Dual representation of spatial rational PH curves

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Abstract

In this paper, the dual representation of parametric curves is generalized to the spatial case, and its properties are studied. Rational curves have a polynomial dual representation, which turns out to be both theoretically and computationally appropriate to tackle the main goal of the paper: spatial rational Pythagorean-hodograph curves (PH curves). The dual representation of a rational PH curve is generated here by a quaternion polynomial which defines the Euler-Rodrigues frame of a curve. Conditions that imposed on this polynomial assure low degree dual form representation are considered in detail. In particular, a linear quaternion polynomial leads to cubic or reparameterized cubic polynomial PH curves. A quadratic quaternion polynomial generates a wider class of rational PH curves, and perhaps the most useful is the ten-parameter family of cubic rational PH curves, determined here in the closed form.

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1. Introduction

Polynomial Pythagorean-hodograph curves are characterized by the property that the Euclidean norm of their hodograph is a polynomial, not a square root of a polynomial. These curves thus have a rational unit vector field of tangents, rational offset curves, and a polynomial arc length what makes them an important practical

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tool that finds its applications in robotics, in CAD/CAM systems, in animations, etc. Polynomial PH curves were introduced in [1] and have widely been studied since then (see [2] and the references therein). The usual approach to obtain a polynomial PH curve is to integrate the appropriate hodograph constructed with the help of the complex or the quaternion polynomials in the plane or in the space case respectively (see e.g. [2], [3], [4]).

But the natural extension of the PH property to rational curves turned out to be quite a task, and only few results are known in this direction. The main obstacle is the fact that the polynomial preimage approach can not be applied here since the integral of a rational curve is not a rational curve in general. Planar rational PH curves were derived in [5], and independently in [6]. The suggested construction determines a planar rational curve as the envelope of a one-parameter family of tangent lines, given by a rational unit vector field of tangents, and a rational support function which defines the distance of the tangent line from the origin. As observed in [5], the introduced dual form of a planar rational PH curve turns out to be more appropriate from the computational point of view than the corresponding point representation. Interpolation schemes involving planar PH curves can be found in [7].

The step to spatial rational PH curves has been carried out in [8] just recently. The construction of rational space PH curves is presented, and it is further justified and illuminated from several equivalent geometric viewpoints. Basically, the approach originates from the implicit curve representation involving the curve binormal direction and a rational function that determines the signed distance of the osculating plane from the origin. In order to assure the PH property of the curve, its binormal directions are generated from a rational vector field of unit length tangents.

In this paper, a further insight into rational spatial PH curves is provided, with emphasis to their construction in a form that could be used in practical applications. First of all, since the construction in [8] leads to rational PH curves of a high degree in general, we borrow the dual form of a planar rational curve from [5]. We show that this curve representation can be naturally extended to the space case, actually to any dimension setup. The de Casteljau algorithm that evaluates a curve point in the dual representation turns out simple and efficient too. In the PH case, the dual form enables one to deal in general with polynomials of a significantly lower degree as in the curve closed form point representation. The exception is the cubic case where degrees of both representations are equal. Equipped with the dual approach, we obtain the dual PH curve form the Euler-Rodrigues frame in a similar way as already in [8]. The E-R frame is generated by quaternion polynomials, and we focus on the question, how one should choose superfluous parameters to assure that the corresponding dual form would be of a low degree. Only some, but significant answers are given here. A linear quaternion polynomial gives rise to cubic or reparameterized cubic polynomial PH curves. Based on a quadratic quaternion polynomial cubic rational PH curves with nonconstant denominator can be constructed. A closed form, depending on ten free parameters, is presented. This comes somewhat as a surprise since there are no additional free parameters in comparison to the cubic polynomial case.

The paper is organized as follows. In the next section the dual form of a parametric space curve is introduced and its properties with the emphasis on rational curves analysed. Section 3 introduces rational PH curves with the dual form based upon E-R frame. The degree of the dual form of a rational PH curve and its reduction is considered in Section 4. In the next section curves that arise from a linear quaternion polynomial are treated. Cubic rational PH curves are derived in Section 6 together with a few examples. In the end, we discuss possible future work directions.

2. Spatial curves in dual form

In [8], the key step to the construction of rational PH curves was a nice implicit representation of a parametric curve. Since the approach works in the planar case too, and it could be extended to more than three dimensions if needed, we briefly recall it. Let $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ be a smooth parametric curve such that the derivatives $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are linearly independent on the parameter interval $[\alpha, \beta]$. Then the corresponding Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is well defined for each $t \in [\alpha, \beta]$. Here, the vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ denote the unit tangent, the principal normal and the unit binormal respectively. Further, a point $\boldsymbol{r}(t)$ can be uniquely recovered as the intersection of the osculating, the rectifying, and the normal plane at a particular parameter value $t \in [\alpha, \beta]$. This gives $\boldsymbol{r}$ as a set of points $\boldsymbol{p} \in \mathbb{R}^3$ that satisfy the linear system

$$\boldsymbol{b}(t)\cdot(\boldsymbol{p}-\boldsymbol{r}(t)) = 0,\ \boldsymbol{n}(t)\cdot(\boldsymbol{p}-\boldsymbol{r}(t)) = 0,\ \boldsymbol{t}(t)\cdot(\boldsymbol{p}-\boldsymbol{r}(t)) = 0, \quad t \in [\alpha, \beta]. \tag{1}$$

If the torsion τ of the curve $\boldsymbol{r}$ does not vanish on $[\alpha, \beta]$, as observed in [8], the one-parametric family of linear systems (1) is by Frenet-Serret formulas equivalent to

$$\boldsymbol{b}^{(\ell)}(t) \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})^{(\ell)}(t) = 0, \quad \ell = 0, 1, 2, \quad t \in [\alpha, \beta]. \tag{2}$$

These systems can further be simplified by any nonzero function $\phi \in C^2([\alpha, \beta])$ to

$$\boldsymbol{u}^{(\ell)}(t) \cdot \boldsymbol{p} - f^{(\ell)}(t) = 0, \quad \ell = 0, 1, 2, \quad t \in [\alpha, \beta], \tag{3}$$

where

$$\boldsymbol{u} := \phi\, \boldsymbol{b}, \quad f := \phi\, \boldsymbol{b} \cdot \boldsymbol{r}.$$

Namely, (2) and (3) are equivalent since by the Leibniz rule one has

$$\begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{p} - f \\ \boldsymbol{u}' \cdot \boldsymbol{p} - f' \\ \boldsymbol{u}'' \cdot \boldsymbol{p} - f'' \end{pmatrix} = \begin{pmatrix} \phi & 0 & 0 \\ \phi' & \phi & 0 \\ \phi'' & 2\phi' & \phi \end{pmatrix} \begin{pmatrix} \boldsymbol{b} \cdot \boldsymbol{p} - \boldsymbol{b} \cdot \boldsymbol{r} \\ \boldsymbol{b}' \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})' \\ \boldsymbol{b}'' \cdot \boldsymbol{p} - (\boldsymbol{b} \cdot \boldsymbol{r})'' \end{pmatrix}.$$

Note that $\frac{f}{\phi}$ denotes the signed distance of the osculating plane with the normal vector $\boldsymbol{u}$ from the origin. If $\boldsymbol{u}$ and f are given, and $\det(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}'') \neq 0$, the curve $\boldsymbol{r}$ may be determined from (3). Rather then using the closed form solution ([8]) that gives the point representation

$$\boldsymbol{r} = \frac{1}{\det(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}'')} \left(f\, \boldsymbol{u}' \times \boldsymbol{u}'' + f'\, \boldsymbol{u}'' \times \boldsymbol{u} + f''\, \boldsymbol{u} \times \boldsymbol{u}' \right),$$

where $\times$ denotes the cross product, we proceed to the construction of curves in the implicit form (3). The coefficients of the first equation in (3) where $\ell = 0$ that pin down the family of the osculating planes, determine the corresponding curve uniquely since the second and the third equation of the system (3) follow from the first one. Following [5] we call them *dual coordinates* of the parametric curve. In order to shorten the notation we introduce an imbedding shortcut $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^4$ that will be used throughout the paper,

$$(u_0; \boldsymbol{u}) := \left(u_0; (u_1, u_2, u_3)^T \right) := (u_0, u_1, u_2, u_3)^T .$$

With $\boldsymbol{u} = (u_1, u_2, u_3)^T$ we order the dual coordinates in the curve *dual form* $\boldsymbol{L}$

$$\boldsymbol{L} := (-f; \boldsymbol{u}) = (-f, u_1, u_2, u_3)^T .$$

Quite clearly, the dual form $\boldsymbol{L}$ is homogeneous. It determines the same curve if multiplied by any smooth nonzero function. More generally, let $\sim$ denote this type of equivalence between vector fields. Two vector fields $\boldsymbol{Q}_i : [\alpha, \beta] \to \mathbb{R}^d, i = 1, 2,$ are equivalent, $\boldsymbol{Q}_1 \sim \boldsymbol{Q}_2$, iff $\boldsymbol{Q}_1 = \zeta \boldsymbol{Q}_2$ for some smooth function ζ that does not vanish on this interval. Since the dual form is homogeneous, it is convenient to rewrite the original curve $\boldsymbol{r}$ in a homogeneous form too,

$$\boldsymbol{P} := (P_0, P_1, P_2, P_3)^T \sim (1; \boldsymbol{r}), \quad \boldsymbol{r} = \frac{1}{P_0} (P_1, P_2, P_3)^T, \quad P_0 \neq 0. \tag{4}$$

Relations between $\boldsymbol{P}$ and $\boldsymbol{L}$ could be written in a very compact form. Let $\cdot \wedge \cdot \wedge \cdot$ denote a particular wedge product, defined as

$$\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3 := \left((-1)^i \det V^{[i]}\right)_{i=1}^{4}, \quad \boldsymbol{v}_j \in \mathbb{R}^4,$$

where $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \in \mathbb{R}^{4\times 3}$ and $V^{[i]} \in \mathbb{R}^{3\times 3}$ is a submatrix of V with i-th row of the original matrix omitted.

Theorem 1. *Suppose that $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ is a smooth parametric curve such that $\boldsymbol{r}'$, $\boldsymbol{r}''$ are linearly independent and the corresponding torsion τ doesn't vanish on the domain interval. Let $\boldsymbol{P}$ be the homogeneous curve representation and let $\boldsymbol{L}$ be its dual form. Then, on $[\alpha, \beta]$,*

$$\boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P}^{(r)} = 0,\ 0 \le \ell + r \le 2, \quad \boldsymbol{L}^{(r)} \cdot \boldsymbol{P}^{(3-r)} + \boldsymbol{L}^{(3-r)} \cdot \boldsymbol{P}^{(r)} = 0,\ r = 0, 1, \tag{5}$$

and

$$\boldsymbol{P} \sim \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'', \quad \boldsymbol{L} \sim \boldsymbol{P} \wedge \boldsymbol{P}' \wedge \boldsymbol{P}''. \tag{6}$$

Also, any regular reparameterization $\varphi : [\alpha, \beta] \to [\gamma, \delta]$ of one form reparametrises the other one also,

$$\boldsymbol{P} \circ \varphi \sim (\boldsymbol{L} \circ \varphi) \wedge (\boldsymbol{L} \circ \varphi)' \wedge (\boldsymbol{L} \circ \varphi)'' ,\ \boldsymbol{L} \circ \varphi \sim (\boldsymbol{P} \circ \varphi) \wedge (\boldsymbol{P} \circ \varphi)' \wedge (\boldsymbol{P} \circ \varphi)'' . \tag{7}$$

PROOF. The system (3) in the homogeneous form reads $\boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P} = 0,\ \ell = 0, 1, 2.$ If we differentiate these equations consecutively, we obtain

$$0 = \left(\boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P}^{(r)}\right)' = \boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P}^{(r+1)}, \quad \ell = 0, \ldots, 1 - r;\ r = 0, 1,$$
$$0 = \left(\boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P}^{(2-\ell)}\right)' = \boldsymbol{L}^{(\ell+1)} \cdot \boldsymbol{P}^{(2-\ell)} + \boldsymbol{L}^{(\ell)} \cdot \boldsymbol{P}^{(3-\ell)}, \quad \ell = 0, 1, 2,$$

what implies (5). By the assumption on the derivatives of $\boldsymbol{r}$ and τ vectors $\boldsymbol{L}, \boldsymbol{L}'$, and $\boldsymbol{L}''$ span a three dimensional subspace of $\mathbb{R}^4$ for any $t \in [\alpha, \beta]$. But $\boldsymbol{P}$ is by (5) orthogonal to any of them, so it should be proportional to the wedge product $\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}''$. The first assertion in (6) is proved. The second one follows similarly. To prove (7) observe $(\boldsymbol{L} \circ \varphi) \wedge (\boldsymbol{L} \circ \varphi)' \wedge (\boldsymbol{L} \circ \varphi)'' = \varphi'^3 \, (\boldsymbol{P} \circ \varphi)\,.$ $\square$

The relations (6) emphasize the symmetry among the homogeneous form of a curve and the dual one. The form $\boldsymbol{L}$ is dual to $\boldsymbol{P}$, and *vice versa*. So dual to dual of $\boldsymbol{P}$ or $\boldsymbol{L}$ should be equivalent to the original form. The following corollary states this fact precisely.

Corollary 2. *Let $\boldsymbol{Q}_1 := \boldsymbol{P}$, $\boldsymbol{Q}_2 := \boldsymbol{L}$ be the homogenuous and the dual representation of a smooth parametric curve $\boldsymbol{r} : [\alpha, \beta] \to \mathbb{R}^3$ that satisfies the assumptions of Theorem 1. Let $i \in \{1, 2\}$ be fixed. If the representation $\boldsymbol{Q}_{3-i}$ is determined by*

$$\boldsymbol{Q}_{3-i} = \boldsymbol{Q}_i \wedge \boldsymbol{Q}_i' \wedge \boldsymbol{Q}_i'', \tag{8}$$

then

$$\boldsymbol{Q}_{3-i} \wedge \boldsymbol{Q}_{3-i}' \wedge \boldsymbol{Q}_{3-i}'' = \det\left(\boldsymbol{Q}_i, \boldsymbol{Q}_i', \boldsymbol{Q}_i'', \boldsymbol{Q}_i^{(3)}\right)^2 \boldsymbol{Q}_i. \tag{9}$$

PROOF. Since by Theorem 1, $\boldsymbol{Q}_i \sim \boldsymbol{Q}_{3-i} \wedge \boldsymbol{Q}_{3-i}' \wedge \boldsymbol{Q}_{3-i}''$, there exists a smooth function $\zeta \neq 0$ such that $\zeta \boldsymbol{Q}_i = \boldsymbol{Q}_{3-i} \wedge \boldsymbol{Q}_{3-i}' \wedge \boldsymbol{Q}_{3-i}''$. If we apply the scalar product $\cdot\, \boldsymbol{Q}_{3-i}^{(3)}$ on both sides of the equation, we obtain, with the help of (5),

$$\zeta = -\frac{\det\left(\boldsymbol{Q}_{3-i}, \boldsymbol{Q}_{3-i}', \boldsymbol{Q}_{3-i}'', \boldsymbol{Q}_{3-i}^{(3)}\right)}{\boldsymbol{Q}_i^{(3)} \cdot \boldsymbol{Q}_{3-i}} = \frac{\det\left(\boldsymbol{Q}_{3-i}, \boldsymbol{Q}_{3-i}', \boldsymbol{Q}_{3-i}'', \boldsymbol{Q}_{3-i}^{(3)}\right)}{\det\left(\boldsymbol{Q}_i, \boldsymbol{Q}_i', \boldsymbol{Q}_i'', \boldsymbol{Q}_i^{(3)}\right)}. \tag{10}$$

The relations (5) give also

$$\begin{aligned}
&\left(\boldsymbol{Q}_{3-i}, \boldsymbol{Q}_{3-i}', \boldsymbol{Q}_{3-i}'', \boldsymbol{Q}_{3-i}^{(3)}\right)^T \left(\boldsymbol{Q}_i, \boldsymbol{Q}_i', \boldsymbol{Q}_i'', \boldsymbol{Q}_i^{(3)}\right) \\
&\quad = \begin{pmatrix}
0 & 0 & 0 & \boldsymbol{Q}_{3-i} \cdot \boldsymbol{Q}_i^{(3)} \\
0 & 0 & \boldsymbol{Q}_{3-i}' \cdot \boldsymbol{Q}_i'' & \boldsymbol{Q}_{3-i}' \cdot \boldsymbol{Q}_i^{(3)} \\
0 & -\boldsymbol{Q}_{3-i}' \cdot \boldsymbol{Q}_i'' & \boldsymbol{Q}_{3-i}'' \cdot \boldsymbol{Q}_i'' & \boldsymbol{Q}_{3-i}'' \cdot \boldsymbol{Q}_i^{(3)} \\
-\boldsymbol{Q}_{3-i} \cdot \boldsymbol{Q}_i^{(3)} & \boldsymbol{Q}_{3-i}^{(3)} \cdot \boldsymbol{Q}_i' & \boldsymbol{Q}_{3-i}^{(3)} \cdot \boldsymbol{Q}_i'' & \boldsymbol{Q}_{3-i}^{(3)} \cdot \boldsymbol{Q}_i^{(3)}
\end{pmatrix}.
\end{aligned}$$

From here and (5) we obtain

$$\begin{aligned}
&\det\left(\boldsymbol{Q}_{3-i}, \boldsymbol{Q}_{3-i}', \boldsymbol{Q}_{3-i}'', \boldsymbol{Q}_{3-i}^{(3)}\right) \det\left(\boldsymbol{Q}_i, \boldsymbol{Q}_i', \boldsymbol{Q}_i'', \boldsymbol{Q}_i^{(3)}\right) \\
&\quad = \left(\boldsymbol{Q}_{3-i} \cdot \boldsymbol{Q}_i^{(3)}\right)^2 \left(\boldsymbol{Q}_{3-i}' \cdot \boldsymbol{Q}_i''\right)^2 = \det\left(\boldsymbol{Q}_i, \boldsymbol{Q}_i', \boldsymbol{Q}_i'', \boldsymbol{Q}_i^{(3)}\right)^4,
\end{aligned}$$

what together with (10) concludes the proof. □

Suppose now that the curve $\boldsymbol{r}$ that satisfies the assumptions of Theorem 1 is a rational one,

$$\boldsymbol{r} = \frac{1}{q}\boldsymbol{p} = \frac{1}{q}\left(p_1, p_2, p_3\right)^T,$$

The polynomials q, p_1, p_2, p_3 are assumed to be relatively prime, and the degree of the curve is the highest degree of the polynomials involved. Quite clearly, the degree should be at least 3 since otherwise the torsion of $\boldsymbol{r}$ would vanish. The homogeneous representation of a rational curve is naturally a polynomial one, $\boldsymbol{P} = (q; \boldsymbol{p})$, and the degree of $\boldsymbol{P}$ equals to the degree of $\boldsymbol{r}$. For the dual representation we take (8) with $i = 0$, $\boldsymbol{L} = \boldsymbol{P} \wedge \boldsymbol{P}' \wedge \boldsymbol{P}''$, what by (9) implies

$$\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'' = \det\Big(\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}'', \boldsymbol{P}^{(3)}\Big)^2 \boldsymbol{P}. \tag{11}$$

If the components of $\boldsymbol{L}$ are not relatively prime, we may divide $\boldsymbol{L}$ by the greatest common polynomial divisor to obtain relatively prime components. This simplifies the polynomial dual representation as much as possible. Except for the cubic case, the degrees of $\boldsymbol{P}$ and $\boldsymbol{L}$ will be different in general. The following definition makes a distinction between the degree of the curve and the degree of its dual form.

Definition 1. *Rational curves with relatively prime polynomial dual form $\boldsymbol{L}$ of degree m are called* class m curves.

Let us reveal the relation between degrees of the curve representations. But first, we need the following theorem.

Theorem 3. *Let $\boldsymbol{p} : \mathbb{R} \to \mathbb{R}^d$ be a polynomial vector field of the degree n, and let $0 \leq r < d$, $r \leq n$. Any $(r+1) \times (r+1)$ minor of the matrix $P = \big(\boldsymbol{p}^{(j)}\big)_{j=0}^{r}$ is a polynomial of the degree $\leq (r+1)(n-r)$.*

PROOF. Polynomial vector field can be expressed as $\boldsymbol{p} = A\,(t^i)_{i=0}^{n}$ where $A \in \mathbb{R}^{d\times(n+1)}$ is a matrix of its coefficients. Then $P = AM$, where the matrix that doesn't depend on the polynomial field coefficients reads $M = \big((t^i)^{(j)}\big)_{i=0;j=0}^{n;r}$. So it is enough to consider the minors of M only. For this purpose, let us introduce polynomials

$$\pi_j(t) := \frac{(n-r)!}{(n-j)!} \prod_{k=0}^{r-j-1} (n-j-k-t), \quad j = 0, 1, \ldots, r. \tag{12}$$

Since the polynomial π_j is of the degree $r-j$, it can be written in the Newton form as

$$\pi_j(t) = \sum_{k=0}^{r-j} [0, 1, \ldots, k]\pi_j \prod_{\ell=0}^{k-1} (t-\ell).$$

With divided differences, involved in this polynomial representations, we build a lower triangular matrix

$$C = (c_{ij})_{i=0,j=0}^{r,r} \in \mathbb{R}^{r+1,r+1}, \quad c_{ij} := \begin{cases} t^{i-j}\,[0,1,\ldots,i-j]\pi_j, & i \geq j, \\ 0, & i < j. \end{cases}$$

Since $\det C = \prod_{j=0}^{r} c_{jj} = \prod_{j=0}^{r} \pi_j(0) = 1$, the minors of M are equal to the minors of the product $MC = (\widetilde{m}_{i,j})_{i=0,j=0}^{n,r}$. Note that $\widetilde{m}_{ij} = 0$ for $i < j$, and

$$\begin{aligned} \frac{(i-j)!}{i!} t^{j-i} \widetilde{m}_{ij} &= \sum_{k=j}^{\min\{i,r\}} \left(\frac{(i-j)!}{i!} t^{j-i} \right) \left(\frac{i!}{(i-k)!} t^{i-k} \right) \left(t^{k-j} [0,1,\ldots,k-j]\pi_j \right) \\ &= \sum_{k=j}^{\min\{i,r\}} \frac{(i-j)!}{(i-k)!} [0,1,\ldots,k-j]\pi_j = \pi_j(i-j), \quad i \geq j. \end{aligned}$$

From (12) it follows now that

$$\widetilde{m}_{ij} = 0, \quad i = n-r+j+1, n-r+j+2, \ldots, n, \; j = 0,1,\ldots,r-1.$$

All the nonzero elements of $\widetilde{M}$ are thus of the degree $\leq n-r$ and the conclusion follows. $\square$

Let $n = \deg \boldsymbol{P}$, $m = \deg \boldsymbol{L}$ and $\boldsymbol{L} = \boldsymbol{P} \wedge \boldsymbol{P}' \wedge \boldsymbol{P}''$. Theorem 3, applied to $\boldsymbol{L}$ shows that $m \leq 3(n-2)$. Further, it reveals $\deg \left(\det \left(\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}'', \boldsymbol{P}^{(3)} \right) \right) \leq 4(n-3)$, and if the equality is reached, then (11) implies $m = 3(n-2)$ since

$$3(m-2) \geq \deg \left(\boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}'' \right) = 2(4(n-3)) + n = 3(3(n-2)-2).$$

Similarly, if $\widetilde{m} = \deg \boldsymbol{L}$, and $\widetilde{n} = \deg \boldsymbol{P}$, $\boldsymbol{P} = \boldsymbol{L} \wedge \boldsymbol{L}' \wedge \boldsymbol{L}''$ the bound reads $\widetilde{n} \leq 3(\widetilde{m}-2)$ where the equality holds if $\deg \left(\det \left(\boldsymbol{L}, \boldsymbol{L}', \boldsymbol{L}'', \boldsymbol{L}^{(3)} \right) \right) = 4(\widetilde{m}-3)$. This observation simply says that a switch from a known dual representation $\boldsymbol{L}$ to a closed form representation $\boldsymbol{P}$ might significantly increase complexity of the curve representation. If one is forced to start with the dual representation as it is the case with PH curves, low class curves will be the first to consider. However, point evaluation is a basic task one encounters, but it can be done efficiently from the dual form too. The wedge product requires $\mathcal{O}(1)$ operations only, and the values $\boldsymbol{L}(t)$, $\boldsymbol{L}'(t)$ and $\boldsymbol{L}''(t)$ can be computed simultaneously. Let us demonstrate this in the case the dual representation is expressed in the Bézier form.

Suppose that $\boldsymbol{L}$ is a dual representation of the *class* m curve given in the Bézier form $\boldsymbol{L}(t) = \sum_{i=0}^{m} \boldsymbol{L}_i B_{m,i}(t)$, where $B_{m,i}(t) := \binom{m}{i} t^i (1-t)^{m-i}$ are the Bernstein basis polynomials, and $\boldsymbol{L}_i$ the corresponding control points of the dual representation. The de Casteljau algorithm evaluates the dual form as

$$\begin{aligned} \boldsymbol{L}_i^0(t) &:= \boldsymbol{L}_i, \quad i = 0,1,\ldots,m, \\ \boldsymbol{L}_i^r(t) &= (1-t)\boldsymbol{L}_i^{r-1}(t) + t\boldsymbol{L}_{i+1}^{r-1}(t),\ i = 0,1,\ldots,m-r,\ r = 1,2,\ldots,m. \end{aligned} \tag{13}$$

It seems that the last three columns $r = m-2, m-1, m$ of the triangular array (13) are needed to compute the required three values $\boldsymbol{L}(t)$, $\boldsymbol{L}'(t)$ and $\boldsymbol{L}''(t)$. But the following observation shortens the algorithm.

Theorem 4. *Let $\boldsymbol{L}$ be the dual representation of a* class m *curve, and let $\boldsymbol{P}$ be the homogeneous curve representation. Then*

$$\boldsymbol{P}(t) \sim \boldsymbol{L}_0^{m-2}(t) \wedge \boldsymbol{L}_1^{m-2}(t) \wedge \boldsymbol{L}_2^{m-2}(t), \tag{14}$$

with $\boldsymbol{L}_i^r(t)$ determined by the de Casteljau algorithm (13).

PROOF. Note that

$$\begin{aligned} \boldsymbol{L}(t) &= \boldsymbol{L}_0^m(t) = (1-t)^2\boldsymbol{L}_0^{m-2}(t) + 2(1-t)t\boldsymbol{L}_1^{m-2}(t) + t^2\boldsymbol{L}_2^{m-2}(t), \\ \boldsymbol{L}'(t) &= m\left(\boldsymbol{L}_1^{m-1}(t) - \boldsymbol{L}_0^{m-1}(t)\right) \\ &= m\left((t-1)\boldsymbol{L}_0^{m-2}(t) + (1-2t)\boldsymbol{L}_1^{m-2}(t) + t\boldsymbol{L}_2^{m-2}(t)\right), \\ \boldsymbol{L}''(t) &= m(m-1)\left(\boldsymbol{L}_2^{m-2}(t) - 2\boldsymbol{L}_1^{m-2}(t) + \boldsymbol{L}_0^{m-2}(t)\right). \end{aligned}$$

Since the wedge product is linear in all operands involved, we obtain from (6)

$$\boldsymbol{P}(t) \sim \boldsymbol{L}(t) \wedge \boldsymbol{L}'(t) \wedge \boldsymbol{L}''(t) = m^2(m-1)\left(\boldsymbol{L}_0^{m-2}(t) \wedge \boldsymbol{L}_1^{m-2}(t) \wedge \boldsymbol{L}_2^{m-2}(t)\right).$$

□

The de Casteljau algorithm used on dual coordinates has a nice geometric interpretation. Every coefficient in the array defines the plane in homogeneous coordinates. Every linear interpolation (13) gives a new plane that passes through the intersection line of two given planes. The last coefficient $\boldsymbol{L}_0^m(t)$ defines the osculating plane of the spatial curve at parameter t. Intersection of $\boldsymbol{L}_0^{m-1}(t)$ and $\boldsymbol{L}_1^{m-1}(t)$ corresponds to the tangent line of the curve at parameter t. Finally, the intersection of $\boldsymbol{L}_0^{m-2}(t)$, $\boldsymbol{L}_1^{m-2}(t)$ and $\boldsymbol{L}_2^{m-2}(t)$ gives the homogeneous coordinates (14) of the point on the curve.

3. Rational PH curves in dual form

Let us turn our attention to rational PH curves. The PH conditions are satisfied if one starts with a rational unit length tangent $\boldsymbol{t} = \frac{1}{\rho}\boldsymbol{h}$, where $\boldsymbol{h}$ is a polynomial curve, and $\rho = \|\boldsymbol{h}\|$ is its polynomial norm. This gives the hodograph $\boldsymbol{r}' = \psi\boldsymbol{h}$, where ψ is any rational function such that

$$\boldsymbol{r} = \int \boldsymbol{r}' = \int \psi\boldsymbol{h} =: \frac{1}{q}\boldsymbol{p} \tag{15}$$

is a rational curve too. Based upon a homogeneous curve representation $\boldsymbol{P} = (q;\boldsymbol{p})$ we compute the polynomial dual form $\boldsymbol{L} = (-f;\boldsymbol{u})$ as

$$\begin{aligned}\boldsymbol{P}\wedge\boldsymbol{P}'\wedge\boldsymbol{P}'' = \boldsymbol{P}\wedge q\left(\frac{\boldsymbol{P}}{q}\right)'\wedge q\left(\frac{\boldsymbol{P}}{q}\right)'' = q^2\psi^2\boldsymbol{P}\wedge(0;\boldsymbol{h})\wedge(0;\boldsymbol{h}') \sim\\ \sim \boldsymbol{P}\wedge(0;\boldsymbol{h})\wedge(0;\boldsymbol{h}') = \boldsymbol{L}.\end{aligned}$$

From (15) it follows that $\boldsymbol{p} = q\int\psi\boldsymbol{h}$, and the wedge product evaluates components of $\boldsymbol{L}$ to

$$f = q\det\left(\int\psi\boldsymbol{h},\boldsymbol{h},\boldsymbol{h}'\right), \quad \boldsymbol{u} = q\boldsymbol{h}\times\boldsymbol{h}'. \tag{16}$$

Quite clearly, choosing ψ in (15) as a polynomial is only one of the possible choices which implies $q = 1$ and leads to polynomial PH curves. More generally, the curve $\boldsymbol{r}$ is reproduced from its dual form (16) where polynomials f and q are in a particular relation that involves also a rational function ψ. But, the possible candidates for ψ are hard to be determined, especially if $\boldsymbol{h}$ depends on some unknown coefficients. More or less, all reduces to a question, when the integral of a rational is rational too. Thus we tear apart the relation between the polynomials f and q, and we consider them as independent. To emphasize this, we rename q to g. Of course, a curve obtained from such a dual form will still be a rational PH curve, but its denominator would be equal to g only for particular pairs f and g that would produce a common factor in the homogeneous curve representation such as one determined in (11). Theorem 5 summarizes the discussion and reveals the connection between a chosen dual form and the rational function ψ.

Theorem 5. *Suppose that $\boldsymbol{h}:\mathbb{R}\to\mathbb{R}^3$ is a polynomial curve such that $\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}''$ are linearly independent, and its norm $\|\boldsymbol{h}\|$ is polynomial too. Let $\boldsymbol{v} := \boldsymbol{h}\times\boldsymbol{h}'$, and let f, g be relatively prime polynomials. The dual form*

$$\boldsymbol{L} = (-f;g\,\boldsymbol{v}) \tag{17}$$

determines a rational PH curve $\boldsymbol{r}$ with the denominator

$$g^3\,\det(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}'') = g^3\det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')^2 \tag{18}$$

and the hodograph $\boldsymbol{r}' = \psi\boldsymbol{h}$, where

$$\psi = \left(\frac{f}{g}\frac{\det(\boldsymbol{h},\boldsymbol{h}'',\boldsymbol{h}''')}{\det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')^2} + \left(\left(\frac{f}{g}\right)'\frac{1}{\det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')}\right)'\right)' + \frac{f}{g}\frac{\det(\boldsymbol{h}',\boldsymbol{h}'',\boldsymbol{h}''')}{\det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')^2}. \tag{19}$$

Moreover, if g is constant and the polynomial f is chosen as

$$f = g\det\left(\int\psi\boldsymbol{h},\boldsymbol{h},\boldsymbol{h}'\right), \tag{20}$$

for some arbitrary polynomial ψ, then the curve reduces to a polynomial one.

PROOF. The reduction of a rational PH curve to a polynomial one and formula (20) follow from the previous discussion. Using a cross product identity

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c}) = \det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\boldsymbol{a}, \quad \boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\mathbb{R}^3,$$

one obtains

$$\boldsymbol{v}\times\boldsymbol{v}' = (\boldsymbol{h}\times\boldsymbol{h}')\times(\boldsymbol{h}\times\boldsymbol{h}'') = \det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')\,\boldsymbol{h}. \tag{21}$$

Furthermore,

$$\det(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}'') = (\boldsymbol{v}\times\boldsymbol{v}')\cdot(\boldsymbol{h}'\times\boldsymbol{h}''+\boldsymbol{h}\times\boldsymbol{h}''') = \det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')^2 \tag{22}$$

and

$$\det\big(g\boldsymbol{v},(g\boldsymbol{v})',(g\boldsymbol{v})''\big) = g^3\det(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}'') = g^3\det(\boldsymbol{h},\boldsymbol{h}',\boldsymbol{h}'')^2,$$

which confirms (18). In order to prove the hodograph assertion and (19), let $\boldsymbol{L}$ be the curve dual form (17), and let $\widetilde{\boldsymbol{L}} := \frac{1}{g}\boldsymbol{L} = \left(-\frac{f}{g};\boldsymbol{h}\times\boldsymbol{h}'\right)$. Then equivalent homogeneous representations of the rational curve $\boldsymbol{r}$ are derived as

$$\boldsymbol{L}\wedge\boldsymbol{L}'\wedge\boldsymbol{L}'' \sim \widetilde{\boldsymbol{L}}\wedge\widetilde{\boldsymbol{L}}'\wedge\widetilde{\boldsymbol{L}}'' =: \boldsymbol{R} =: (R_0,R_1,R_2,R_3)^T \sim \frac{1}{R_0}\boldsymbol{R} =: \widetilde{\boldsymbol{R}} = (1;\boldsymbol{r}),$$

and the derivative $\widetilde{\boldsymbol{R}}'$ reads

$$\widetilde{\boldsymbol{R}}' = (0;\boldsymbol{r}') = \frac{1}{R_0}\boldsymbol{R}' - \frac{R_0'}{R_0^2}\boldsymbol{R} = \widetilde{\boldsymbol{L}}\wedge\widetilde{\boldsymbol{L}}'\wedge\left(\frac{1}{R_0}\widetilde{\boldsymbol{L}}''\right)',$$

where $R_0 = \det(\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{h}'')^2$ by (22). Further, we observe from (5)

$$\widetilde{\boldsymbol{R}}' \cdot \boldsymbol{L} = 0 = \boldsymbol{r}' \cdot (\boldsymbol{h} \times \boldsymbol{h}') \,, \quad \widetilde{\boldsymbol{R}}' \cdot \boldsymbol{L}' = 0 = \boldsymbol{r}' \cdot (\boldsymbol{h} \times \boldsymbol{h}')' = \boldsymbol{r}' \cdot (\boldsymbol{h} \times \boldsymbol{h}'') \,.$$

This shows that $\boldsymbol{r}'$ is orthogonal to $\boldsymbol{v}$ and $\boldsymbol{v}'$. By (21) we conclude that $\boldsymbol{r}' = \psi \boldsymbol{h}$ for some function ψ. Applying a linear functional $\det(\cdot, \boldsymbol{h}', \boldsymbol{h}'')$ on both sides of this equation gives

$$\psi = \frac{\det(\boldsymbol{r}', \boldsymbol{h}', \boldsymbol{h}'')}{\det(\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{h}'')}.$$

The proof of (19) is completed by a straightforward evaluation of the determinant

$$(\boldsymbol{h}' \times \boldsymbol{h}'') \cdot \boldsymbol{r}' = (0; \boldsymbol{h}' \times \boldsymbol{h}'') \cdot \widetilde{\boldsymbol{R}}' = (0; \boldsymbol{h}' \times \boldsymbol{h}'') \cdot \left(\widetilde{\boldsymbol{L}} \wedge \widetilde{\boldsymbol{L}}' \wedge \left(\frac{1}{R_0} \widetilde{\boldsymbol{L}}'' \right) \right)' .$$

$\square$

There are two equivalent common ways to obtain a rational unit vector field. The first one is to use the stereographic projection, which defines a bi-rational correspondence between points in the plane and points on the unit sphere in $\mathbb{R}^3$ (see [8]). The other approach which will be used in this paper is based upon quaternion polynomials.

Space of quaternions $\mathbb{H}$ is a 4–dimensional vector space with a standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Quaternions can be written as $\mathcal{A} = (a, \boldsymbol{a})$ where the first component is called a scalar part, and the remaining three components form a vector part of the quaternion. A quaternion with a zero scalar part is called a pure quaternion, and such quaternions are identified with vectors in $\mathbb{R}^3$, *i.e.*, $\mathcal{A} \equiv \boldsymbol{a}$ for $\mathcal{A} = (0, \boldsymbol{a})$.

To construct a rational unit vector field of tangents we start by a quaternion polynomial $\mathcal{A} \in \mathbb{H}[t]$, which components are relatively prime. With quaternion polynomial $\mathcal{A}$ we associate the orthonormal Euler-Rodrigues frame $(\boldsymbol{e}_i)_{i=1}^3$, defined by

$$\boldsymbol{e}_1 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A} \, \mathbf{i} \, \mathcal{A}^*, \quad \boldsymbol{e}_2 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A} \, \mathbf{j} \, \mathcal{A}^*, \quad \boldsymbol{e}_3 := \frac{1}{\|\mathcal{A}\|^2} \mathcal{A} \, \mathbf{k} \, \mathcal{A}^*, \tag{23}$$

where $\mathcal{A}^* = (a, -\boldsymbol{a})$ denotes the conjugate of $\mathcal{A}$, and $\|\mathcal{A}\| = \sqrt{\mathcal{A}\mathcal{A}^*}$ the norm. A vector field $\boldsymbol{\omega}$ that satisfies $\boldsymbol{e}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i$ determines the rotation axis of the frame. If we express it in the E-R frame moving coordinate system, $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$, the relation $\boldsymbol{e}_j \cdot \boldsymbol{e}_i' = \boldsymbol{e}_j \cdot (\boldsymbol{\omega} \times \boldsymbol{e}_i) = \det(\boldsymbol{e}_j, \boldsymbol{\omega}, \boldsymbol{e}_i)$ gives the coefficients

$$\omega_1 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_2' = -\boldsymbol{e}_2 \cdot \boldsymbol{e}_3', \; \omega_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3' = -\boldsymbol{e}_3 \cdot \boldsymbol{e}_1', \; \omega_3 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' = -\boldsymbol{e}_1 \cdot \boldsymbol{e}_2', \tag{24}$$

and the frame speed $(\boldsymbol{e}_i')_{i=1}^3$ follows from

$$\boldsymbol{e}_1' = \omega_3\boldsymbol{e}_2 - \omega_2\boldsymbol{e}_3, \quad \boldsymbol{e}_2' = -\omega_3\boldsymbol{e}_1 + \omega_1\boldsymbol{e}_3, \quad \boldsymbol{e}_3' = \omega_2\boldsymbol{e}_1 - \omega_1\boldsymbol{e}_2. \tag{25}$$

The polynomial elements needed in Theorem 5 are introduced by

$$\rho := \|\mathcal{A}\|^2, \quad \boldsymbol{h}_i := \rho\, \boldsymbol{e}_i,\ i = 1,2,3, \quad \boldsymbol{h} := \boldsymbol{h}_1. \tag{26}$$

Since the components of $\mathcal{A}$ are relatively prime, one has $\rho > 0$ ([2], p. 483). Moreover, the vector field $\boldsymbol{v} = \boldsymbol{h} \times \boldsymbol{h}'$ of Theorem 5 can be expressed as

$$\boldsymbol{v} = \nu_2\boldsymbol{h}_2 + \nu_3\boldsymbol{h}_3, \quad \boldsymbol{\nu} := (\nu_i)_{i=1}^3 := \rho\ (\omega_i)_{i=1}^3\,, \tag{27}$$

where ν_i is the numerator of ω_i. Given f in g, the procedure outlined *generates* the corresponding dual form $\boldsymbol{L} = (-f; g\boldsymbol{v})$ from $\mathcal{A}$ by (23), (24), (26), and (27) completely.

As an example, let us apply Theorem 5 to a curve generated by the quadratic quaternion polynomial

$$\mathcal{A}(t) = \left(1, (0,0,0)^T\right) + \left(0, (1,0,1)^T\right) t + \left(0, (1,-1,1)^T\right) t^2.$$

The components of $\mathcal{A}$ are $1, t(t+1), -t^2, t(t+1)$, thus relatively prime. One obtains

$$\boldsymbol{h} = \begin{pmatrix} 1-t^4 \\ 2(1-t^2)t(t+1) \\ 2\left(t^2 + (t^2+t)^2\right) \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} 4t^2\left(t^4+6t^3+7t^2+4t+2\right) \\ -4t\left(t^5+2t^4+2t^2+3t+2\right) \\ 2(t+1)^2\left(-t^4+4(t-1)t^2+1\right) \end{pmatrix}.$$

The dual form (17) is of the degree 6 provided g is constant and $\deg(f) \leq 6$. The choice $g(t) = 1$ and $f(t) = 1 + t^3 - t^6$ determines a class 6 rational PH curve $\boldsymbol{r}$ of the degree 12 with a hodograph $\boldsymbol{r}' = \psi\boldsymbol{h}$ where

$$\begin{aligned}\psi(t) =& \frac{3}{8\left(2t^6+3t^5-12t^3-9t^2-3t-1\right)^3}\big(22t^{12}+42t^{11}+6t^{10}+559t^9 \\ &+1152t^8+1035t^7+423t^6+81t^5+189t^4+547t^3+363t^2+96t+5\big).\end{aligned}$$

Singular points of $\boldsymbol{r}$ are determined by ψ only. The derivative $\boldsymbol{r}'$ vanishes at zeroes of ψ: $t = -3.13475$, $t = -0.0676759$, $t = \pm\infty$, and it is unbounded at $t = -0.59708$, $t = 1.70123$.

4. Low class rational PH curves

The degree of a dual PH curve form (17) depends on degrees of $\mathcal{A}$, f and g. The degree of quaternion polynomial affects it most. Namely, if the degree of $\mathcal{A}$ is equal to k, it is easy to verify that

$$\deg(\nu_i) \leq 2k-2, \quad \deg(\boldsymbol{h}_i) \leq 2k, \quad i = 1,2,3. \tag{28}$$

Theorem 3 implies then $\deg(\boldsymbol{v}) \leq 4k-2$. If the components of $\boldsymbol{v}$ are relatively prime and the equality holds, the class m equals to

$$m = \max\left\{4k-2+\deg(g), \deg(f)\right\}.$$

So a quest for low class rational PH curves must start with $k=1$, and g of a low degree. Further, a quadratic $\mathcal{A}$ yields curves of the class ≥ 6, and cubic one results in the class ≥ 10 provided the components of $\boldsymbol{v}$ are relatively prime. But if they are not, the class could be quite clearly reduced by a proper choice of f. Suppose that

$$\boldsymbol{v} = \vartheta \boldsymbol{v}_R = \vartheta\left(v_{R,1}, v_{R,2}, v_{R,3}\right)^T$$

for some nonconstant polynomial ϑ. Then for any f which is divisible by ϑ a class m reduces by $\deg(\vartheta)$. Relations between quaternion polynomial $\mathcal{A}$ coefficients that give rise to a common polynomial factor ϑ in $\boldsymbol{v}$ are rather complicated. The next lemma reveals some necessary ones. The notation $p\big|q$ will be used to denote that a polynomial p divides a polynomial q.

Lemma 1. *Suppose that $\boldsymbol{v} = \vartheta \boldsymbol{v}_R$. Then ϑ must satisfy*

$$\vartheta\big|\left(\nu_2\,\rho^2\right), \quad \vartheta\big|\left(\nu_3\,\rho^2\right). \tag{29}$$

Moreover,

$$\vartheta^2\big|\lambda, \tag{30}$$

where

$$\lambda := \det(\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{h}'') = \nu_1\left(\nu_2^2+\nu_3^2\right) + \rho\left(\nu_2\nu_3' - \nu_3\nu_2'\right). \tag{31}$$

PROOF. Since $\boldsymbol{h}_i \cdot \boldsymbol{h}_j = \delta_{i,j}\rho^2$, the assertion (29) follows from (27). The expansion (31) is computed from (24), (25),(26), and (27). From (21) we obtain

$$\boldsymbol{v}\times\boldsymbol{v}' = \vartheta^2 \boldsymbol{v}_R\times\boldsymbol{v}_R' = \lambda\boldsymbol{h}.$$

The condition (30) clearly holds if the components of $\boldsymbol{h}$ are relatively prime. To prove that it always holds we need to examine the case when $\boldsymbol{h} = \vartheta_h \boldsymbol{h}_R$, where ϑ_h denotes the greatest common factor. In this case $\lambda = \vartheta_h^3 \det(\boldsymbol{h}_R, \boldsymbol{h}'_R, \boldsymbol{h}''_R)$. This shows that any function that divides ϑ_h divides also λ which implies the assertion and completes the proof. $\square$

The next lemma shows that a multiplication of $\mathcal{A}$ by a constant quaternion does not change the class of the corresponding curve or the degree of a common factor in $\boldsymbol{v}$. It induces a rotation and the scaling of the original curve only.

Lemma 2. *Suppose that $\mathcal{Q} = \left(q_0, (q_1, q_2, q_3)^T\right) \in \mathbb{H}$, $\|\mathcal{Q}\| > 0$, $\mathcal{A} \in \mathbb{H}[t]$, and polynomials f, g are prescribed. Let dual forms $\boldsymbol{L} = (-f; g\boldsymbol{v})$, $\boldsymbol{L}_\mathcal{Q} = (-f; g\boldsymbol{v}_\mathcal{Q})$ be generated by the quaternion polynomials $\mathcal{A}$ and $\mathcal{A}_\mathcal{Q} := \mathcal{Q}\mathcal{A}$ respectively and let $\boldsymbol{r}$ and $\boldsymbol{r}_\mathcal{Q}$ be the corresponding rational curves. Then*

$$\boldsymbol{v}_\mathcal{Q} = \|\mathcal{Q}\|^4 \mathcal{R}(\mathcal{Q})\boldsymbol{v}, \quad \boldsymbol{r}_\mathcal{Q} = \frac{1}{\|\mathcal{Q}\|^4}\mathcal{R}(\mathcal{Q})\boldsymbol{r},$$

where

$$\mathcal{R}(\mathcal{Q}) := \frac{1}{\|\mathcal{Q}\|^2}\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

is the rotation matrix that corresponds to the quaternion $\mathcal{Q}$.

PROOF. For any vector $\boldsymbol{s} \in \mathbb{R}^3$ a quaternion product $\mathcal{Q}(0, \boldsymbol{s})\mathcal{Q}^*$ gives a pure quaternion which is equivalent to a multiplication of $\boldsymbol{s}$ by a rotation matrix (see e.g. [9, Chapter 29]), *i.*e.,

$$\mathcal{Q}(0, \boldsymbol{s})\mathcal{Q}^* \equiv \|\mathcal{Q}\|^2 \mathcal{R}(\mathcal{Q})\boldsymbol{s}.$$

Therefore,

$$\boldsymbol{h}_\mathcal{Q} := \mathcal{A}_\mathcal{Q}\mathbf{i}\mathcal{A}_\mathcal{Q}^* = \|\mathcal{Q}\|^2 \mathcal{R}(\mathcal{Q})\mathcal{A}\mathbf{i}\mathcal{A}^* = \|\mathcal{Q}\|^2 \mathcal{R}(\mathcal{Q})\boldsymbol{h}$$

and

$$\boldsymbol{v}_\mathcal{Q} = \boldsymbol{h}_\mathcal{Q} \times \boldsymbol{h}'_\mathcal{Q} = \|\mathcal{Q}\|^4 (\mathcal{R}(\mathcal{Q})\boldsymbol{h}) \times (\mathcal{R}(\mathcal{Q})\boldsymbol{h}') = \|\mathcal{Q}\|^4 \mathcal{R}(\mathcal{Q})\boldsymbol{v}.$$

To prove the second assertion note that

$$(M\boldsymbol{s}_1) \wedge (M\boldsymbol{s}_2) \wedge (M\boldsymbol{s}_3) = \det M \left(M^{-T}(\boldsymbol{s}_1 \wedge \boldsymbol{s}_2 \wedge \boldsymbol{s}_3)\right)$$

for any nonsingular matrix $M \in \mathbb{R}^{4\times 4}$ and any $\boldsymbol{s}_i \in \mathbb{R}^4$, $i = 1, 2, 3$. If we choose M as $M = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \|\mathcal{Q}\|^4 \mathcal{R}(\mathcal{Q}) \end{pmatrix}$, then $\boldsymbol{L}_{\mathcal{Q}} = M\boldsymbol{L}$ and $\boldsymbol{P}_{\mathcal{Q}} = \det M \left(M^{-T}\boldsymbol{P}\right)$. Using (4) and $M^{-T} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \frac{1}{\|\mathcal{Q}\|^4}\mathcal{R}(\mathcal{Q}) \end{pmatrix}$ concludes the proof. □

Lemma 2 simplifies the rational PH curve generation. Without loss of generality, we may write the quaternion polynomial $\mathcal{A} \in \mathbb{H}[t]$ involved in (23) as a product $\mathcal{Q}\mathcal{A}$ where the leading quaternion coefficient of $\mathcal{A}$ equals $\mathbf{1} = \left(1, (0,0,0)^T\right)$. The quaternion $\mathcal{Q}$ represents the final curve rotation, but it adds only two free parameters to its representation. Quite clearly, since the E-R frame (23) is homogeneous, we can assume $\|\mathcal{Q}\| = 1$. The rotation around $\boldsymbol{e}_1$, implied by a quaternion $\mathcal{U} = \left(\cos\varphi, (\sin\varphi, 0, 0)^T\right)$, pins $\mathcal{Q}$ down to two degrees of freedom. So it is enough to investigate dual forms $\boldsymbol{L} = (-f; g\mathcal{R}\boldsymbol{v})$, where $\boldsymbol{v}$ is generated by the quaternion polynomial with the leading coefficient equal to $\mathbf{1}$. For the final rotation $\mathcal{R}$ one may choose any rotation in $\mathbb{R}^3$ depending on two free parameters such as

$$\mathcal{R} = \mathcal{R}(\theta, \varrho) = \begin{pmatrix} \cos\theta\cos\varrho & \cos\varrho\sin\theta & \sin\varrho \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\varrho & -\sin\theta\sin\varrho & \cos\varrho \end{pmatrix}.$$

Since a rotation doesn't influence the basic curve generation, we shall stick to the case $\mathcal{R} = I$ from now on only.

5. Rational PH curves generated by linear quaternion polynomials

Let us examine rational PH curves generated by a linear quaternion polynomial

$$\mathcal{A}(t) = \mathcal{A}_0 + \mathcal{A}_1 t, \quad \mathcal{A}_i := \left(a_{0,i}, (a_{1,i}, a_{2,i}, a_{3,i})^T\right), \tag{32}$$

with $\mathcal{A}_1 = \mathbf{1}$. The functions and the fields needed are generated by (23), (24), (26), and (27). From (28) we observe

$$\deg(\nu_i) = 0, \quad \deg(\boldsymbol{h}_i) \le 2, \quad i = 1, 2, 3,$$

and thus (27) implies that the vector field $\boldsymbol{v}$ is of degree ≤ 2. Further, from (31) we compute $\lambda = -8a_{1,0}\left(a_{2,0}^2 + a_{3,0}^2\right)$, and the curve generated is well defined provided $a_{1,0}, a_{2,0}, a_{3,0} \ne 0$. The dual form $\boldsymbol{L} = (-f; g\boldsymbol{v})$ is of the degree

$\max\{2+\deg(g), \deg(f)\}$ and the class 3 curve $\boldsymbol{r}$ is obtained by the choice

$$f(t) = f_0 + f_1 t + f_2 t^2 + f_3 t^3, \quad g(t) = g_0 + g_1 t,\ g \not\equiv 0.$$

Let us reparameterize an equivalent dual form $\widetilde{\boldsymbol{L}} := \frac{1}{g^3}\boldsymbol{L} = \left(-\frac{f}{g^3}; \frac{1}{g^2}\boldsymbol{v}\right)$ by

$$t = \varphi(\tau) := \frac{g_1 + g_0\tau}{g_0 - g_1\tau}. \tag{33}$$

From

$$\frac{t^i}{g(t)^k} = \frac{\varphi(\tau)^i}{g(\varphi(\tau))^k} = \frac{1}{(g_0^2+g_1^2)^k}\left(g_0\tau + g_1\right)^i \left(g_0 - g_1\tau\right)^{k-i},\ 0 \le i \le k,\ k = 2, 3,$$

we observe that the polynomial $\frac{f}{g^3} \circ \varphi$ and the polynomial field $\frac{1}{g^2}\boldsymbol{v} \circ \varphi$ are of the degree ≤ 3 and ≤ 2 respectively. But then, by (7), (18), and Theorem 3, the reparameterized cubic curve $\boldsymbol{r} \circ \varphi$ is a polynomial one. This proves the following observation.

Theorem 6. *Suppose that $\boldsymbol{v}$ is generated by a linear quaternion* (32)*, with $\mathcal{A}_1 = \mathbf{1}$, and $a_{1,0}, a_{2,0}, a_{3,0} \neq 0$. Suppose that f is a cubic polynomial, and g is a linear one. Then a rational curve obtained from a dual representation $\boldsymbol{L} = (-f; g\boldsymbol{v})$ is a reparameterized polynomial cubic PH curve.*

6. Rational PH curves generated by quadratic quaternion polynomials

The previous section showed that we must start with at least a quadratic quaternion polynomial

$$\mathcal{A}(t) = \mathcal{A}_0 + \mathcal{A}_1 t + \mathcal{A}_2 t^2, \quad \mathcal{A}_i = \left(a_{0,i}, (a_{1,i}, a_{2,i}, a_{3,i})^T\right), \quad i = 0, 1, 2, \tag{34}$$

to obtain a rational PH curve that can not be reparameterized to a polynomial one in the case $\deg(g) \le 1$. Without loos of generality we simplify (34) by $\mathcal{A}_2 = \mathbf{1}$. Again, the functions and the fields needed are generated by (23), (24), (26), and (27). Also, (28) gives $\deg(\nu_i) \le 2,\ \deg(\boldsymbol{h}_i) \le 4,\ i = 1, 2, 3$. It is straightforward to obtain

$$\boldsymbol{\nu} = -2\begin{pmatrix} a_{1,1}t^2 + 2a_{1,0}t + a_{0,1}a_{1,0} - a_{0,0}a_{1,1} - a_{2,1}a_{3,0} + a_{2,0}a_{3,1} \\ a_{2,1}t^2 + 2a_{2,0}t + a_{0,1}a_{2,0} - a_{0,0}a_{2,1} + a_{1,1}a_{3,0} - a_{1,0}a_{3,1} \\ a_{3,1}t^2 + 2a_{3,0}t - a_{1,1}a_{2,0} + a_{1,0}a_{2,1} + a_{0,1}a_{3,0} - a_{0,0}a_{3,1} \end{pmatrix}, \tag{35}$$

$$\rho = t^4 + 2a_{0,1}t^3 + \left(\sum_{i=0}^{3} a_{i,1}^2 + 2a_{0,0}\right)t^2 + 2\left(\sum_{i=0}^{3} a_{i,0}a_{i,1}\right)t + \sum_{i=0}^{3} a_{i,0}^2,$$

$$\lambda = -8\left(a_{2,1}a_{3,0} - a_{2,0}a_{3,1} + a_{1,1}\left(a_{2,1}^2 + a_{3,1}^2\right)\right)t^6 + \dots\ ,$$

and the key vector field

$$
\boldsymbol{v} = \begin{pmatrix} -4\left(a_{2,1}a_{3,0} - a_{2,0}a_{3,1} + a_{1,1}\left(a_{2,1}^2 + a_{3,1}^2\right)\right)t^4 + \dots \\ -2a_{2,1}t^6 - 4\left(a_{2,0} + a_{0,1}a_{2,1} - a_{1,1}a_{3,1}\right)t^5 + \dots \\ -2a_{3,1}t^6 - 4\left(a_{3,0} + a_{1,1}a_{2,1} + a_{0,1}a_{3,1}\right)t^5 + \dots \end{pmatrix}, \tag{36}
$$

which is of the degree 6 provided $a_{2,1}^2 + a_{3,1}^2 > 0$. We assume $g = 1$ and $\deg(f) \leq \deg(\boldsymbol{v})$ to decrease the class of the curve too. The class 6 curves are right at hand. Their standard rational form $\boldsymbol{r}$ will be by Theorem 3 of the degree 12 in general. But to obtain lower class curves some further relations between quaternion parameters have to be imposed. There are two ways to decrease the curve class: one could simply annihilate the leading coefficients of $\boldsymbol{v}$ or one may determine conditions that imply polynomial factorization $\boldsymbol{v} = \vartheta \boldsymbol{v}_R$. In the latter case, we write f as ϑf to make it divisible by ϑ too, and we obtain the reduced dual form $\boldsymbol{L} = (-f; \boldsymbol{v}_R)$. From (36) we observe that additional assumptions on the quaternion coefficients

$$
a_{2,1} = a_{3,1} = 0 \tag{37}
$$

generate class 5 curves with the standard rational form of the degree 9, with $\boldsymbol{v}$ from (36) reduced to

$$
\boldsymbol{v} = \begin{pmatrix} -4\left(a_{2,0}^2 + a_{3,0}^2\right)\left(a_{1,1}t^2 + 2a_{1,0}t + a_{0,1}a_{1,0} - a_{0,0}a_{1,1}\right) \\ -4a_{2,0}t^5 + \left(6a_{1,1}a_{3,0} - 10a_{0,1}a_{2,0}\right)t^4 + \dots \\ -4a_{3,0}t^5 - 2\left(3a_{1,1}a_{2,0} + 5a_{0,1}a_{3,0}\right)t^4 + \dots \end{pmatrix}, \tag{38}
$$

and further

$$
\begin{aligned}
\boldsymbol{\nu} &= -2 \begin{pmatrix} a_{1,1}t^2 + 2a_{1,0}t + a_{0,1}a_{1,0} - a_{0,0}a_{1,1} \\ 2a_{2,0}t + a_{0,1}a_{2,0} + a_{1,1}a_{3,0} \\ 2a_{3,0}t - a_{1,1}a_{2,0} + a_{0,1}a_{3,0} \end{pmatrix}, \\
\lambda &= -\left(a_{2,0}^2 + a_{3,0}^2\right)\left(24a_{1,1}t^4 + 16\left(4a_{1,0} + a_{0,1}a_{1,1}\right)t^3 + \dots\right).
\end{aligned} \tag{39}
$$

Note that the degree of the denominator will only be equal to $2\deg(\lambda) = 2*4 = 8$ in general. As an example, take $\mathcal{A} = \left(t^2 - t, (-t, -2, 2)^T\right)$, and $f = t^3$. Then we obtain

$$
\boldsymbol{h} = \begin{pmatrix} (t-2)\left(t^3 + 2t + 4\right) \\ 4t^2 \\ 4(t-2)t \end{pmatrix}, \ \boldsymbol{v} = \begin{pmatrix} 32t^2 \\ 8(t-2)^2\left(t^3 - 2\right) \\ -8t\left(t^4 - t^3 + 8\right) \end{pmatrix}, \ \boldsymbol{L} = \left(-t^3; \boldsymbol{v}\right), \tag{40}
$$

and the homogeneous curve form equals

$$\boldsymbol{P} = \begin{pmatrix} -4096\left(-3t^4+2t^3+8\right)^2 \\ -128t\left(3t^8-6t^6+2t^5-48t^4+48t^3-16t^2+192\right) \\ 512t^3\left(3t^4-t^3+8\right) \\ 512t^2\left(3t^5-4t^4+8t-24\right) \end{pmatrix}.$$

The hodograph $\boldsymbol{r}' = \psi\, \boldsymbol{h}$ follows from (40) and

$$\psi = \frac{3\left(\left(3t^5+96t-32\right)t^3+64\right)}{32\left(t^3(3t-2)-8\right)^3}.$$

The second approach seems to lead to a different type of the class 5 curves, but they only differ in the parameterization. Suppose that $\vartheta := \vartheta(t) := \theta_0 + \theta_1 t$, $\theta_1 \neq 0$, divides the field $\boldsymbol{v}$, $\boldsymbol{v} = \vartheta \boldsymbol{v}_R$. The denominator of the curve obtained from $\boldsymbol{L} = (-f; \boldsymbol{v}_R)$ equals

$$\det(\boldsymbol{v}_R, \boldsymbol{v}_R', \boldsymbol{v}_R'') = \frac{\lambda^2}{\vartheta^3} = \left(\frac{\lambda}{\vartheta^2}\right)^2 \vartheta, \tag{41}$$

but the last term is by (30) the product of polynomials. So the curve has a singular point at the parameter value $t = -\frac{\theta_0}{\theta_1}$. However, the reparameterization (33) with $g_i \to \theta_i$, $i = 0, 1$, maps this singular point to infinity, and the reparameterized dual form

$$\left(\theta_0 - \theta_1 \tau\right)^9 \boldsymbol{L}\left(-f \circ \varphi; \boldsymbol{v}_R \circ \varphi\right)$$

is of the same type as the one obtained by assuming (37).

If we carry on the degree reduction of $\boldsymbol{v}$ from (37) further, we don't obtain class 4 curves. Indeed, the reduction of (38) to the degree ≤ 4 implies $a_{2,0} = a_{3,0} = 0$, but then the curve generated is not defined since the corresponding λ equals 0. This leaves two ways to obtain the class 4 curves. The first one is to determine conditions such that $\boldsymbol{v}$ in (38) would admit a common linear factor ϑ. But then, as in (41), the corresponding curve would inevitably have a singular point at some finite parameter value, and the other one at infinity. The second way is to determine a quadratic divisor ϑ of the field (36) where we assume $a_{2,1}, a_{3,1} \neq 0$. Since $\rho = \|\mathcal{A}\|^2 = \|\boldsymbol{h}\|$ and a common factor of $\boldsymbol{h}$ cannot have real roots provided $\mathcal{A}$ has relatively prime components, it is clear that any polynomial that divides ρ can only be factorized to irreducible quadratic factors over the real field $\mathbb{R}$. But then by (29) the polynomial ϑ must divide either ν_2 and ν_3, or ρ.

The second possibility is rather complex, and we consider the first one only. Note (35). Since ϑ, ν_2 and ν_3 are all quadratic, $\vartheta\big|\nu_2$ and $\vartheta\big|\nu_3$ imply

$$\frac{\nu_2}{a_{2,1}} = \frac{\nu_3}{a_{3,1}} \implies a_{1,0} = \frac{a_{1,1}a_{3,0}}{a_{3,1}},\ a_{2,0} = \frac{a_{2,1}a_{3,0}}{a_{3,1}}.$$

This gives the coefficient relations of the quadratic quaternion polynomial (34) that generates the class 4 curves as

$$\mathcal{A}_0 = \left(a_{0,0}, \left(\frac{a_{1,1}a_{3,0}}{a_{3,1}}, \frac{a_{2,1}a_{3,0}}{a_{3,1}}, a_{3,0}\right)^T\right).$$

It is easy to verify that this relation is equivalent to $\mathcal{A}_0, \mathcal{A}_1$, and $\mathcal{A}_2 = \mathbf{1}$ being linearly independent. As an example, $f = t^3$ and the quaternion polynomial $\mathcal{A} = \left(t^2 + 2t + 2, (-2t-2, -2t-2, 2t+2)^T\right)$ generates the curve determined by either one of the forms

$$\boldsymbol{L} = \begin{pmatrix} -t^3 \\ -32(t+1)^2 \\ -2t^4 \\ 2(t+2)^4 \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} 2048t^3(t+2)^3 \\ 192t^4(t+2)^2 \\ 128t^2(t+2)^2\left(t^2+2t-6\right) \\ 128t^4\left(t^2+6t+6\right) \end{pmatrix}.$$

The class 3 PH curves seems to be the most promising in practical applications. As with the class 5 we send with (37) one singular point to infinity, and only a quadratic reduction factor ϑ has to be decided upon. But from (38) and (39) we observe that ϑ should be proportional to ν_1, so we choose

$$\vartheta = -\frac{1}{2a_{1,1}}\nu_1 = t^2 + 2\,\frac{a_{1,0}}{a_{1,1}}\,t + \frac{a_{0,1}a_{1,0} - a_{0,0}a_{1,1}}{a_{1,1}}.$$

By (29), it is necessary that ϑ divides ρ. The remainder that should vanish is a linear polynomial

$$\begin{aligned}\frac{1}{a_{1,1}^3}\left(a_{1,0}^2 - a_{0,1}a_{1,1}a_{1,0} + a_{0,0}a_{1,1}^2\right)&\left(4\left(a_{0,1}a_{1,1} - 2a_{1,0}\right)t\right.\\ &\left.+\left(a_{0,1}^2 + 4a_{0,0}\right)a_{1,1} - 4a_{0,1}a_{1,0} + a_{1,1}^3\right) + a_{2,0}^2 + a_{3,0}^2.\end{aligned}$$

The term $a_{1,0}^2 - a_{0,1}a_{1,1}a_{1,0} + a_{0,0}a_{1,1}^2$ must not vanish since obviously $a_{2,0}^2 + a_{3,0}^2 > 0$. This implies $a_{0,1}a_{1,1} - 2a_{1,0} = 0$, and we obtain equations

$$a_{2,0}^2 + a_{3,0}^2 + \frac{1}{4}\left(a_{0,1}^2 - 4a_{0,0}\right)\left(a_{0,1}^2 - a_{1,1}^2 - 4a_{0,0}\right) = 0, \quad a_{0,1}a_{1,1} - 2a_{1,0} = 0,$$

simplified to

$$a_{0,1}=\frac{2a_{1,0}}{a_{1,1}},\quad a_{0,0}=\frac{a_{1,0}^2}{a_{1,1}^2}-\frac{a_{1,1}^2}{8}\pm\frac{1}{8}\sqrt{a_{1,1}^4-16\left(a_{2,0}^2+a_{3,0}^2\right)}.$$

If these equations are satisfied, ϑ divides not only ρ, but $\boldsymbol{v}$ too. Namely, it is straightforward to evaluate $\boldsymbol{v}_R=\frac{1}{\vartheta}\boldsymbol{v}$, and to prove the following conclusion.

Theorem 7. *Suppose that coefficients of the quadratic quaternion are given as*

$$\mathcal{A}_0=\left(a_{0,0},(a_{1,0},a_{2,0},a_{3,0})^T\right),\ \mathcal{A}_1=\left(2\frac{a_{1,0}}{a_{1,1}},(a_{1,1},0,0)^T\right),\ \mathcal{A}_2=\mathbf{1},$$

where $a_{1,1}^4\geq 16\left(a_{2,0}^2+a_{3,0}^2\right)>0$ *and*

$$a_{0,0}=\frac{a_{1,0}^2}{a_{1,1}^2}-\frac{a_{1,1}^2}{8}\pm\frac{1}{8}\sqrt{a_{1,1}^4-16\left(a_{2,0}^2+a_{3,0}^2\right)}.$$

If f is a polynomial of the degree ≤ 3, then the dual form $\boldsymbol{L}=(-f;\boldsymbol{v}_R)$, with $\boldsymbol{v}_R=(v_{R,1},v_{R,2},v_{R,3})^T$, and

$$\begin{aligned}
v_{R,1}&=-4a_{1,1}\left(a_{2,0}^2+a_{3,0}^2\right),\\
v_{R,2}&=\frac{2}{a_{1,1}^3}\left(-2a_{2,0}a_{1,1}^3t^3+\left(3a_{1,1}^4a_{3,0}-6a_{1,0}a_{1,1}^2a_{2,0}\right)t^2+6a_{1,1}^3\left(a_{1,0}a_{3,0}-a_{0,0}a_{2,0}\right)t\right.\\
&\quad\left.+\left(a_{1,1}^4+5a_{0,0}a_{1,1}^2-2a_{1,0}^2\right)a_{3,0}a_{1,1}^2+2a_{1,0}\left(2a_{1,0}^2-3a_{0,0}a_{1,1}^2\right)a_{2,0}\right),\\
v_{R,3}&=-\frac{2}{a_{1,1}^3}\left(2a_{3,0}a_{1,1}^3t^3+\left(3a_{2,0}a_{1,1}^4+6a_{1,0}a_{3,0}a_{1,1}^2\right)t^2+6a_{1,1}^3\left(a_{1,0}a_{2,0}+a_{0,0}a_{3,0}\right)t\right.\\
&\quad\left.+\left(a_{1,1}^4+5a_{0,0}a_{1,1}^2-2a_{1,0}^2\right)a_{2,0}a_{1,1}^2+2a_{1,0}\left(3a_{0,0}a_{1,1}^2-2a_{1,0}^2\right)a_{3,0}\right),
\end{aligned}$$

determines the rational cubic PH curve with the denominator

$$576a_{1,1}^2\left(a_{2,0}^2+a_{3,0}^2\right)^2\vartheta=576\left(a_{2,0}^2+a_{3,0}^2\right)^2\left(a_{1,1}^2t^2+2a_{1,1}a_{1,0}t+2a_{1,0}^2-a_{0,0}a_{1,1}^2\right).$$

Let us now determine the homogeneous point representation $\boldsymbol{P}=(P_0,P_1,P_2,P_3)^T$ for the curve, given by Theorem 7. If we extract the common factor

$$\frac{36\left(a_{2,0}^2+a_{3,0}^2\right)}{a_{1,1}^3},$$

we obtain the simplified homogeneous coordinates

$$
\begin{aligned}
P_0 &= -16a_{1,1}^3\left(a_{2,0}^2+a_{3,0}^2\right)\left(t^2a_{1,1}^2+2ta_{1,1}a_{1,0}+2a_{1,0}^2-a_{0,0}a_{1,1}^2\right),\\
P_1 &= \left(\left(8a_{1,0}^2a_{1,1}^2-4a_{0,0}a_{1,1}^4\right)t^2+4a_{1,0}a_{1,1}^3\left(a_{1,1}^2+2a_{0,0}\right)t\right.\\
&\qquad \left.-4a_{1,0}^4+\left(5a_{1,1}^4+8a_{0,0}a_{1,1}^2\right)a_{1,0}^2-a_{1,1}^4\left(a_{0,0}a_{1,1}^2+5\left(a_{2,0}^2+a_{3,0}^2\right)\right)\right)t,\\
P_2 &= 4a_{1,1}^3\left(\left(a_{2,0}a_{1,1}^2+2a_{1,0}a_{3,0}\right)t+2a_{1,1}\left(a_{1,0}a_{2,0}+a_{0,0}a_{3,0}\right)\right)t,\\
P_3 &= 4a_{1,1}^3\left(\left(a_{3,0}a_{1,1}^2-2a_{1,0}a_{2,0}\right)t+2a_{1,1}\left(a_{1,0}a_{3,0}-a_{0,0}a_{2,0}\right)\right)t.
\end{aligned}
$$

As an example,

$$
f=t^3,\ a_{1,0}=0, a_{1,1}=3, a_{2,0}=2, a_{3,0}=1, a_{0,0}=-\frac{5}{4},-1, \tag{42}
$$

generates curves

$$
-\frac{1}{240\left(4t^2+5\right)}\left(5t\left(4t^2-11\right),8t(12t-5),16t(3t+5)\right)^T,
$$

and

$$
-\frac{1}{60\left(t^2+1\right)}\left(t\left(t^2-4\right),2t(3t-1),t(3t+4)\right)^T.
$$

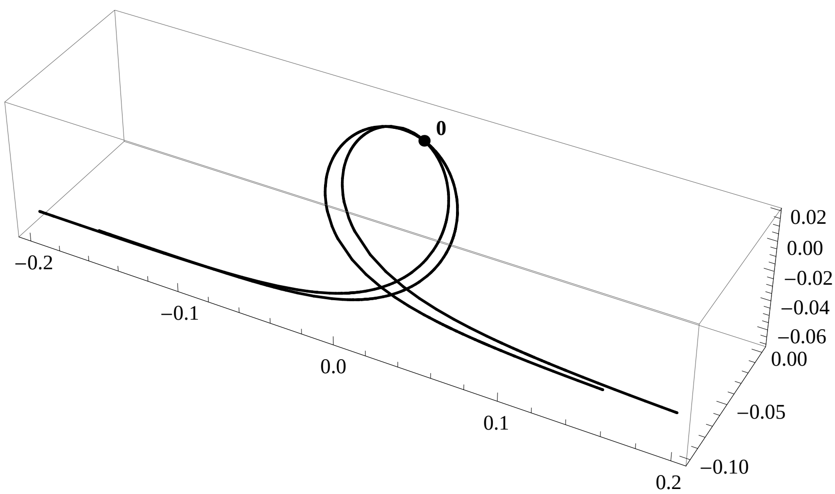

Figure 1: Spatial rational cubic PH curve pair, generated by parameters (42).

7. Conclusion and future work

Recently, the Pythagorean-hodograph property has been extended from planar rational curves to spatial ones ([8]). In this paper, we carry the approach a step further, but based upon the dual curve representation mainly.

There are many questions left concerning spatial rational PH curves. Of course, one has to state and analyse interpolation schemes suited for this purpose. Analysis of singular points and potential alternative ways how to present rational PH curves might also be two interesting issues. But perhaps the most valuable would be the answer to the question, where and when rational PH curves do better than their polynomial counterpart.

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