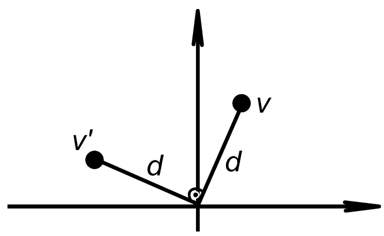
6 Homework 6 (Change of basis and similarity)

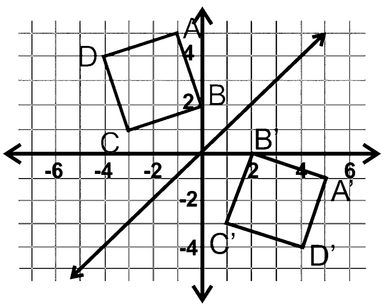
1. Let R_{90} denote rotation of $90°$ with centre of rotation in origin $(0,0)$, so that point $v \in \mathbb{R}^2$ is mapped to point $v' \in \mathbb{R}^2$ (as is illustrated at figure right).

(a) Find coordinates of R_{90} with respect to standard basis.

(b) Determine what is rotation of point $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $90°$ about origin.

c) Find coordinates of R_{90} with respect to basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.





2. Let T denote linear operator on $\mathbb{R}^2$ which is reflection symmetry about line $y = x$ (for illustration what is reflection symmetry about line $y = x$ see $T(\square ABCD) = \square A'B'C'D'$ on figure left).

(a) Find coordinate matrix of T with respect to the standard basis.

(b) Compute $T(v)$, if we have that $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

(c) Find coordinate matrix representation of T with respect to basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

3. Let T denote linear operator defined on space $\mathbb{R}^2$ which first rotate vector for angle $\pi/3$ around origin in positive direction, and after that do reflection symmetry about line $y = x$. Find coordinate matrix representation of T with respect to basis $\mathcal{B} = \{(1,1)^\top, (1,-1)^\top\}$ (in another words find $[T]_\mathcal{B}$). Find coordinates of vector $T(v)$ with respect to same basis $\mathcal{B}$, where v is arbitrary element from $\mathbb{R}^2$.

4. Let T denote linear operator defined on space $\mathbb{R}^2$ which do three things: first make reflection symmetry about line $y = -x$, then do rotation for angle $\frac{\pi}{4}$ around origin in negative direction, and finally make reflection symmetry about line $y = x$. Find coordinate matrix representation of T with respect to the basis $\mathcal{B} = \left\{ 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

5. Let

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

be coordinate matrix representation of $T : \mathcal{V}^2(0) \to \mathcal{V}^2(0)$ with respect to the canonical basis $\left\{ \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Find coordinate matrix representation of T with respect to the basis $\{\vec{i} + 2\vec{j}, \vec{i} + 3\vec{j}\}$. Does there exists a vector $\vec{v} \in \mathcal{V}^2(0)$ such that $T(\vec{v}) = 3\vec{i} + 5\vec{j}$?

6. **(Challenge)** For $A \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ let λ_1, λ_2 and λ_3 denote three different real numbers such that

$$p(x) = \det(A - xI) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

(i) Show that there exist nonzero real numbers c_1, c_2 and c_3 such that

$$c_1(x - \lambda_2)(x - \lambda_3) + c_2(x - \lambda_1)(x - \lambda_3) + c_3(x - \lambda_1)(x - \lambda_2) = 1 \tag{1}$$

holds.

(ii) Let c_1, c_2 and c_3 be nonzero real numbers for which (1) holds. Define matrices S_1, S_2 and S_3 on the following way

$$S_1 = c_1(A - \lambda_2 I)(A - \lambda_3 I), \qquad S_2 = c_2(A - \lambda_1 I)(A - \lambda_3 I), \qquad S_3 = c_3(A - \lambda_1 I)(A - \lambda_2 I).$$

Show that

(a) $\dim \mathrm{im}(S_i) = 1$ for all i $(1 \le i \le 3)$.
(b) $\mathbb{R}^3 = \mathrm{im}(S_1) + \mathrm{im}(S_2) + \mathrm{im}(S_3)$.
(c) $\forall w \in \mathrm{im}(S_i)$ we have $Aw = \lambda_i w$ $(1 \le i \le 3)$.