(x^2 \pm y^2\right) \left(x_n(t) - x\right)^2 + 2\,x \left(x^2 + y^2\right) \left(x_n(t) - x\right) - x^2 \varepsilon(t) = 0. \tag{41}$$

But $x^2 \pm y^2 = 1$, and the solutions of (41) are

$$x_n(t) - x = \frac{x^2 \varepsilon(t)}{x\left(x^2+y^2\right) \pm |x| \sqrt{\left(x^2+y^2\right)^2 + \varepsilon(t)}}.$$

Since only one solution is needed, it is obvious to choose the one, that satisfies $x_n(t) - x \to 0$ when $|\varepsilon(t)| \to 0$, as the basis for the reparameterization. This confirms (37), and the equations (38) and (39) follow from (40) and (13). The proof is completed.

Let us recall that $\varepsilon(t) = \pm t^{2n}$. Lemma 6 clearly implies that the parameter t must be limited to $[-1,1]$, otherwise the normal distance grows as $n \to \infty$. The question left is to find a set $\mathcal{D} \subset \mathbb{R}^2$, such that a solution $t(x,y) \in [-1,1]$ of (36) exists for each $(x,y) \in \mathcal{D}$, and that the obtained reparameterization is regular. A conic will therefore be written in a parametric form, and the parameter interval that implies a regular reparameterization for the best solution will be determined.

Let us parameterize the unit circle as

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \quad s \in \mathbb{R}. \tag{42}$$

From (36) we obtain the equation

$$\frac{y_n(t)}{x_n(t)} = \tan s. \tag{43}$$

By expressing each factor of (21) in the polar form and assuming that $t \in [-1,1]$, the polynomial $p_e(t, \boldsymbol{\sigma}^*)$ in (21) can be rewritten in the polar form as

$$x_n(t) + \mathrm{i}\, y_n(t) = (-1)^n \sqrt{1+t^{2n}}\, \exp\left(\mathrm{i} \sum_{k=0}^{n-1} \arctan \frac{t \sin \varphi_k}{1 - t\cos\varphi_k} \right),$$

with $\varphi_k := \frac{2k+1}{2n}\pi$. Therefore (43) can be simplified to

$$\psi_n(t) = s, \qquad \psi_n(t) := \sum_{k=0}^{n-1} \arctan\left(\frac{t\sin\varphi_k}{1 - t\cos\varphi_k} \right). \tag{44}$$

Since $\sin\varphi_k = \sin\varphi_{n-1-k}$ and $\cos\varphi_k = -\cos\varphi_{n-1-k}$, the function ψ_n is odd. For $t \in [-1,1]$,

$$\psi_n'(t) = \sum_{k=0}^{n-1} \frac{\sin\varphi_k}{t^2 - 2t\cos\varphi_k + 1} > 0,$$

which implies that ψ_n is strictly increasing on $[-1,1]$. Moreover,

$$\psi_n(1) = \sum_{k=0}^{n-1} \arctan\left(\cot\frac{\varphi_k}{2}\right) = \frac{n\pi}{4}.$$

Thus, for any $s \in \left[-\frac{n\pi}{4}, \frac{n\pi}{4}\right]$, there exists a unique solution

$$t = t(s) := t(x(s), y(s)) \in [-1,1]$$

of (44). The monotonicity of ψ_n also implies the regularity of the reparameterization. The length of the interval for s, divided by 2π, defines the number of winds of the approximating curve around the origin, namely $\left\lfloor \frac{n}{4} \right\rfloor$ (see Fig. 4). The equation (44) provides also the series expansion of the reparameterization

$$\begin{aligned} t(s) =& \psi_n^{-1}(s) = \omega_n\, s - \frac{\omega_n^3 s^3}{9 - 12\,\omega_n^2} \\ & - \frac{2\left(1 + 14\omega_n^2 - 16\omega_n^4\right)\omega_n^5 s^5}{15\left(3 - 4\omega_n^2\right)^2\left(5 - 20\omega_n^2 + 16\omega_n^4\right)} + \mathcal{O}\left((\omega_n s)^7\right). \end{aligned}$$

In order to approximate the unit circle it is enough to take the nearest loop of the polynomial curve only. The preceding discussion, and the equations (39) and (14) establish the following useful consequence.

Corollary 7 *Let x_n and y_n be given by* (27)*. The polynomial curve segment*

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix}, \qquad t \in [-h_n, h_n], \qquad h_n := \psi_n^{-1}(\pi), \tag{45}$$

approximates the unit circle with the normal distance bounded by

$$h_n^{2n} + \mathcal{O}\left(h_n^{4n}\right) = \left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{4n}\right).$$

6.2 Hyperbolic case

It is enough to consider only the right-hand side branch $x \geq 1$ of a hyperbola, parameterized by

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix}, \qquad s \in \mathbb{R}. \tag{46}$$

From the symmetry of the best hyperbolic solution $(x_n, y_n)^T$, determined in (28), we can further assume that $s \geq 0$, which narrows the interval of interest for t to $[0, 1]$. Since $(x_n(0), y_n(0))^T = (x(0), y(0))^T = (1, 0)^T$ and $t = t(s) := t(x(s), y(s))$, it follows $t(0) = 0$. Thus only $s > 0$ needs to be considered. Suppose that n is odd. Let us show that the value $t(s)$ is uniquely determined by the equation (37) for s small enough. Since n is odd, $\varepsilon(t) = -t^{2n}$, and (37) corresponds to $\phi_n(t, s) = 0$, with ϕ_n defined as

$$\phi_n(t, s) := x_n(t) - \cosh s + \frac{t^{2n} \cosh s}{\cosh(2s) + \sqrt{\cosh^2(2s) - t^{2n}}}.$$

Recall that the coefficients of the polynomials x_n and y_n are nonnegative. Thus

$$\frac{\partial \phi_n}{\partial t}(t, s) = x_n'(t) + \frac{n t^{2n-1} \cosh s}{\sqrt{\cosh^2(2s) - t^{2n}}} > 0,$$

and at $t = 0$ we obtain the inequality

$$\phi_n(0, s) = x_n(0) - \cosh s = 1 - \cosh s < 0.$$

Consequently, for any $s > 0$ such that $\phi_n(1, s) \geq 0$, there exists precisely one t, such that $\phi_n(t, s) = 0$. Since

$$\frac{\partial \phi_n}{\partial s}(1, s) = (\sinh s - \cosh s)(2\cosh 2s - \sinh 2s) < 0,$$

$\phi_n(1, s)$ is strictly decreasing. The boundary s_n^* is determined uniquely by the equation $\phi_n(1, s_n^*) = 0$, and the regular reparameterization $t = t(s)$ is available for all $s \in [-s_n^*, s_n^*]$. So far for odd n only, but for even n a similar argument carries through, with the equation (38) replacing (37). Let us derive now the asymptotic behavior of the reparameterization $t(s)$, valid for n large enough. First of all, observe the expansion

$$\ln\left(p_h\left(t, \mathcal{I}_n^*\right)\right) = \frac{1}{\omega_n}\left(t - \frac{t^3}{9 - 12\omega_n^2} + \frac{t^5}{25 - 100\omega_n^2 + 80\omega_n^4}\right) + \mathcal{O}\left(t^7\right).$$

But then (28) yields

$$\begin{aligned} x_n(t) &= \frac{1}{2}\left(p_h\left(t, \mathcal{I}_n^*\right) + p_h\left(-t, \mathcal{I}_n^*\right)\right) \\ &= \cosh\left(\frac{1}{\omega_n}\left(t - \frac{t^3}{9 - 12\omega_n^2} + \frac{t^5}{25 - 100\omega_n^2 + 80\omega_n^4}\right)\right) + \mathcal{O}\left(t^7\right). \end{aligned}$$

We insert this expansion in the parametric form of the equation (37), and obtain

$$t(s) = \omega_n s + \frac{\omega_n^3 s^3}{9 - 12\omega_n^2} - \frac{2\left(1 + 14\omega_n^2 - 16\omega_n^4\right)\omega_n^5 s^5}{15\left(3 - 4\omega_n^2\right)^2\left(5 - 20\omega_n^2 + 16\omega_n^4\right)} + \mathcal{O}\left(\left(\omega_n s\right)^7\right).$$

This expansion reveals also an approximation of s_n^*, that satisfies $t(s_n^*) = 1$. Namely, a detailed examination of the expansion indicates

$$s_n^* = G\frac{1}{\omega_n} + \mathcal{O}\left(\omega_n\right), \tag{47}$$

where G denotes the Catalan's constant,

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.9159656.$$

Corollary 8 *Let (x_n, y_n) be given by (28). The polynomial curve segment*

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix}, \qquad t \in [t\left(\underline{s}\right), t\left(\overline{s}\right)],$$

approximates a hyperbola segment

$$\begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix}, \quad s \in [\,\underline{s}, \overline{s}\,] \subset [\,-s_n^*, s_n^*\,], \quad \phi_n(1, s_n^*) = 0,$$

with the parametric distance bounded by

$$(\omega_n h)^{2n} + \mathcal{O}\left((\omega_n h)^{4n}\right) = \left(\frac{\pi h}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi h}{2n}\right)^{4n}\right), \quad h := \max\left\{|\underline{s}|, |\overline{s}|\right\}.$$

7 Höllig-Koch's conjecture

In [11], a conjecture has been stated that a planar parametric polynomial curve of degree n can approximate a smooth regular parametric curve with the approximation order $2n$. Many authors considered this problem, but the conjecture remains unproven in general. Planar curves are studied in [14] where a particular nonlinear system is derived and the optimal approximation order is confirmed provided the system has at least one real solution. The existence of a solution is established for degree $n \leq 5$ for general curves. In [9] the results are extended to general degree n for so-called circle-like curves. Here, the Höllig-Koch's conjecture will be proved for a general degree n for all conic sections.

A general conic section is given as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Without a loss of generality, we can assume that the point $(0,0)$ lies on the conic, and that the normal at that point is $(0,1)$. This implies $f = d = 0$ and $e = 1$. In the vicinity of the point $(0,0)$, the conic

$$C(x,y) := ax^2 + bxy + cy^2 + y = 0$$

can be parameterized by the first component:

$$\begin{pmatrix} x \\ \dfrac{-(1+bx) + \sqrt{(1+b\,x)^2 - 4ac\,x^2}}{2c} \end{pmatrix} =: \begin{pmatrix} x \\ u(x) \end{pmatrix}.$$

Recall [14, Thm. 4.5]. To prove the existence of the Lagrange geometric interpolant of degree n that approximates $C(x,y)$ with the approximation order $2n$, it is enough to show that the Taylor expansion of $u(x_n(t))$ for $x_n(t) =$

$t + \sum_{i=2}^n \alpha_i t^i$ is of the form

$$u(x_n(t)) = y_n(t) + \sum_{i=2n}^{\infty} \beta_i t^i, \quad y_n(t) := \sum_{i=2}^{n} \beta_i t^i.$$

This relation is equivalent to

$$\sqrt{(1 + b\, x_n(t))^2 - 4\, a\, c\, x_n^2(t)} = 1 + b\, x_n(t) + 2\, c \left(y_n(t) + \sum_{i=2n}^{\infty} \beta_i t^i \right),$$

and after it is squared, it can be rewritten to

$$a\, x_n^2(t) + b\, x_n(t) y_n(t) + c\, y_n^2(t) + y(t) = \mathcal{O}(t^{2n}).$$

The confirmation of Höllig-Koch's conjecture for conic sections now follows from the next theorem.

Theorem 9 *For any $a, b, c \in \mathbb{R}$ there exist polynomials*

$$x_n(t) = t + \sum_{i=2}^{n} \alpha_i t^i, \quad y_n(t) = \sum_{i=2}^{n} \beta_i t^i,$$

of degree n, such that

$$C(x_n(t), y_n(t)) = (a\, \alpha_n^2 + b\, \alpha_n\, \beta_n + c\, \beta_n^2) t^{2n}. \tag{48}$$

PROOF. By a diagonalization of the matrix

$$\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} = U^T \Lambda U, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2), \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = \frac{1}{2}\mathrm{arccot}\left(\frac{a-c}{b}\right)$, and introduction of new variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix},$$

a conic is transformed into a canonical form

$$C(\tilde{x}, \tilde{y}) = \lambda_1 \left(\tilde{x} + \frac{\sin\theta}{2\lambda_1} \right)^2 + \lambda_2 \left(\tilde{y} + \frac{\cos\theta}{2\lambda_2} \right)^2 - \frac{\lambda_2\, \sin^2\theta + \lambda_1\, \cos^2\theta}{4\lambda_1\lambda_2} = 0.$$

The cases with $\lambda_1 = 0$ or $\lambda_2 = 0$ can be excluded since they correspond to a parabola or two lines. Let

$$\tilde{x}_n(t) := \sum_{i=1}^{n} \tilde{\alpha}_i t^i := \cos\theta\, x_n(t) + \sin\theta\, y_n(t),$$
$$\tilde{y}_n(t) := \sum_{i=1}^{n} \tilde{\beta}_i t^i := -\sin\theta\, x_n(t) + \cos\theta\, y_n(t).$$

The system (48) then becomes

$$\frac{\lambda_1}{D}\left(\tilde{x}_n(t) + \frac{\sin\theta}{2\lambda_1}\right)^2 + \frac{\lambda_2}{D}\left(\tilde{y}_n(t) + \frac{\cos\theta}{2\lambda_2}\right)^2 = 1 + \frac{\left(\lambda_1\tilde{\alpha}_n^2 + \lambda_2\tilde{\beta}_n^2\right)}{D} t^{2n},$$

where $D := \dfrac{\lambda_2\, \sin^2\theta + \lambda_1\, \cos^2\theta}{4\lambda_1\lambda_2}$. A reparameterization

$$t = t(s) = \sqrt[2n]{\left|\frac{D}{\lambda_1\tilde{a}_n^2 + \lambda_2\tilde{b}_n^2}\right|}\, s$$

transforms the problem into finding the solution of the system

$$\operatorname{sign}\left(\frac{\lambda_1}{D}\right) X_n^2(s) + \operatorname{sign}\left(\frac{\lambda_2}{D}\right) Y_n^2(s) = 1 \pm s^{2n},$$

where

$$X_n(s) = \sqrt{\left|\frac{\lambda_1}{D}\right|}\left(\tilde{x}_n(t(s)) + \frac{\sin\theta}{2\lambda_1}\right), \quad Y_n(s) = \sqrt{\left|\frac{\lambda_2}{D}\right|}\left(\tilde{y}_n(t(s)) + \frac{\cos\theta}{2\lambda_2}\right).$$

Since $\tilde{x}_n(0) = \tilde{y}_n(0) = 0$, the solutions must satisfy

$$X_n(0) = \left|1 + \frac{\lambda_1}{\lambda_2\tan^2\theta}\right|^{-\frac{1}{2}}, \quad Y_n(0) = \left|1 + \frac{\lambda_2\tan^2\theta}{\lambda_1}\right|^{-\frac{1}{2}}.$$

The existence now follows from Section 4 with a slight modification. Namely, for the elliptic case $p_e(t;\boldsymbol{\sigma})$ in (21) must be multiplied by $X_n(0) + \mathrm{i}\, Y_n(0)$, while in the hyperbolic case, polynomials p_h and q_h must be multiplied by $X_n(0)+Y_n(0)$ and $X_n(0)-Y_n(0)$, respectively. The best solutions are obtained as explained in Thm. 2. This completes the proof.

8 Examples

Let us conclude the paper with some examples. Consider first the approximation of the unit circle (42) and the unit hyperbola (46) with the best approximants, defined in Thm. 2.

In order to approximate the whole circle, the parameter interval for s is $[-\pi, \pi]$. For each n the corresponding interval $[-h_n, h_n]$, with h_n defined in (45), must be determined. Tab. 3 shows the values h_n for $n = 3, 4, \ldots, 15$, and the normal distance between the circle and the best approximant. Note that h_n decreases with a growing n. To compare the normal distance between the hyperbola and the best approximants of different degrees n, let us limit the parameter s to $[-\ln 10, \ln 10]$. In this case the boundary points on the hyperbola are $(5.05, \pm 4.95)$. The corresponding parameter interval $[-h, h]$, for the approximating curve is given in the fifth column of Tab. 3, and in the last column the normal distance is shown. In both cases the normal distance decreases to zero very fast with a growing n, which confirms the results of Section 6. In Fig. 3, elliptic approximants for $n = 4, 5, 6$ are shown, and the curvature profile demonstrates the shape preserving property.

Table 3
The normal distance for the polynomial approximation of the unit circle and the unit hyperbola.

	elliptic case		hyperbolic case		
n	h_n	error	s_n^*	h	error
3	1.41421	2	1.79253	1.36452	0.45670
4	1	0.41421	2.36459	0.96918	0.05504
5	0.84612	0.08999	2.94151	0.75580	0.00430
6	0.74225	0.01389	3.52038	0.62125	0.00023
7	0.65658	0.00138	4.10045	0.52817	$9.3 \cdot 10^{-6}$
8	0.58526	$9.5 \cdot 10^{-5}$	4.68126	0.45973	$2.8 \cdot 10^{-7}$
9	0.52643	$4.8 \cdot 10^{-6}$	5.26259	0.40719	$6.7 \cdot 10^{-9}$
10	0.47766	$1.9 \cdot 10^{-7}$	5.84427	0.36555	$1.3 \cdot 10^{-10}$
11	0.43680	$6.1 \cdot 10^{-9}$	6.42621	0.33169	$2.0 \cdot 10^{-12}$
12	0.40217	$1.6 \cdot 10^{-10}$	7.00835	0.30362	$2.7 \cdot 10^{-14}$
13	0.37249	$3.5 \cdot 10^{-12}$	7.59064	0.27996	$3.0 \cdot 10^{-16}$
14	0.34681	$6.6 \cdot 10^{-14}$	8.17304	0.25974	$2.9 \cdot 10^{-18}$
15	0.32438	$1.1 \cdot 10^{-15}$	8.75554	0.24225	$2.4 \cdot 10^{-20}$

In the hyperbola case the parameter interval $[-s_n^*, s_n^*]$, that corresponds to $t \in [-1, 1]$, is determined as the unique solution of $\phi_n(1, s_n^*) = 0$. The values s_n^* for $n = 3, 4, \ldots, 15$, are shown in the fourth column of Tab. 3, and they numerically confirm (47).

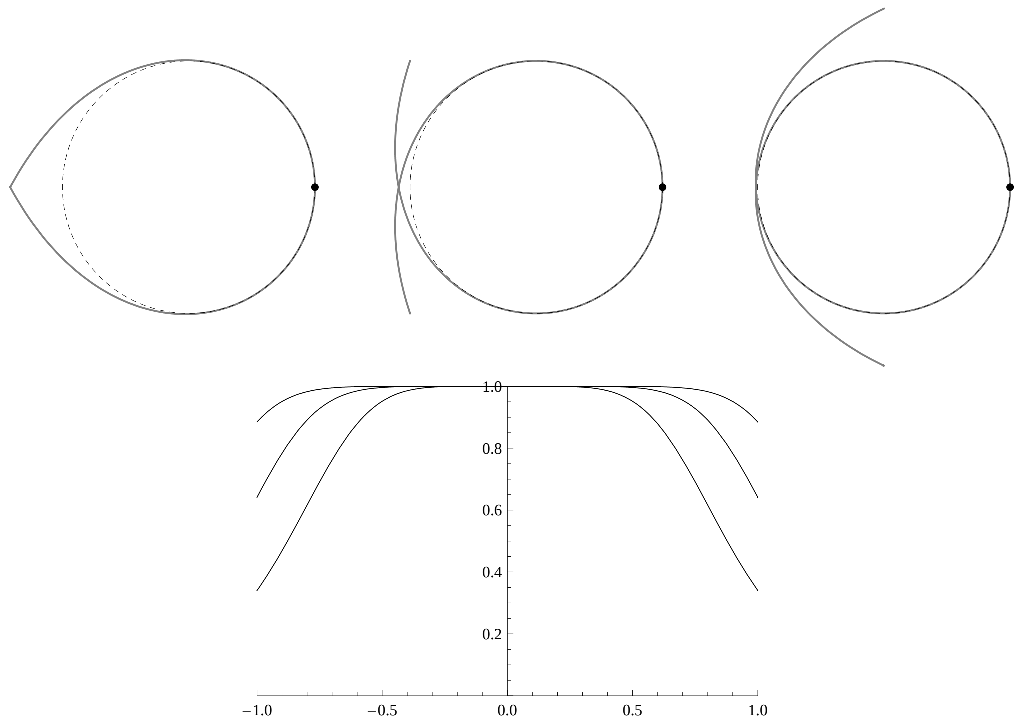

Fig. 3. Polynomial approximants of the unit circle for $n = 4, 5, 6$ (top) and their corresponding curvatures on $[-h_n, h_n]$, linearly reparameterized to $[-1, 1]$ (bottom).

In the circle case this interval is simply $\left[-\frac{n\pi}{4}, \frac{n\pi}{4}\right]$. This further implies a very interesting property of the circle approximant, shown in Fig. 4. For n large enough the approximant cycles the unit circle several times before the error becomes significantly large. The plot is obtained as a lift of the planar curve in space, i.e., $(x_{20}(t), y_{20}(t), t)$.

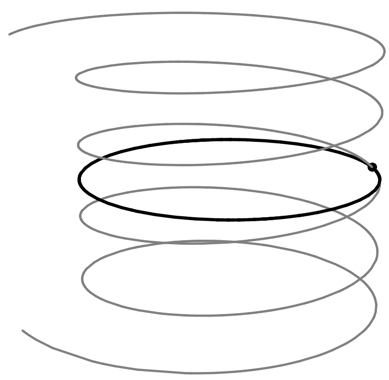

Fig. 4. Unit circle together with the cycles of the approximant for $n = 20$ and $t \in [-1, 1]$.

To conclude the paper, some examples of the approximation of conic sections, where the interpolation point is chosen arbitrarily, are shown in Fig. 5 and

Fig. 6.

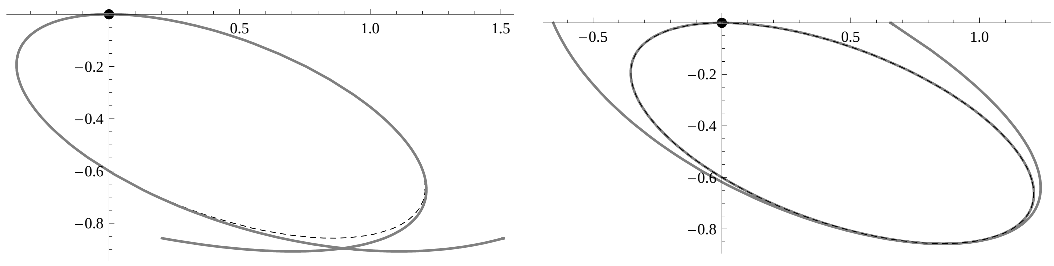

Fig. 5. Approximation of the ellipse $\frac{1}{2}x^2+xy+\frac{5}{3}y^2+y=0$ with the best approximant of degree $n=5,7$.

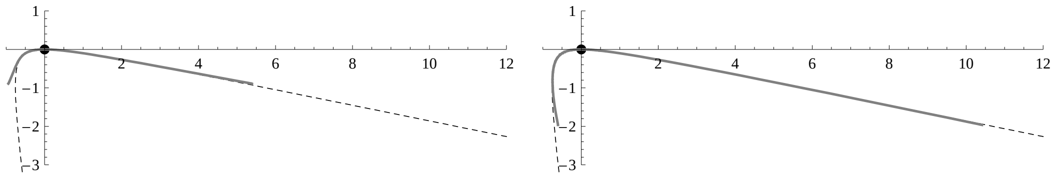

Fig. 6. Approximation of the hyperbola $\frac{1}{5}x^2+xy+\frac{1}{8}y^2+y=0$ with the best approximant of degree $n=3,4$.

References

[1] M. Floater, High-order approximation of conic sections by quadratic splines, Comput. Aided Geom. Design 12 (6) (1995) 617–637.

[2] M. S. Floater, An $O(h^{2n})$ Hermite approximation for conic sections, Comput. Aided Geom. Design 14 (2) (1997) 135–151.

[3] T. Dokken, M. Dæhlen, T. Lyche, K. Mørken, Good approximation of circles by curvature-continuous Bézier curves, Comput. Aided Geom. Design 7 (1-4) (1990) 33–41, Curves and surfaces in CAGD '89 (Oberwolfach, 1989).

[4] M. Goldapp, Approximation of circular arcs by cubic polynomials, Comput. Aided Geom. Design 8 (3) (1991) 227–238.

[5] K. Mørken, Best approximation of circle segments by quadratic Bézier curves, in: Curves and surfaces (Chamonix-Mont-Blanc, 1990), Academic Press, Boston, MA, 1991, pp. 331–336.

[6] L. Fang, Circular arc approximation by quintic polynomial curves, Comput. Aided Geom. Design 15 (8) (1998) 843–861.

[7] L. Fang, G^3 approximation of conic sections by quintic polynomial curves, Comput. Aided Geom. Design 16 (8) (1999) 755–766.

[8] T. Lyche, K. Mørken, A metric for parametric approximation, in: Curves and surfaces in geometric design (Chamonix-Mont-Blanc, 1993), A K Peters, Wellesley, MA, 1994, pp. 311–318.

[9] G. Jaklič, J. Kozak, M. Krajnc, E. Žagar, On geometric interpolation of circle-like curves, Comput. Aided Geom. Design 24 (5) (2007) 241–251.

[10] M. S. Floater, High order approximation of rational curves by polynomial curves, Comput. Aided Geom. Design 23 (8) (2006) 621–628.

[11] K. Höllig, J. Koch, Geometric Hermite interpolation with maximal order and smoothness, Comput. Aided Geom. Design 13 (8) (1996) 681–695.

[12] W. L. F. Degen, High accuracy approximation of parametric curves, in: Mathematical methods for curves and surfaces (Ulvik, 1994), Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 83–98.

[13] E. J. Barbeau, Pell's equation, Problem Books in Mathematics, Springer-Verlag, New York, 2003.

[14] G. Jaklič, J. Kozak, M. Krajnc, E. Žagar, On geometric interpolation by planar parametric polynomial curves, Math. Comp. 76 (260) (2007) 1981–1993.