Construction of G^3 rational motion of degree eight

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Abstract

The paper presents a construction of a rigid body motion with point trajectories being rational spline curves of degree eight joining together with G^3 smoothness. The motion is determined through interpolation of positions and derivative data up to order three in the geometric sense. Nonlinearity in the spherical part of construction results in a single univariate quartic equation which yields solutions in a closed form. Sufficient conditions on the regions for the curvature data are derived, implying the existence of a real admissible solution. The algorithm how to choose appropriate data is proposed too. The theoretical results are substantiated with numerical examples.

Keywords: motion design, geometric interpolation, rational spline motion, geometric continuity

1. Introduction

Rational spline motions are motions of a rigid body where each point of a body travels along a trajectory which is a rational spline curve. Construction of such motions, that match a given sequence of positions, i.e., points and orientations of a moving object is needed to manipulate objects in Computer Graphics, for path planning in Robotics, etc.

The construction of a rigid body motion is usually divided into two parts, a rotational (spherical) part and a translational part of the motion. Many interpolation

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approaches from curve design can directly be generalized to the motion design. However the standard algorithms have a drawback since they imply rational motions of a relatively high degree (see [2], [3], [4], [5]), which is a consequence of high degree of a rotational part of the motion. One of the possible remedies to get motions of a lower degree is to use geometric interpolation techniques instead (see [6], [7], [8], [9], [10], [11]). The first steps in this direction were proposed in [12] and [13]. Geometric interpolation by parabolic splines was used to construct G^1 quartic rational spline motions. Later the generalization of this approach to cubic interpolation, which leads to G^2 rational spline motions of degree six, was considered in [14]. Other geometrically continuous motions of degree six can be found in [15] and [16]. The difficulty when using geometric interpolation methods is that nonlinear equations are involved and the existence and uniqueness of an interpolant are not assured for all given data.

In practical applications G^3 constructions are sometimes preferable (see e.g., [1]). Namely, such motions imply that also torsion of each trajectory is continuous. Furthermore, if some general motion is approximated with the interpolatory G^3 rational spline motion then the order of approximation is expected to be eight while for the G^2 case it is only six. In the present paper a construction of a G^3 continuous Hermite rational spline motion of degree eight, which on every segment interpolates two positions and three derivative data at each position, is considered. As expected, since geometric interpolation is involved, the explicit conditions stated only on given data configurations which would assure an G^3 interpolant to exist are nontrivial to be found. Instead, some solvability conditions on the second order curvature quaternions are derived, which show that if only positions, velocity quaternions and the first order curvature quaternions for the spline motion construction are given, then with some possible slight modifications of the first order curvature quaternions we can always choose such second order curvature quaternions that the admissible G^3 spline interpolant exists.

The remaining of the paper is organized as follows. In the next section the main properties of rational motions are presented. In Section 3 equations for the G^3 interpolatory spline motion are derived. The starting 32 nonlinear equations that determine a spherical part of the motion reduce to a fourth order polynomial equation in one variable. This enables us to express admissible solutions of the interpolation problem explicitly. Next section considers conditions on the second order curvature data, which assure the existence of an admissible solution. The paper is concluded with some numerical examples.

2. Preliminaries

A motion of a rigid body can be determined by a trajectory $\boldsymbol{c} = (c_1, c_2, c_3)^\top$ of the origin of the moving system that describes a translational part of the motion and by a 3×3 matrix $\mathcal{R} = \mathcal{R}(t)$ needed for a spherical (rotational) part of the motion. A trajectory of an arbitrary point $\widehat{\boldsymbol{p}}$ of a rigid body is given as

$$(\widehat{\boldsymbol{p}}, t) \mapsto \boldsymbol{p}(t) = \boldsymbol{c}(t) + \mathcal{R}(t)\,\widehat{\boldsymbol{p}}.$$

The motion is called *rational (spline) motion*, if the elements of the vector $\boldsymbol{c}$ and the matrix $\mathcal{R}$ are rational (spline) functions. The *degree of the motion* is the maximal degree of the functions involved.

We will use the space of quaternions to present rotation matrices. The space of quaternions $\mathbb{H}$ is a 4–dimensional vector space with the standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

$$\mathbf{1} = (1,0,0,0), \quad \mathbf{i} = (0,1,0,0), \quad \mathbf{j} = (0,0,1,0), \quad \mathbf{k} = (0,0,0,1).$$

Quaternions can be written as $\boldsymbol{A} = (a, \boldsymbol{a})$, where the first component is called scalar part, and the remaining three components form the vector part of the quaternion, i.e., $\operatorname{scal}(\boldsymbol{A}) = a$, $\operatorname{vec}(\boldsymbol{A}) = \boldsymbol{a}$. Quaternion sum and product are defined as

$$\boldsymbol{A} + \boldsymbol{B} = (a + b, \boldsymbol{a} + \boldsymbol{b}), \quad \boldsymbol{A}\boldsymbol{B} = (ab - \boldsymbol{a} \cdot \boldsymbol{b}, a\boldsymbol{b} + b\boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}),$$

where $\boldsymbol{B} := (b, \boldsymbol{b})$, and $\times$, $\cdot$ denote the standard vector and scalar products in $\mathbb{R}^3$. Equipped with these two operations $\mathbb{H}$ becomes an algebra. By using quaternions $\boldsymbol{Q} = (q_0, q_1, q_2, q_3) \in \mathbb{H}$ and the *kinematical mapping* $\chi : \mathbb{H} \setminus \{\mathbf{0}\} \to \mathrm{SO}_3$,

$$\boldsymbol{Q} = (q_i)_{i=0}^3 \mapsto \chi(\boldsymbol{Q}) :=$$
$$\frac{1}{q_0^2 + q_1^2 + q_2^2 + q_3^2}, \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix},$$

every rotation $\mathcal{R}$ can be represented as $\chi(\boldsymbol{Q})$ for some quaternion $\boldsymbol{Q}$. Note that

$$\chi(\lambda \boldsymbol{Q}) = \chi(\boldsymbol{Q}), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

The kinematical mapping defines a bijective correspondence between the set of three dimensional rotations and the unit quaternion sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with identified antipodal points (see [17]). More generally, applying mapping χ to a polynomial (spline) curve of degree n gives a spherical rational (spline) motion of degree $2n$.

Since the construction of the translational part $\boldsymbol{c} = (c_i)_{i=1}^3$ of the motion should not increase a degree of the motion, the functions c_i should be chosen as

$$c_i = \frac{w_i}{r}, \quad r = \sum_{j=0}^{3} q_j^2, \quad i = 1, 2, 3,$$

where $\boldsymbol{w} := (w_i)_{i=1}^3$ is a polynomial (spline) curve of degree $\leq 2n$.

Let us now shortly recall geometric continuity conditions for motions, defined in [14]. Suppose that two trajectories of an arbitrary point $\widehat{\boldsymbol{p}}$,

$$\begin{aligned}
\boldsymbol{p}(t) &= \boldsymbol{c}(t) + \mathcal{R}(t)\, \widehat{\boldsymbol{p}}, \quad t \in [t_0, t_1], \\
\widetilde{\boldsymbol{p}}(s) &= \widetilde{\boldsymbol{c}}(s) + \widetilde{\mathcal{R}}(s)\, \widehat{\boldsymbol{p}}, \quad s \in [s_0, s_1],
\end{aligned}$$

are given and let quaternion curves $\boldsymbol{q}$, $\widetilde{\boldsymbol{q}}$ represent the rotations $\mathcal{R}$, $\widetilde{\mathcal{R}}$. The trajectories join with geometric continuity of order k (or shortly with G^k continuity) at a common point $\boldsymbol{p}(\tau) = \widetilde{\boldsymbol{p}}(\sigma)$ iff there exists a regular reparameterization $\varphi\colon [t_0, t_1] \to [s_0, s_1]$, such that $\varphi' > 0$, $\varphi(\tau) = \sigma$ and

$$\frac{d^j \boldsymbol{p}(t)}{dt^j}\Big|_{t=\tau} = \frac{d^j \left(\widetilde{\boldsymbol{p}} \circ \varphi\right)(t)}{dt^j}\Big|_{t=\tau}, \quad j = 0, 1, \ldots, k,$$

or equivalently

$$\frac{d^j \boldsymbol{c}(t)}{dt^j}\Big|_{t=\tau} = \frac{d^j \left(\widetilde{\boldsymbol{c}} \circ \varphi\right)(t)}{dt^j}\Big|_{t=\tau}, \quad \frac{d^j \mathcal{R}(t)}{dt^j}\Big|_{t=\tau} = \frac{d^j \left(\widetilde{\mathcal{R}} \circ \varphi\right)(t)}{dt^j}\Big|_{t=\tau}. \tag{1}$$

The G^k continuity conditions (1) for a spherical part are equivalent to

$$\frac{d^j \boldsymbol{q}(t)}{dt^j}\Big|_{t=\tau} = \frac{d^j}{dt^j} \left(\lambda(t) \widetilde{\boldsymbol{q}}(\varphi(t))\right)\Big|_{t=\tau}, \quad j = 0, 1, \ldots, k, \tag{2}$$

where $\lambda : [t_0, t_1] \to \mathbb{R}$ is a zero free scalar function, arising from the equivalence relation in the 3-dimensional projective space. More details are given in [14].

3. G^3 interpolatory spline motion

In this section the interpolation problem that results in a construction of a G^3 continuous rational spline motion of degree eight is presented. Let a sequence of $(N+1)$ rigid body positions $\mathrm{Pos}_i = [\boldsymbol{C}_i, \mathcal{R}_i]$, $i = 0, 1, \ldots, N$, composed of a center position $\boldsymbol{C}_i$ and a rotation matrix $\mathcal{R}_i$, such that $\mathcal{R}_i \neq \mathcal{R}_{i+1}$, be given.

With every rotation matrix $\mathcal{R}_i$ we associate a unit quaternion $\boldsymbol{Q}_i$ in such a way that the standard scalar product $\langle \boldsymbol{Q}_{i-1}, \boldsymbol{Q}_i \rangle$ of vectors $\boldsymbol{Q}_{i-1}, \boldsymbol{Q}_i \in \mathbb{R}^4$ is nonnegative for $i = 1, 2, \ldots, N$. Additionaly, every position is equipped with derivative information. More precisely, for a rotational part of the motion sets of Euler velocity quaternions $(\boldsymbol{U}_i)_{i=0}^N$ and curvature quaternions $\left(\boldsymbol{U}_i^{[2]}\right)_{i=0}^N$, $\left(\boldsymbol{U}_i^{[3]}\right)_{i=0}^N$ are given. Similarly, we assume that positions of a center are equipped with sets $(\boldsymbol{f}_i)_{i=0}^N$, $\left(\boldsymbol{f}_i^{[2]}\right)_{i=0}^N$, $\left(\boldsymbol{f}_i^{[3]}\right)_{i=0}^N$ of tangent and curvature vectors, prescribed by a user. The task is to construct a spline motion with equidistant breakpoints $0, 1, \ldots, N$, described by a quaternion spline curve $\boldsymbol{q} : [0, N] \to \mathbb{H}$ and by a trajectory $\boldsymbol{c} : [0, N] \to \mathbb{R}^3$ in such a way that the given data are interpolated in a geometric sense and that the resulting motion is G^3 continuous.

Every segment of the spline can be expressed by rational motions, parameterized on the interval $[0, 1]$, as

$$\begin{aligned} \boldsymbol{q}(u) &= \boldsymbol{q}_\ell \left(u - \ell + 1\right), \\ \boldsymbol{c}(u) &= \boldsymbol{c}_\ell \left(u - \ell + 1\right), \end{aligned} \quad u \in [\ell - 1, \ell], \quad \ell = 1, 2, \ldots, N,$$

where $\boldsymbol{q}_\ell$ are determined by interpolation conditions derived from (2) (see also [14]),

$$\begin{aligned} \boldsymbol{q}_\ell(j) =& \lambda_{j,0}^\ell \boldsymbol{Q}_{\ell-1+j}, \\ \boldsymbol{q}_\ell'(j) =& \lambda_{j,1}^\ell \boldsymbol{Q}_{\ell-1+j} + \lambda_{j,0}^\ell \varphi_{j,1}^\ell \boldsymbol{U}_{\ell-1+j}, \\ \boldsymbol{q}_\ell''(j) =& \lambda_{j,2}^\ell \boldsymbol{Q}_{\ell-1+j} + \left(2\lambda_{j,1}^\ell \varphi_{j,1}^\ell + \lambda_{j,0}^\ell \varphi_{j,2}^\ell\right) \boldsymbol{U}_{\ell-1+j} + \lambda_{j,0}^\ell \left(\varphi_{j,1}^\ell\right)^2 \boldsymbol{U}_{\ell-1+j}^{[2]}, \\ \boldsymbol{q}_\ell'''(j) =& \lambda_{j,3}^\ell \boldsymbol{Q}_{\ell-1+j} + \left(3\lambda_{j,2}^\ell \varphi_{j,1}^\ell + 3\lambda_{j,1}^\ell \varphi_{j,2}^\ell + \lambda_{j,0}^\ell \varphi_{j,3}^\ell\right) \boldsymbol{U}_{\ell-1+j} + \\ & 3\left(\lambda_{j,1}^\ell \left(\varphi_{j,1}^\ell\right)^2 + \lambda_{j,0}^\ell \varphi_{j,1}^\ell \varphi_{j,2}^\ell\right) \boldsymbol{U}_{\ell-1+j}^{[2]} + \lambda_{j,0}^\ell \left(\varphi_{j,1}^\ell\right)^3 \boldsymbol{U}_{\ell-1+j}^{[3]}, \quad j = 0, 1, \end{aligned} \tag{3}$$

and the center trajectory is of the form $\boldsymbol{c}_\ell = \frac{\boldsymbol{w}_\ell}{r_\ell}$ with $r_\ell = \|\boldsymbol{q}_\ell\|^2$, where $\boldsymbol{w}_\ell$ must by (1) satisfy

$$\begin{aligned} \boldsymbol{w}_\ell(j) =& r_\ell(j) \boldsymbol{C}_{\ell-1+j}, \\ \boldsymbol{w}_\ell'(j) =& r_\ell'(j) \boldsymbol{C}_{\ell-1+j} + r_\ell(j) \varphi_{j,1}^\ell \boldsymbol{f}_{\ell-1+j}, \\ \boldsymbol{w}_\ell''(j) =& r_\ell''(j) \boldsymbol{C}_{\ell-1+j} + \left(2r_\ell'(j) \varphi_{j,1}^\ell + r_\ell(j) \varphi_{j,2}^\ell\right) \boldsymbol{f}_{\ell-1+j} + r_\ell(j) \left(\varphi_{j,1}^\ell\right)^2 \boldsymbol{f}_{\ell-1+j}^{[2]}, \\ \boldsymbol{w}_\ell'''(j) =& r_\ell'''(j) \boldsymbol{C}_{\ell-1+j} + \left(3r_\ell''(j) \varphi_{j,1}^\ell + 3r_\ell'(j) \varphi_{j,2}^\ell + r_\ell(j) \varphi_{j,3}^\ell\right) \boldsymbol{f}_{\ell-1+j} + \\ & 3\left(r_\ell'(j) \left(\varphi_{j,1}^\ell\right)^2 + r_\ell(j) \varphi_{j,1}^\ell \varphi_{j,2}^\ell\right) \boldsymbol{f}_{\ell-1+j}^{[2]} + r_\ell(j) \left(\varphi_{j,1}^\ell\right)^3 \boldsymbol{f}_{\ell-1+j}^{[3]}, \; j = 0, 1, \end{aligned} \tag{4}$$

for $\ell = 1, 2, \dots, N$. We will assume that the quaternion curves are written in the standard form, i.e.,

$$\lambda_{0,0}^{\ell} = \lambda_{1,0}^{\ell} = 1. \tag{5}$$

This assumption is similar to assuming the standard form of a Bézier rational curve, i.e., normalized weights at the first and the last control point which can be obtained by a bilinear reparameterization (see e.g., [18]). The remaining six free parameters $\lambda_{0,i}^{\ell}$ and $\lambda_{1,i}^{\ell}$, $i = 1, 2, 3$, represent derivatives of the local scalar function λ^{ℓ} at values $t = 0$ and $t = 1$, while $\varphi_{0,i}^{\ell}$ and $\varphi_{1,i}^{\ell}$, $i = 1, 2, 3$, correspond to derivatives of the local reparameterization function φ^{ℓ} at $t = 0$ and $t = 1$. Note that, in order to obtain a regular reparameterization, the inequalities

$$\varphi_{0,1}^{\ell} > 0, \quad \varphi_{1,1}^{\ell} > 0$$

must be fulfilled. Such a solution will be called admissible solution and the corresponding interpolating polynomial will be called admissible interpolating polynomial. These twelve free parameters can be used to decrease the degree of the motion. Namely, for each $\ell \in \{1, 2, \dots, N\}$ conditions (3) represent a system of 32 equations for the unknown coefficients of a quaternion $\boldsymbol{q}_{\ell}$. Clearly, a degree seven quaternion curve would be sufficient for the solution to exist if all the additional parameters would be fixed in advance. But, considering them to be free, one can try to use $\boldsymbol{q}_{\ell}$ of degree four with $4 \times 5 = 20$ degrees of freedom to fulfill (3). With such a quaternion, equations (3) represent a nonlinear system of 32 equations for 32 unknowns, i.e., coefficients of $\boldsymbol{q}_{\ell}$ and $\lambda_{j,i}^{\ell}$, $\varphi_{j,i}^{\ell}$, $j = 0, 1$, $i = 1, 2, 3$. The corresponding rational motion is of degree eight which is much less then the degree 14 motion derived from a degree seven quaternion polynomial.

To simplify the further analysis, let us assume that

$$\det(\boldsymbol{Q}_{\ell-1}, \boldsymbol{Q}_{\ell}, \boldsymbol{U}_{\ell-1}, \boldsymbol{U}_{\ell}) \neq 0, \quad \ell = 1, \dots, N. \tag{6}$$

Then on each spline segment ℓ, curvature quaternions can be expressed as

$$\boldsymbol{U}_{\ell+j-1}^{[2]} = \alpha_{j,0}^{\ell} \boldsymbol{Q}_{\ell-1} + \alpha_{j,1}^{\ell} \boldsymbol{Q}_{\ell} + \alpha_{j,2}^{\ell} \boldsymbol{U}_{\ell-1} + \alpha_{j,3}^{\ell} \boldsymbol{U}_{\ell}, \tag{7}$$

$$\boldsymbol{U}_{\ell+j-1}^{[3]} = \beta_{j,0}^{\ell} \boldsymbol{Q}_{\ell-1} + \beta_{j,1}^{\ell} \boldsymbol{Q}_{\ell} + \beta_{j,2}^{\ell} \boldsymbol{U}_{\ell-1} + \beta_{j,3}^{\ell} \boldsymbol{U}_{\ell}, \tag{8}$$

for $j = 0, 1$.

In the next section a solvability of equations (3) that determine a rotational part of the interpolatory motion defined by quaternion polynomials $\boldsymbol{q}_{\ell}$ of degree four is revealed. Once a spherical part of the motion is determined, the trajectory

$\boldsymbol{c}$ can be computed using polynomials $\boldsymbol{w}_\ell$, $\ell = 1, 2, \dots, N$, that fulfill (4). In order for $\boldsymbol{w}_\ell$ to be uniquely defined, we must choose them to be of degree seven. Since the final motion would be of degree eight, one can choose also $\boldsymbol{w}_\ell$ as polynomial curves of degree eight which gives additional parameters of freedom that can be used to adjust a shape of the motion. More details on a construction of trajectory $\boldsymbol{c}$ can be found in [16].

4. Analysis of a rotational part of the motion

From the previous section it is clear that the construction of the motion is completely local once all the interpolation data are provided. The analysis of the equations can thus be limited only to one segment of the spline. Let us choose this to be the first segment, i.e., $\ell = 1$. To simplify the notation, the upper indices that denote the segment will be omitted throughout this section.

The unknown quaternion polynomial $\boldsymbol{q}_1$ can be expressed in a Bernstein–Bézier basis as

$$\boldsymbol{q}_1(t) = \sum_{i=0}^{4} \boldsymbol{B}_i b_i^4(t), \quad t \in [0,1],$$

where $\boldsymbol{B}_i$ are control points/quaternions and $b_i^4(t) = \binom{4}{i} t^i (1-t)^{4-i}$ is the i-th Bernstein basis polynomial of degree four. Using some basic properties of Bézier curves, one obtains from (3) the equations

$$\begin{aligned}
&\boldsymbol{B}_0 = \boldsymbol{Q}_0,\\
&\boldsymbol{B}_1 = \boldsymbol{B}_0 + \frac{1}{4}(\varphi_{0,1}\boldsymbol{U}_0 + \lambda_{0,1}\boldsymbol{Q}_0) = \frac{1}{4}\left(\lambda_{0,1} + 4\right)\boldsymbol{Q}_0 + \frac{1}{4}\varphi_{0,1}\boldsymbol{U}_0, \qquad (9)\\
&\boldsymbol{B}_2 = 2\boldsymbol{B}_1 - \boldsymbol{B}_0 + \frac{1}{12}\Big(\lambda_{0,2}\boldsymbol{Q}_0 + 2\lambda_{0,1}\boldsymbol{U}_0\varphi_{0,1} + \varphi_{0,1}^2\boldsymbol{U}_0^{[2]} + \varphi_{0,2}\boldsymbol{U}_0\Big)\\
&= \frac{1}{12}\left(6\lambda_{0,1} + \lambda_{0,2} + 12\right)\boldsymbol{Q}_0 + \frac{1}{12}\left(2\left(\lambda_{0,1} + 3\right)\varphi_{0,1} + \varphi_{0,2}\right)\boldsymbol{U}_0 + \frac{1}{12}\varphi_{0,1}^2\boldsymbol{U}_0^{[2]}, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{B}_3 = 3\boldsymbol{B}_2 - 3\boldsymbol{B}_1 + \boldsymbol{B}_0 + \frac{1}{24}\big(\lambda_{0,3}\boldsymbol{Q}_0 + 3\lambda_{0,2}\varphi_{0,1}\boldsymbol{U}_0 + 3\lambda_{0,1}\varphi_{0,1}^2\boldsymbol{U}_0^{[2]}\\
&+ 3\lambda_{0,1}\varphi_{0,2}\boldsymbol{U}_0 + \varphi_{0,1}^3\boldsymbol{U}_0^{[3]} + 3\varphi_{0,2}\varphi_{0,1}\boldsymbol{U}_0^{[2]} + \varphi_{0,3}\boldsymbol{U}_0\big) \qquad (11)\\
&= \frac{1}{24}\left(18\lambda_{0,1} + 6\lambda_{0,2} + \lambda_{0,3} + 24\right)\boldsymbol{Q}_0 + \frac{1}{24}\big(3\left(4\lambda_{0,1} + \lambda_{0,2} + 6\right)\varphi_{0,1}\\
&+3\left(\lambda_{0,1} + 2\right)\varphi_{0,2} + \varphi_{0,3}\big)\boldsymbol{U}_0 + \frac{1}{8}\varphi_{0,1}\left(\left(\lambda_{0,1} + 2\right)\varphi_{0,1} + \varphi_{0,2}\right)\boldsymbol{U}_0^{[2]} + \frac{1}{24}\varphi_{0,1}^3\boldsymbol{U}_0^{[3]},
\end{aligned}$$

which guarantee that $\boldsymbol{q}_1$ interpolates given quaternion data at the left end point, and the equations

$$\begin{aligned}
&\boldsymbol{B}_4 = \boldsymbol{Q}_1,\\
&\boldsymbol{B}_3 = \boldsymbol{B}_4 - \frac{1}{4}\left(\varphi_{1,1}\boldsymbol{U}_1 + \lambda_{1,1}\boldsymbol{Q}_1\right) = \frac{1}{4}\left(4-\lambda_{1,1}\right)\boldsymbol{Q}_1 - \frac{1}{4}\varphi_{1,1}\boldsymbol{U}_1, && (12)\\
&\boldsymbol{B}_2 = 2\boldsymbol{B}_3 - \boldsymbol{B}_4 + \frac{1}{12}\Big(\lambda_{1,2}\boldsymbol{Q}_1 + 2\lambda_{1,1}\varphi_{1,1}\boldsymbol{U}_1 + \varphi_{1,1}^2\boldsymbol{U}_1^{[2]} + \varphi_{1,2}\boldsymbol{U}_1\Big) && (13)\\
&= \frac{1}{12}\left(-6\lambda_{1,1} + \lambda_{1,2} + 12\right)\boldsymbol{Q}_1 + \frac{1}{12}\left(2\left(\lambda_{1,1}-3\right)\varphi_{1,1} + \varphi_{1,2}\right)\boldsymbol{U}_1 + \frac{1}{12}\varphi_{1,1}^2\boldsymbol{U}_1^{[2]},\\
&\boldsymbol{B}_1 = 3\boldsymbol{B}_2 - 3\boldsymbol{B}_3 + \boldsymbol{B}_4 - \frac{1}{24}\big(\lambda_{1,3}\boldsymbol{Q}_1 + 3\lambda_{1,2}\varphi_{1,1}\boldsymbol{U}_1 + 3\lambda_{1,1}\varphi_{1,1}^2\boldsymbol{U}_1^{[2]}\\
&+ 3\lambda_{1,1}\varphi_{1,2}\boldsymbol{U}_1 + \varphi_{1,1}^3\boldsymbol{U}_1^{[3]} + 3\varphi_{1,2}\varphi_{1,1}\boldsymbol{U}_1^{[2]} + \varphi_{1,3}\boldsymbol{U}_1\big) && (14)\\
&= \frac{1}{24}\left(-18\lambda_{1,1} + 6\lambda_{1,2} - \lambda_{1,3} + 24\right)\boldsymbol{Q}_1 - \frac{1}{24}\left(-3\left(4\lambda_{1,1} - \lambda_{1,2} - 6\right)\varphi_{1,1}\right.\\
&\left.+3\left(\lambda_{1,1}-2\right)\varphi_{1,2} + \varphi_{1,3}\right)\boldsymbol{U}_1 - \frac{1}{8}\varphi_{1,1}\left(\left(\lambda_{1,1}-2\right)\varphi_{1,1} + \varphi_{1,2}\right)\boldsymbol{U}_1^{[2]} - \frac{1}{24}\varphi_{1,1}^3\boldsymbol{U}_1^{[3]}
\end{aligned}$$

for the interpolation at the right end point. Identifying the equations for the inner control points, i.e., (9) and (14), (10) and (13), (11) and (12), we derive three quaternion equations

$$\begin{aligned}
0 =\, & 6\left(\lambda_{0,1}+4\right)\boldsymbol{Q}_0 + \left(18\lambda_{1,1} - 6\lambda_{1,2} + \lambda_{1,3} - 24\right)\boldsymbol{Q}_1\\
& + 6\varphi_{0,1}\boldsymbol{U}_0 + \left(-3\left(4\lambda_{1,1} - \lambda_{1,2} - 6\right)\varphi_{1,1} + 3\left(\lambda_{1,1}-2\right)\varphi_{1,2} + \varphi_{1,3}\right)\boldsymbol{U}_1\\
& + 3\varphi_{1,1}\left(\left(\lambda_{1,1}-2\right)\varphi_{1,1} + \varphi_{1,2}\right)\boldsymbol{U}_1^{[2]} + \varphi_{1,1}^3\boldsymbol{U}_1^{[3]},\\
0 =\, & \left(6\lambda_{0,1} + \lambda_{0,2} + 12\right)\boldsymbol{Q}_0 + \left(6\lambda_{1,1} - \lambda_{1,2} - 12\right)\boldsymbol{Q}_1 + \varphi_{0,1}^2\boldsymbol{U}_0^{[2]} - \varphi_{1,1}^2\boldsymbol{U}_1^{[2]}\\
& + \left(2\left(\lambda_{0,1}+3\right)\varphi_{0,1} + \varphi_{0,2}\right)\boldsymbol{U}_0 - \left(2\left(\lambda_{1,1}-3\right)\varphi_{1,1} + \varphi_{1,2}\right)\boldsymbol{U}_1,\\
0 =\, & \left(18\lambda_{0,1} + 6\lambda_{0,2} + \lambda_{0,3} + 24\right)\boldsymbol{Q}_0 + 6\left(\lambda_{1,1}-4\right)\boldsymbol{Q}_1\\
& + \left(3\left(4\lambda_{0,1} + \lambda_{0,2} + 6\right)\varphi_{0,1} + 3\left(\lambda_{0,1}+2\right)\varphi_{0,2} + \varphi_{0,3}\right)\boldsymbol{U}_0 + 6\varphi_{1,1}\boldsymbol{U}_1\\
& + 3\varphi_{0,1}\left(\left(\lambda_{0,1}+2\right)\varphi_{0,1} + \varphi_{0,2}\right)\boldsymbol{U}_0^{[2]} + \varphi_{0,1}^3\boldsymbol{U}_0^{[3]}.
\end{aligned}$$

Recall assumptions (6) and insert expansions (7) and (8) for $\ell = 1$ and $j = 0, 1$ into these equations. Since a linear combination of linearly independent quaternions can be zero only in the case when all the coefficients in the combination are zero, we obtain twelve scalar equations for twelve unknown parameters,

$$
\begin{aligned}
0 =&3\alpha_{j,j}\varphi_{j,1}\vartheta_j + \beta_{j,j}\varphi_{j,1}^3 + 18\lambda_{j,1} + \lambda_{j,3} + (-1)^j\left(24 + 6\lambda_{j,2}\right),\\
0 =&3\alpha_{j,1-j}\varphi_{j,1}\vartheta_j + \beta_{j,1-j}\varphi_{j,1}^3 + 6\lambda_{1-j,1} + 24(-1)^{j+1},\\
0 =&3\alpha_{j,3-j}\varphi_{j,1}\vartheta_j + \beta_{j,3-j}\varphi_{j,1}^3 + 6\varphi_{1-j,1},\\
0 =&3\alpha_{j,2+j}\varphi_{j,1}\vartheta_j + \beta_{j,2+j}\varphi_{j,1}^3 + 3\varphi_{j,1}\left(6 + \lambda_{j,2} + 4(-1)^j\lambda_{j,1}\right) +\\
&3\left(\lambda_{j,1} + 2(-1)^j\right)\varphi_{j,2} + \varphi_{j,3},\\
0 =&\alpha_{0,j}\varphi_{0,1}^2 - \alpha_{1,j}\varphi_{1,1}^2 + 6\lambda_{j,1} + (-1)^j\left(\lambda_{j,2} + 12\right),\\
0 =&\alpha_{0,2+j}\varphi_{0,1}^2 - \alpha_{1,2+j}\varphi_{1,1}^2 + (-1)^j\left(\varphi_{j,2} + 2\varphi_{j,1}(\lambda_{j,1} + 3(-1)^j)\right),
\end{aligned}
\tag{15}
$$

for $j = 0, 1$, where

$$\vartheta_j := \vartheta_j(\lambda_{j,1}, \varphi_{j,1}, \varphi_{j,2}) := (\lambda_{j,1} + 2(-1)^j)\varphi_{j,1} + \varphi_{j,2}, \qquad j = 0, 1.$$

Although this is a system of nonlinear equations, it turns out that from the first ten equations ten unknowns can be explicitly expressed in terms of $\varphi_{0,1}$ and $\varphi_{1,1}$ as

$$
\begin{aligned}
\lambda_{j,1} =&\frac{1}{6}\left((-1)^{1-j}24 - \varphi_{1-j,1}^3\beta_{1-j,j}\right) + \frac{\alpha_{1-j,j}}{6\,\alpha_{1-j,2+j}}\eta_j,\\
\lambda_{j,2} =&12 + \alpha_{1-j,j}\varphi_{1-j,1}^2 + (-1)^j\beta_{1-j,j}\varphi_{1-j,1}^3 - \alpha_{j,j}\varphi_{j,1}^2 - \frac{(-1)^j\alpha_{1-j,j}}{\alpha_{1-j,2+j}}\eta_j,\\
\lambda_{j,3} =&\frac{\alpha_{j,j}\,\eta_{1-j}}{\alpha_{j,3-j}} + \frac{3\alpha_{1-j,j}\,\eta_j}{\alpha_{1-j,2+j}} + 6(\alpha_{0,j}\varphi_{0,1}^2 - \alpha_{1,j}\varphi_{1,1}^2) - \beta_{j,j}\varphi_{j,1}^3 -\\
&3\beta_{1-j,j}\varphi_{1-j,1}^3 + 24(-1)^{j+1},\\
\varphi_{j,2} =& -\frac{\eta_{1-j}}{3\varphi_{j,1}\alpha_{j,3-j}} - \frac{\varphi_{j,1}\alpha_{1-j,j}\,\eta_j}{6\,\alpha_{1-j,2+j}} + \frac{1}{6}\varphi_{j,1}\left(\beta_{1-j,j}\varphi_{1-j,1}^3 + 12(-1)^j\right),\\
\varphi_{j,3} =&\varphi_{j,1}\Bigg(3\alpha_{j,j}\varphi_{j,1}^2 + 6 - \beta_{j,2+j}\varphi_{j,1}^2 + \frac{1}{12}\beta_{1-j,j}^2\varphi_{1-j,1}^6 - 3\varphi_{1-j,1}^2\alpha_{1-j,j} + (-1)^j\cdot\\
&\beta_{1-j,j}\varphi_{1-j,1}^3\Bigg) + \frac{\varphi_{j,1}\alpha_{1-j,j}^2\,\eta_j^2}{12\alpha_{1-j,2+j}^2} + \frac{\varphi_{j,1}\left((-1)^{j+1}\alpha_{1-j,j}\,\eta_j(6 + (-1)^j\beta_{1-j,j}\varphi_{1-j,1}^3)\right)}{6\alpha_{1-j,2+j}} +\\
&\frac{\alpha_{1-j,j}\,\eta_0\,\eta_1}{6\alpha_{1,2}\alpha_{0,3}\varphi_{j,1}} + \frac{\eta_{1-j}}{6\alpha_{j,3-j}\varphi_{j,1}}\left(6\varphi_{j,1}\alpha_{j,2+j} - \beta_{1-j,j}\varphi_{1-j,j}^3 - 12(-1)^j\right),
\end{aligned}
\tag{16}
$$

for $j = 0, 1$, where

$$\eta_\ell := \eta_\ell(\varphi_{0,1}, \varphi_{1,1}, \beta_{1-\ell,2+\ell}) := 6\varphi_{\ell,1} + \beta_{1-\ell,2+\ell}\,\varphi_{1-\ell,1}^3, \qquad \ell = 0, 1,$$

under the assumption that $\alpha_{0,3} \neq 0, \alpha_{1,2} \neq 0$. Moreover, by inserting the expressions for $\lambda_{j,1}$, $\varphi_{j,2}$, $j = 0, 1$, into (15) one obtains two polynomial equations of total degree five for the remaining two unknowns $\varphi_{0,1}$ and $\varphi_{1,1}$:

$$\begin{aligned}
0 =& \varphi_{1,1}^3 \varphi_{0,1}^2 \alpha_{0,3} \left(\alpha_{1,0}\beta_{1,2} - \alpha_{1,2}\beta_{1,0}\right) - 2\varphi_{0,1}^3 \left(\alpha_{1,2}\beta_{0,3} - 3\alpha_{0,3}(\alpha_{1,0} + \alpha_{0,2}\alpha_{1,2})\right) \\
&- 6\varphi_{1,1}^2 \varphi_{0,1} \alpha_{0,3} \alpha_{1,2}^2 - 12\varphi_{1,1}\alpha_{1,2}, \qquad (17) \\
0 =& \varphi_{1,1}^2 \varphi_{0,1}^3 \alpha_{1,2} \left(\alpha_{0,3}\beta_{0,1} - \alpha_{0,1}\beta_{0,3}\right) + 2\varphi_{1,1}^3 \left(\alpha_{0,3}\beta_{1,2} - 3\alpha_{1,2}(\alpha_{0,1} + \alpha_{0,3}\alpha_{1,3})\right) \\
&+ 6\varphi_{1,1}\varphi_{0,1}^2 \alpha_{0,3}^2 \alpha_{1,2} + 12\varphi_{0,1}\alpha_{0,3}. \qquad (18)
\end{aligned}$$

Recall that we are interested only in finding positive solutions $\varphi_{0,1}$ and $\varphi_{1,1}$. To further simplify equations (17) and (18), we divide them by $\varphi_{1,1}$ and $\varphi_{0,1}$, respectively, and we introduce new unknowns u and v by

$$u := \varphi_{0,1}\varphi_{1,1}, \quad v := \frac{\varphi_{0,1}^3}{\varphi_{1,1}}. \qquad (19)$$

Note that there is a one-to-one correspondence between positive solutions $(\varphi_{0,1}, \varphi_{1,1})$ and positive (u, v). The obtained equations for the unknowns u and v have the advantage that the first equation is linear in the variable v which gives

$$v = \frac{u^2\alpha_{0,3}\left(\alpha_{1,2}\beta_{1,0} - \alpha_{1,0}\beta_{1,2}\right) + 6u\alpha_{1,2}^2\alpha_{0,3} + 12\alpha_{1,2}}{2\alpha_{1,2}\left(3\alpha_{0,2}\alpha_{0,3} - \beta_{0,3}\right) + 6\alpha_{1,0}\alpha_{0,3}} =: V(u). \qquad (20)$$

Moreover, using (20), the second equation reduces to a quartic equation for u

$$\begin{aligned}
0 = & u^4 \left(\alpha_{1,2}\beta_{1,0} - \alpha_{1,0}\beta_{1,2}\right)\left(\alpha_{0,3}\beta_{0,1} - \alpha_{0,1}\beta_{0,3}\right) + 6u^3 \big(\alpha_{1,2}^2 \left(\alpha_{0,3}\beta_{0,1} - \alpha_{0,1}\beta_{0,3}\right) \\
&+ \alpha_{0,3}^2\alpha_{1,2}\beta_{1,0} - \alpha_{1,0}\alpha_{0,3}^2\beta_{1,2}\big) - 4u^2 A + 144u\,\alpha_{1,2}\alpha_{0,3} + 144, \qquad (21)
\end{aligned}$$

where

$$\begin{aligned}
A = & -3\alpha_{0,3}\beta_{1,0} - 3\alpha_{0,2}\alpha_{0,3}\beta_{1,2} - 3\alpha_{1,2}\beta_{0,1} - 3\alpha_{1,2}\alpha_{1,3}\beta_{0,3} - 9\alpha_{1,2}^2\alpha_{0,3}^2 \\
&+ 9\alpha_{0,2}\alpha_{1,2}\alpha_{1,3}\alpha_{0,3} + 9\alpha_{0,1}\alpha_{0,2}\alpha_{1,2} + 9\alpha_{1,0}\left(\alpha_{0,1} + \alpha_{0,3}\alpha_{1,3}\right) + \beta_{1,2}\beta_{0,3}.
\end{aligned}$$

Thus we managed to reduce the starting nonlinear system of twelve equations to a single quartic equation, for which the solutions could be (theoretically) written explicitly using e.g., Ferrari's formula. Of course for numerical computations one can use any of the stable algorithms to compute all the real roots of a degree four polynomial. The obtained results are summarized in the following theorem.

Theorem 1. *Suppose that* (6) *holds true and let* $\alpha^\ell_{1,2} \neq 0$*,* $\alpha^\ell_{0,3} \neq 0$*. If there exists at least one positive solution* u *of* (21) *for which also* $V(u)$ *in* (20) *is positive, then there exists an admissible quartic polynomial* $\boldsymbol{q}_\ell$ *satisfying* (3) *with* $\lambda^\ell_{0,0} = \lambda^\ell_{1,0} = 1$*, where* $\lambda^\ell_{j,i}$ *and* $\varphi^\ell_{j,i}$*,* $j = 0, 1$*,* $i = 1, 2, 3$*, are determined by* (16) *and* (19)*, where in all unknowns the superscript* ℓ *is omitted.*

If the condition (6) is violated, then one can choose any other four linearly independent quaternions from the set $\left\{\boldsymbol{Q}_{\ell+j-1}, \boldsymbol{U}_{\ell+j-1}, \boldsymbol{U}^{[2]}_{\ell+j-1}, \boldsymbol{U}^{[3]}_{\ell+j-1},\ j = 0, 1\right\}$, and express the remaining four similarly as in (7)–(8). The analysis on the existence of the solution is then quite similar and a final equation is of degree four or less. Since there are really many particular cases to be examined, it is impossible to include them all in the paper. Instead, since $\boldsymbol{Q}_{\ell-1} \neq \boldsymbol{Q}_\ell$ by the assumption, one can slightly change the first order derivatives to achieve (6) to be true.

Another assumption in Theorem 1 is that $\alpha^\ell_{1,2} \neq 0$, $\alpha^\ell_{0,3} \neq 0$. If $\alpha^\ell_{1,2} = 0$, then $\boldsymbol{U}^{[2]}_\ell$ lies in a subspace spanned by $\boldsymbol{Q}_{\ell-1}$, $\boldsymbol{Q}_\ell$ and $\boldsymbol{U}_\ell$. Similarly for $\alpha^\ell_{0,3} = 0$, the quaternion $\boldsymbol{U}^{[2]}_{\ell-1}$ lies in a subspace spanned by $\boldsymbol{Q}_{\ell-1}$, $\boldsymbol{Q}_\ell$ and $\boldsymbol{U}_{\ell-1}$. These two particular cases can again be analysed separately which leads to a different final equation of degree four or less if both of the coefficients are zero.

It is quite clear that even under the suppositions of Theorem 1 a positive pair (u, v) of solutions of (20)-(21) does not exist for all possible data configurations. If this is the case one can try to perturb given data or one can insert additional ones in such a way that the positivity of the solution is guaranteed. For this reason we have to find explicit conditions stated only on the given data configurations, but such an analysis seems to be a hard nut to crack due to all the constants involved in (20) and (21). Instead, some sufficient conditions are provided in the next section.

5. Solvability conditions for a rotational part of the motion

In this section, the existence of an admissible G^3 spline interpolant is considered. First, the problem is analysed locally and some sufficient bounds on positions of the curvature quaternions $\boldsymbol{U}^{[3]}_i$ are derived. Since an admissible interpolant does not exist for all possible data configurations we proceed as follows. We require from a user to prescribe only positions, velocity quaternions and the first order curvature quaternions for the spline motion construction, and we show that after some possible slight modifications of the given first order curvature quaternions the remaining curvature quaternions $\boldsymbol{U}^{[3]}_i$ can always be chosen in such a way that an admissible G^3 spline interpolant exists.

5.1. Solvability conditions for one spline segment

Without loss of generality we can analyse the first segment, i.e., $\ell = 1$, only. Again, the upper indices that denote the segment will be omitted. Let us first examine the sufficient conditions for the solution u of (21) to be positive. As the value of a quartic polynomial from (21) is positive for $u = 0$, a negative leading coefficient of this quartic polynomial would imply the existence of at least one positive root. The leading coefficient is negative if

$$B_0 B_1 < 0, \tag{22}$$

where

$$B_0 := \alpha_{0,3}\beta_{0,1} - \alpha_{0,1}\beta_{0,3}, \quad B_1 := \alpha_{1,2}\beta_{1,0} - \alpha_{1,0}\beta_{1,2}. \tag{23}$$

Thus, (22) is a sufficient criteria for a positive solution u of (21) to exist. It remains to examine the positivity of v in (20) after the solution u of (21) is inserted. To exclude the effect of the magnitude of u on the positivity of v, we proceed to find conditions for v to be positive for every real value u. Since $V(u)$ is a quadratic polynomial in a variable u we first require that its discriminant is negative, i.e.,

$$D = 12\alpha_{1,2}\alpha_{0,3}\left(3\alpha_{1,2}^3\alpha_{0,3} - 4B_1\right) < 0. \tag{24}$$

Under this condition, $V(u)$ will be positive for every u iff

$$\operatorname{sign}\left(2\alpha_{1,2}\left(3\alpha_{0,2}\alpha_{0,3} - \beta_{0,3}\right) + 6\alpha_{1,0}\alpha_{0,3}\right) = \operatorname{sign}\left(\alpha_{1,2}\right). \tag{25}$$

One can observe that the conditions (22), (24) and (25) can be satisfied by manipulating only the unknowns $\beta_{j,i}$, $j = 0, 1$, $i = 0, 1, 2, 3$, that correspond to the second order curvature quaternions. To prove this, four different cases depending on the signs of $\alpha_{1,2}$ and $\alpha_{0,3}$ need to be considered. It it straightforward to see that (22) and (24) are fulfilled iff

$$\frac{3}{4}|\alpha_{1,2}\,\alpha_{0,3}|\alpha_{1,2}^2 < \operatorname{sign}(\alpha_{1,2}\alpha_{0,3})B_1, \quad \operatorname{sign}(\alpha_{1,2}\alpha_{0,3})B_0 < 0.$$

By inserting (23) we derive the following two conditions on $\beta_{0,1}, \beta_{0,3}, \beta_{1,0}, \beta_{1,2}$:

$$\begin{aligned}
&\frac{3}{4}\alpha_{1,2}^2|\alpha_{0,3}| + \operatorname{sign}(\alpha_{0,3})\frac{\alpha_{1,0}}{\alpha_{1,2}}\beta_{1,2} < \operatorname{sign}(\alpha_{0,3})\beta_{1,0},\\
&\operatorname{sign}(\alpha_{1,2})\beta_{0,1} < \operatorname{sign}(\alpha_{1,2})\frac{\alpha_{0,1}}{\alpha_{0,3}}\beta_{0,3}.
\end{aligned}$$

Additional condition on $\beta_{0,3}$ follows from (25) as

$$\beta_{0,3} < \frac{3\alpha_{0,3}}{\alpha_{1,2}} \left(\alpha_{1,0} + \alpha_{1,2}\alpha_{0,2}\right).$$

In the next theorem the derived results are summarized and applied to a general spline segment.

Theorem 2. *Suppose that* (6) *holds true and let* $\alpha_{1,2}^{\ell} \neq 0$*,* $\alpha_{0,3}^{\ell} \neq 0$*. If the coefficients* $\beta_{j,i}^{\ell}$*,* $j = 0,1$*,* $i = 0,1,2,3$*, that determine the second order curvature quaternions* (8) *satisfy the inequalities*

$$\widehat{\alpha}_1^{\ell} + \widehat{\alpha}_2^{\ell}\,\beta_{1,2}^{\ell} < \operatorname{sign}(\alpha_{0,3}^{\ell})\,\beta_{1,0}^{\ell}, \tag{26}$$

$$\operatorname{sign}(\alpha_{1,2}^{\ell})\,\beta_{0,1}^{\ell} < \widehat{\alpha}_3^{\ell}\,\beta_{0,3}^{\ell}, \quad \beta_{0,3}^{\ell} < \widehat{\alpha}_4^{\ell}, \tag{27}$$

where

$$\widehat{\alpha}_1^{\ell} := \frac{3}{4}\left(\alpha_{1,2}^{\ell}\right)^2 \left|\alpha_{0,3}^{\ell}\right|, \quad \widehat{\alpha}_2^{\ell} := \operatorname{sign}(\alpha_{0,3}^{\ell})\frac{\alpha_{1,0}^{\ell}}{\alpha_{1,2}^{\ell}},$$

$$\widehat{\alpha}_3^{\ell} := \operatorname{sign}(\alpha_{1,2}^{\ell})\frac{\alpha_{0,1}^{\ell}}{\alpha_{0,3}^{\ell}}, \quad \widehat{\alpha}_4^{\ell} := \frac{3\alpha_{0,3}^{\ell}}{\alpha_{1,2}^{\ell}}\left(\alpha_{1,0}^{\ell} + \alpha_{1,2}^{\ell}\alpha_{0,2}^{\ell}\right),$$

then there exists an admissible G^3 *interpolant* $\boldsymbol{q}_{\ell}$ *of degree four.*

Theorem 2 states simple sufficient conditions on parameters $\beta_{j,i}^{\ell}$, which are such that the conditions for the left and the right second order curvature quaternions of each segment are separated.

5.2. Solvability conditions for the spline

The usual price to be paid for using geometric interpolation methods instead of some linear schemes is to put some limitations on the given data in order to obtain admissible solution. Theorem 2 shows that under some natural restrictions a local G^3 interpolant $\boldsymbol{q}_{\ell}$ exists provided the second order curvature quaternions $\boldsymbol{U}_{\ell-1}^{[3]}, \boldsymbol{U}_{\ell}^{[3]}$ lie in an appropriate region. Unfortunately, each spline segment has its own admissible region and it is not trivial to see whether the intersections are nonempty. Let us examine this problem more precisely. Namely, let us assume that $\boldsymbol{Q}_i$, $\boldsymbol{U}_i$ and $\boldsymbol{U}_i^{[2]}$, $i = 0,1,\ldots,N$, are given, such that (6) and

$$\alpha_{0,3}^{\ell} \neq 0, \quad \alpha_{1,2}^{\ell} \neq 0 \tag{28}$$

hold true for every $\ell = 1, 2, \ldots, N$, and let us consider regions for $\left(\boldsymbol{U}_i^{[3]}\right)_{i=0}^{N}$ that assure the existence of an admissible G^3 spline motion.

Consider the two neighboring segments ℓ and $\ell+1$. The second order curvature quaternion at the middle position $\boldsymbol{Q}_\ell$ can by (8) be expressed in two different ways. Namely as

$$\boldsymbol{U}_\ell^{[3]} = \beta_{1,0}^{\ell} \boldsymbol{Q}_{\ell-1} + \beta_{1,1}^{\ell} \boldsymbol{Q}_\ell + \beta_{1,2}^{\ell} \boldsymbol{U}_{\ell-1} + \beta_{1,3}^{\ell} \boldsymbol{U}_\ell$$

or as

$$\boldsymbol{U}_\ell^{[3]} = \beta_{0,0}^{\ell+1} \boldsymbol{Q}_\ell + \beta_{0,1}^{\ell+1} \boldsymbol{Q}_{\ell+1} + \beta_{0,2}^{\ell+1} \boldsymbol{U}_\ell + \beta_{0,3}^{\ell+1} \boldsymbol{U}_{\ell+1}.$$

The coefficients $\left(\beta_{1,j}^{\ell}\right)_{j=0}^{3}$ and $\left(\beta_{0,j}^{\ell+1}\right)_{j=0}^{3}$ must thus be connected as

$$\left(\beta_{1,0}^{\ell}, \beta_{1,1}^{\ell}, \beta_{1,2}^{\ell}, \beta_{1,3}^{\ell}\right)^\top = C \left(\beta_{0,0}^{\ell+1}, \beta_{0,1}^{\ell+1}, \beta_{0,2}^{\ell+1}, \beta_{0,3}^{\ell+1}\right)^\top, \tag{29}$$

where

$$C := (c_{i,j})_{i,j=1}^{4} = (\boldsymbol{Q}_{\ell-1}, \boldsymbol{Q}_\ell, \boldsymbol{U}_{\ell-1}, \boldsymbol{U}_\ell)^{-1} (\boldsymbol{Q}_\ell, \boldsymbol{Q}_{\ell+1}, \boldsymbol{U}_\ell, \boldsymbol{U}_{\ell+1}).$$

Note that by (7) the same arguments imply

$$\left(\alpha_{1,0}^{\ell}, \alpha_{1,1}^{\ell}, \alpha_{1,2}^{\ell}, \alpha_{1,3}^{\ell}\right)^\top = C \left(\alpha_{0,0}^{\ell+1}, \alpha_{0,1}^{\ell+1}, \alpha_{0,2}^{\ell+1}, \alpha_{0,3}^{\ell+1}\right)^\top. \tag{30}$$

It is straightforward to see that $(c_{i,1})_{i=1}^{4} = (0,1,0,0)^\top$ and $(c_{i,3})_{i=1}^{4} = (0,0,0,1)^\top$. Therefore by (6) also $c_{1,2}c_{3,4} - c_{1,4}c_{3,2} \neq 0$. Recall that in Theorem 2 the quaternion $\boldsymbol{U}_\ell^{[3]}$ is implicitly involved only in inequality (26) for the left hand side segment and in inequalities (27) for the right hand side one. Using (29) this gives

$$\widehat{\alpha}_1^{\ell} + \widehat{\alpha}_2^{\ell} \left(c_{3,2}\beta_{0,1}^{\ell+1} + c_{3,4}\beta_{0,3}^{\ell+1}\right) < \operatorname{sign}(\alpha_{0,3}^{\ell}) \left(c_{1,2}\beta_{0,1}^{\ell+1} + c_{1,4}\beta_{0,3}^{\ell+1}\right), \tag{31}$$

$$\operatorname{sign}(\alpha_{1,2}^{\ell+1})\, \beta_{0,1}^{\ell+1} < \operatorname{sign}(\alpha_{1,2}^{\ell+1}) \frac{\alpha_{0,1}^{\ell+1}}{\alpha_{0,3}^{\ell+1}}\, \beta_{0,3}^{\ell+1}, \qquad \beta_{0,3}^{\ell+1} < \widehat{\alpha}_4^{\ell+1}. \tag{32}$$

Relations (31) and (32) represent three inequalities for $\beta_{0,1}^{\ell+1}$ and $\beta_{0,3}^{\ell+1}$. Let us consider them more precisely. First note that by (30) we obtain, among others, the following relation

$$\begin{pmatrix} c_{1,2} & c_{1,4} \\ c_{3,2} & c_{3,4} \end{pmatrix} \begin{pmatrix} \alpha_{0,1}^{\ell+1} \\ \alpha_{0,3}^{\ell+1} \end{pmatrix} = \begin{pmatrix} \alpha_{1,0}^{\ell} \\ \alpha_{1,2}^{\ell} \end{pmatrix}. \tag{33}$$

Inequality (31) can be rewritten to

$$\frac{\operatorname{sign}(\alpha_{0,3}^{\ell})}{\alpha_{1,2}^{\ell}} \left(\beta_{0,1}^{\ell+1} \left(c_{3,2}\alpha_{1,0}^{\ell} - c_{1,2}\alpha_{1,2}^{\ell}\right) + \beta_{0,3}^{\ell+1} \left(c_{3,4}\alpha_{1,0}^{\ell} - c_{1,4}\alpha_{1,2}^{\ell}\right)\right) + \widehat{\alpha}_1^{\ell} < 0$$

and further by using (33) to

$$\sigma_1 \left(\alpha_{0,3}^{\ell+1}\beta_{0,1}^{\ell+1} - \alpha_{0,1}^{\ell+1}\beta_{0,3}^{\ell+1} + \nu\right) < 0, \quad \nu := \frac{3\,(\alpha_{1,2}^{\ell})^3\alpha_{0,3}^{\ell}}{4\,(c_{1,4}c_{3,2} - c_{1,2}c_{3,4})} \neq 0, \tag{34}$$

where

$$\sigma_1 = \operatorname{sign}(\alpha_{0,3}^{\ell})\operatorname{sign}(\alpha_{1,2}^{\ell})\operatorname{sign}(c_{1,4}c_{3,2} - c_{1,2}c_{3,4}).$$

Moreover, the first inequality in (32) simplifies to

$$\sigma_2 \left(\alpha_{0,3}^{\ell+1}\beta_{0,1}^{\ell+1} - \alpha_{0,1}^{\ell+1}\beta_{0,3}^{\ell+1}\right) < 0, \quad \sigma_2 = \operatorname{sign}(\alpha_{0,3}^{\ell+1})\operatorname{sign}(\alpha_{1,2}^{\ell+1}). \tag{35}$$

By (34) and (35) one observes that the lines in $\left(\beta_{0,1}^{\ell+1}, \beta_{0,3}^{\ell+1}\right)$–plane that determine the boundaries of both regions are parallel, but by (28) not parallel to the abscissa line $\beta_{0,3}^{\ell+1} = 0$. Also one of them is passing through the origin $(\beta_{0,1}^{\ell+1}, \beta_{0,3}^{\ell+1})^\top = (0,0)^\top$ (see Fig. 1, left). Since the last region given by the second inequality in (32) is bounded by the horizontal line $\beta_{0,3}^{\ell+1} = \widehat{\alpha}_4^{\ell+1}$, an empty intersection of all the three regions could happen only if

$$\sigma_1 = 1,\, \sigma_2 = -1,\, \nu > 0 \quad \text{or} \quad \sigma_1 = -1,\, \sigma_2 = 1,\, \nu < 0. \tag{36}$$

Since it is straightforward to see that $\operatorname{sign}(\nu) = \sigma_1$, one has to assure $\sigma_1 = \sigma_2$ in order to avoid undesired situations (36). More precisely, the following condition has to be fulfilled

$$\operatorname{sign}(\alpha_{0,3}^{\ell})\operatorname{sign}(\alpha_{1,2}^{\ell})\operatorname{sign}(c_{1,4}c_{3,2} - c_{1,2}c_{3,4}) = \operatorname{sign}(\alpha_{0,3}^{\ell+1})\operatorname{sign}(\alpha_{1,2}^{\ell+1}). \tag{37}$$

This proves that if the condition (37) is fulfilled, we can always find appropriate $\beta_{0,1}^{\ell+1}$ and $\beta_{0,3}^{\ell+1}$, while parameters $\beta_{0,0}^{\ell+1}$ and $\beta_{0,2}^{\ell+1}$ are completely free to be chosen. In the case (37) does not hold true, we first have to modify some of the parameters $\alpha_{i,j}^{\ell}$ and $\alpha_{i,j}^{\ell+1}$, which implies modification of the first curvature quaternions. In order to preserve the locality, we can only modify parameters $\alpha_{1,i}^{\ell}$ and $\alpha_{0,i}^{\ell+1}$, $i \in \{0,1,2,3\}$. The whole procedure is presented in Algorithm 1.

For the choice of the parameters in Algorithm 1, we propose the following procedure that provides good numerical results. We choose $\beta_{0,0}^{\ell+1} = \beta_{0,2}^{\ell+1} = 0$ for $\ell = 0,1,\ldots,N-1$, and $\beta_{1,1}^{N} = \beta_{1,3}^{N} = 0$. In Line 10 of the algorithm there are two possible regions, determined by (31) and (32), where the point $\left(\beta_{0,1}^{\ell+1}, \beta_{0,3}^{\ell+1}\right)^\top$ can be chosen from (points P_1 and P_2 in Fig. 1), and the appropriate one is determined by the sign of $\sigma_1 = \sigma_2$. We then choose it on the corresponding angle bisector with the distance γ from the intersection of the two lines determining the appropriate

Algorithm 1 Computation of second order curvature quaternions.

Input: Quaternions $\boldsymbol{Q}_i, \boldsymbol{U}_i, \boldsymbol{U}_i^{[2]}$, $i = 0, 1, \ldots, N$
Output: Quaternions $\boldsymbol{U}_i^{[2]}$, $i = 1, 2, \ldots, N-1$, and $\boldsymbol{U}_i^{[3]}$, $i = 0, 1, \ldots, N$

1: Choose arbitrary $\beta^1_{0,0}$, $\beta^1_{0,2}$;
2: Choose $\beta^1_{0,1}$, $\beta^1_{0,3}$ that satisfy (27);
3: Compute $\boldsymbol{U}_0^{[3]} = \beta^1_{0,0}\boldsymbol{Q}_0 + \beta^1_{0,1}\boldsymbol{Q}_1 + \beta^1_{0,2}\boldsymbol{U}_0 + \beta^1_{0,3}\boldsymbol{U}_1$;
4: **for** $\ell = 1, 2, \ldots, N-1$ **do**
5:     **if** condition (37) is not fulfilled **then**
6:         Modify parameters $\alpha^{\ell}_{1,i}$ and $\alpha^{\ell+1}_{0,i}$, $i \in \{0, 1, 2, 3\}$, in order to fulfill (37);
7:         Modify $\boldsymbol{U}_\ell^{[2]} = \alpha^{\ell+1}_{0,0}\boldsymbol{Q}_\ell + \alpha^{\ell+1}_{0,1}\boldsymbol{Q}_{\ell+1} + \alpha^{\ell+1}_{0,2}\boldsymbol{U}_\ell + \alpha^{\ell+1}_{0,3}\boldsymbol{U}_{\ell+1}$;
8: **for** $\ell = 1, 2, \ldots, N-1$ **do**
9:     Choose arbitrary $\beta^{\ell+1}_{0,0}$ and $\beta^{\ell+1}_{0,2}$
10:     Choose $\beta^{\ell+1}_{0,1}$ and $\beta^{\ell+1}_{0,3}$ that satisfy (31) and (32);
11:     Compute $\boldsymbol{U}_\ell^{[3]} = \beta^{\ell+1}_{0,0}\boldsymbol{Q}_\ell + \beta^{\ell+1}_{0,1}\boldsymbol{Q}_{\ell+1} + \beta^{\ell+1}_{0,2}\boldsymbol{U}_\ell + \beta^{\ell+1}_{0,3}\boldsymbol{U}_{\ell+1}$;
12: Choose arbitrary $\beta^N_{1,1}$, $\beta^N_{1,3}$;
13: Choose $\beta^N_{1,0}$, $\beta^N_{1,2}$ that satisfy (26);
14: Compute $\boldsymbol{U}_N^{[3]} = \beta^N_{1,0}\boldsymbol{Q}_{N-1} + \beta^N_{1,1}\boldsymbol{Q}_N + \beta^N_{1,2}\boldsymbol{U}_{N-1} + \beta^N_{1,3}\boldsymbol{U}_N$;

region (see Fig. 1, left). In practical applications, parameter γ could be determined using some optimization or minimization approach. Perhaps, one could minimize the average of the stretch and the band energies of the quaternion curves (see [16], e.g.). Alternatively, one could try to minimize the perturbation of the computed and the user-given second order curvature quaternions. Similarly we proceed in Line 2 of the algorithm (see Fig. 1, right). In Line 13 of the algorithm we replace the role of the angle bisectors by the line perpendicular to the line determining (26) through the intersection with the ordinate line and two possible candidates are chosen at distance 1 from the intersection point. It remains only to explain how to modify parameters $\alpha^{\ell}_{i,j}$ and $\alpha^{\ell+1}_{i,j}$ in Line 6 of the algorithm. One of the possible ways is the following:

$$(\alpha^{\ell+1}_{0,3})_{\text{new}} := -\alpha^{\ell+1}_{0,3}, \quad (\alpha^{\ell}_{1,i})_{\text{new}} := (\alpha^{\ell}_{1,i}), \quad i = 1, 2, 3,$$

and all the remaining parameters

$$(\alpha^{\ell}_{1,0})_{\text{new}}, \; (\alpha^{\ell+1}_{0,0})_{\text{new}}, \; (\alpha^{\ell+1}_{0,1})_{\text{new}} \;\text{ and }\; (\alpha^{\ell+1}_{0,2})_{\text{new}}$$

change accordingly to (30).

6. Numerical examples

In this section some numerical examples of spline motions of degree eight are presented.

Example 1 Suppose that we are given a smooth motion defined by the quaternion curve $\widetilde{\boldsymbol{q}}$,

$$\widetilde{\boldsymbol{q}} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}, \quad \boldsymbol{q}(s) = \left(s^2+1, 3\sin\left(\frac{\pi s}{4}\right), 2\cos\left(\frac{\pi s}{4}\right), \frac{1}{2}\sqrt{s^2+1}\right)^\top. \tag{38}$$

The orientations and the corresponding derivative data are sampled as

$$\boldsymbol{Q}_i = \widetilde{\boldsymbol{q}}(s_i), \; \boldsymbol{U}_i = \widetilde{\boldsymbol{q}}'(s_i), \; \boldsymbol{U}_i^{[j]} = \widetilde{\boldsymbol{q}}^{(j)}(s_i), \; j = 2, 3. \tag{39}$$

where $s_i = 0, 1, \ldots, 5$. On each spline segment we first solve the quartic equation (21) for u. In this case, we obtain at least one positive solution for every pair of neighboring quaternion data. If there exists more than one admissible solution, u is chosen such that the length of the corresponding quaternion interpolant is minimal. Solving (20), we obtain also positive v on each spline segment. We consider the substitution (19) and from (16) we compute the remaining ten unknowns that determine the control quaternions. Fig. 2 (left) shows the obtained spherical motion of degree eight. The interpolating orientations are denoted by bold cuboids. The trajectory $\mathcal{R}\hat{\boldsymbol{p}}$ of a cuboid center $\hat{\boldsymbol{p}}$, which is a spherical rational curve of degree eight lying on the sphere centered at the origin with radius $\|\hat{\boldsymbol{p}}\|$, is shown in Fig. 2 (right) together with the center trajectory of the original motion (thin curve). Both trajectories almost coincide.

Example 2 Let the data again be sampled from (38) as $\boldsymbol{Q}_i, \boldsymbol{U}_i$ and $\boldsymbol{U}_i^{[2]}$ given by (39) for $i = 0, 1, \ldots, 5$, and let us determine the second order curvature quaternions by Algorithm 1, considering the procedure proposed in the previous section. At the second and fourth position the inequality (37) is not satisfied and $\boldsymbol{U}_1^{[2]}$ and $\boldsymbol{U}_3^{[2]}$ are modified in order to fulfill this condition. We choose $\gamma = 230$ and compute the unknown $\boldsymbol{U}_i^{[3]}$, $i = 0, 1, \ldots, 5$. The corresponding spherical motion of degree eight is shown in Fig. 3 (left) where six interpolating orientations are denoted by bold cuboids. In Fig. 3 (right), a comparison between center trajectories of the original (thin curve) and the obtained G^3 (thick curve) motion, is given.

Example 3 Now, suppose that the interpolation data are listed in the Table 1. To determine the second order curvature quaternions we use Algorithm 1 and the procedure that is proposed in the previous section. The parameter γ is set to 1200. The obtained spherical G^3 motion is shown in Fig. 4 (left), where interpolation orientations are presented by bold cuboids. The corresponding center trajectory is presented in Fig. 4 (right).

Example 4 Finally, let us consider a case where only the orientations and velocity quaternions are given, without curvature quaternions of the first and second order. To construct G^3 spline motion, the remaining derivative data must be estimated from the input data, e.g., by using local polynomials of degree nine. In Table 2, eight positions are listed. The required curvature quaternions $\boldsymbol{U}_i^{[2]}$ and $\boldsymbol{U}_i^{[3]}$, $i = 0, 1, \ldots, 7$, are taken from normalized local polynomials of degree nine, passing through five consecutive orientations and velocity quaternions. Five consecutive positions are considered in order to preserve the symmetry. We use centripetal parameterization to construct local polynomials and on each spline segment we obtain at least one positive pair (u, v), determined by (20) and (21). The obtained curvature data are listed in Table 3. The corresponding spherical motion of degree eight, composed of seven segments (left) and the interpolation orientations (right) are shown in Fig. 5.

7. Conclusion

The paper presents an interpolation scheme to construct a G^3 continuous rational spline motion of degree eight that interpolates eight motion data on each polynomial segment. In contrast to standard interpolation methods, where the parameterization is prescribed in advance, geometric approach allows us to significantly lower the degree of a rotational part of the motion, described here by quaternion polynomials. In particular, the existence of a quaternion polynomial of degree four that interpolates eight quaternions, which represent positions and derivative information up to the order three, is examined and the problem is reduced to the analysis of one quartic equation. Sufficient conditions for an admissible solution to exist are given, that can be used completely locally in a spline setting. An algorithm for modifying curvature quaternions to have the existence of a solution guaranteed is proposed and numerical examples that illustrate the theoretical results are presented.

The analysis of the approximation order, which is expected to be eight, optimization approaches in Algorithm 1 and the determination of free parameters in a translational part of the motion could be interesting topics for future research.

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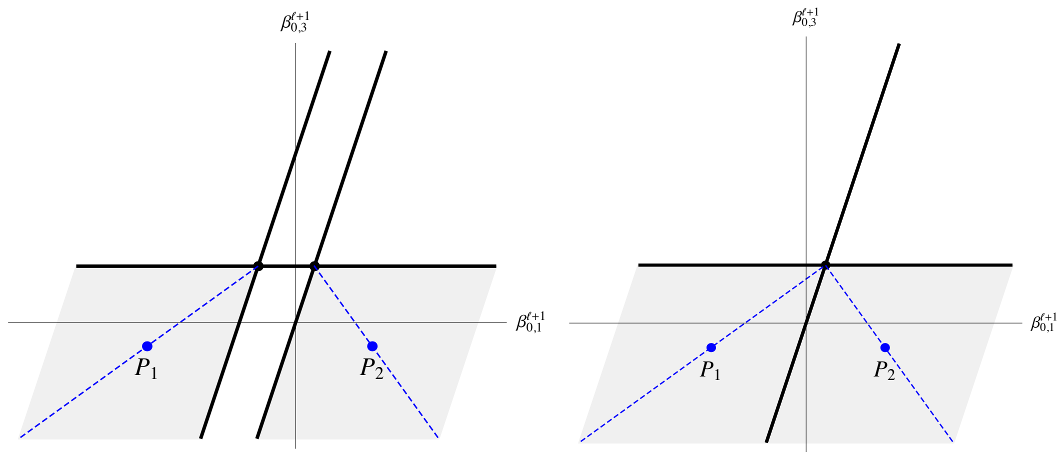

Figure 1: Three lines determining regions in (31) and (32) split the plane into six domains, among which only two are possible candidates where the point $\left(\beta_{0,1}^{\ell+1}, \beta_{0,3}^{\ell+1}\right)^\top$ could be chosen from (left). For the first segment only two lines determine regions in (27) (right).

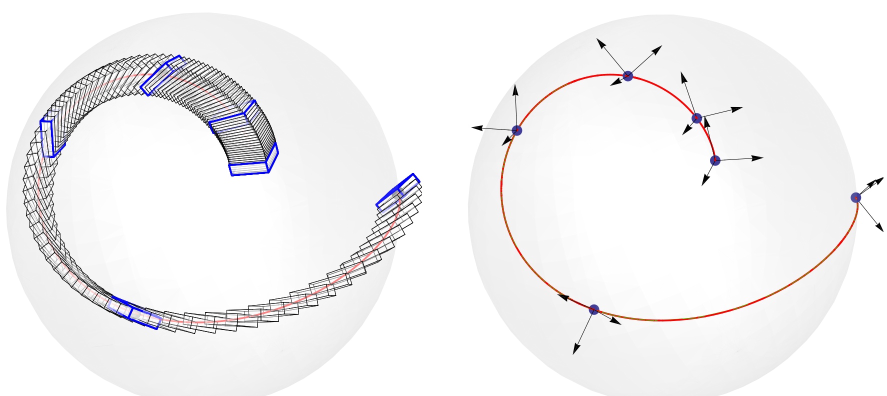

Figure 2: Six orientations of a cuboid interpolated by a spherical motion of degree eight (left) and two trajectories of cuboid center (right), one of the original (thin curve) and one of the obtained G^3 (thick curve) motion.

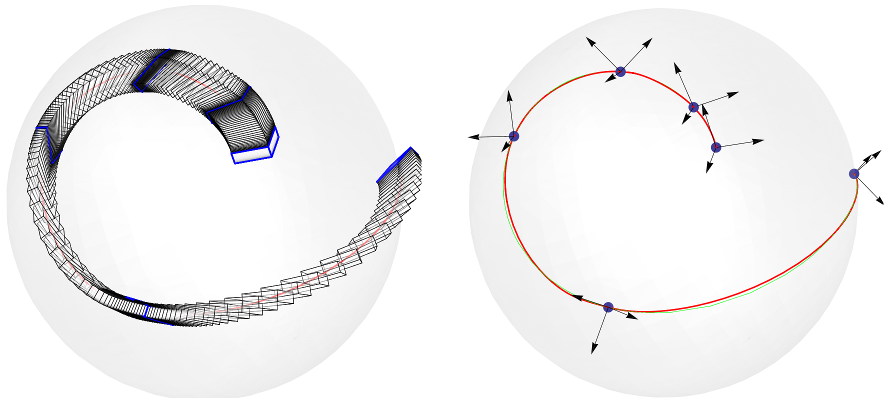

Figure 3: Six orientations of a cuboid interpolated by a spherical motion of degree eight (left) obtained by Algorithm 1 and two trajectories of cuboid center (right), one of the original (thin curve) and one of the obtained G^3 (thick curve) motion.

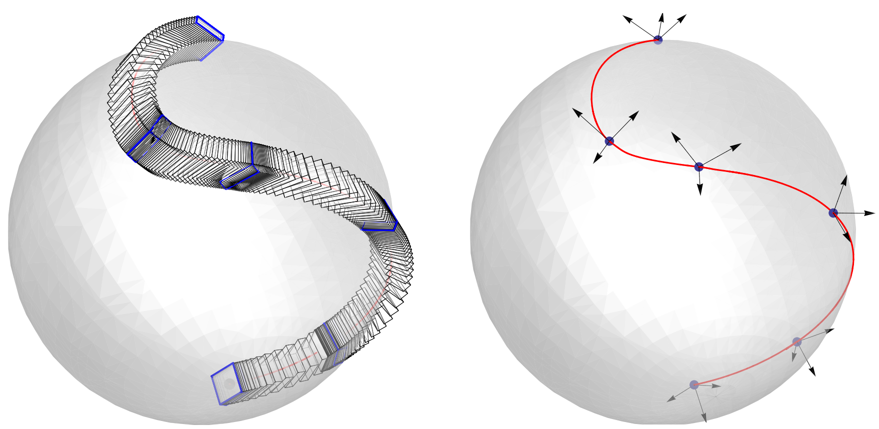

Figure 4: Six orientations of a cuboid interpolated by a spherical motion of degree eight (left) obtained by Algorithm 1 and the corresponding trajectory of a cuboid center (right).

i	$\boldsymbol{Q}_i^\top$	$\boldsymbol{U}_i^\top$	$\left(\boldsymbol{U}_i^{[2]}\right)^\top$
0	(0.82045, 0.54697, 0.13674, 0.094782)	(-1.5, 2.4, -0.26, -0.18)	(-1.353, -13.485, 1.500, 0.039)
1	(0.33602, 0.92828, 0.15154, 0.049157)	(-0.45, 0.13, 0.21, -0.023)	(1.139, -0.787, 0.492, 0.151)
2	(0.19607, 0.92430, 0.32350, 0.050772)	(-0.18, -0.14, 0.51, 0.023)	(0.174, -0.654, 0.797, 0.065)
3	(0.10799, 0.72020, 0.68193, 0.068048)	(-0.20, -0.80, 0.88, 0.036)	(-0.180, -2.024, 0.037, -0.059)
4	(0, 0.15760, 0.98498, 0.070594)	(-0.19, -1.1, 0.18, -0.029)	(0.289, 1.450, -1.552, -0.085)
5	(-0.05841, -0.18604, 0.97933, 0.05368)	(-0.06, -0.30, -0.058, -0.03)	(0.155, 1.255, 0.148, 0.033)

Table 1: Six given orientations, i.e., unit quaternions and the corresponding derivative data.

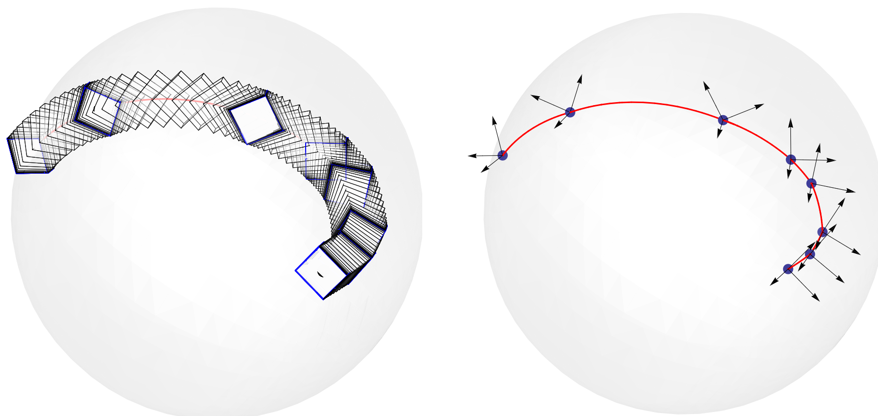

Figure 5: Eight orientations of a cuboid interpolated by spherical spline motion of degree eight (left) and the corresponding trajectory of a cuboid center (right).

i	$\boldsymbol{Q}_i^\top$	$\boldsymbol{U}_i^\top$
0	(0.224851, 0.618311, 0.399232, 0.638551)	(0.530909, -0.443629, 0.930039, -0.338856)
1	(0.333866, 0.495794, 0.586485, 0.546587)	(0.348036, -0.507894, 0.57974, -0.373949)
2	(0.464081, 0.227317, 0.776956, 0.359577)	(0.13301, -0.370011, 0.140488, -0.241315))
3	(0.520945, 0.0580751, 0.812909, 0.253817)	(0.0849161, -0.253027, 0.0112832, -0.152529)
4	(0.549005, -0.0205116, 0.80963, 0.206571)	(0.0769946, -0.197804, -0.0266713, -0.119735)
5	(0.6176956, -0.1455922, 0.7632878, 0.1210241)	(0.0778027, -0.0841554, -0.0667376, -0.0774288)
6	(0.664775, -0.175622, 0.721838, 0.0786243)	(0.0781429, -0.0166773, -0.0689334, -0.0650893)
7	(0.708952, -0.16732, 0.683796, 0.0425932)	(0.0697368, 0.0432317, -0.0581471, -0.05742)

Table 2: Eight given orientations, i.e., unit quaternions and the corresponding velocity quaternions.

i	$\left(\boldsymbol{U}_i^{[2]}\right)^\top$	$\left(\boldsymbol{U}_i^{[3]}\right)^\top$
0	(-266.86, 180.961, -468.308, 132.156)	(-266.86, 180.961, -468.308, 132.156)
1	(-46.933, 20.7791, -89.6946, 18.2155)	(-46.933, 20.7791, -89.6946, 18.2155)
2	(16.3496, -32.5486, 23.7482, -23.5737)	(16.3496, -32.5486, 23.7482, -23.5737))
3	(16.8544, 53.0396, 4.92389, 31.925)	(-16.8544, 53.0396, 4.92389, 31.925)
4	(17.3494, -19.3385, -14.6186, -15.7197)	(17.3494, -19.3385, -14.6186, -15.7197)
5	(41.165, -63.1912, -35.0628, -50.6554)	(41.165, -63.1912, -35.0628, -50.6554)
6	(17.9138, -5.3353, -15.6819, -16.1321)	(17.9138, -5.3353, -15.6819, -16.1321)
7	(-0.12887, 1.04352, 0.357228, 0.0990213)	(27.3592, 16.9671, -22.7922, -23.6025)

Table 3: Eight additional derivative data, i.e., curvature quaternions of the first and second order, estimated by using normalized local polynomials of degree nine.