3 Homework 3 (Linear independence)

1. Without doing any computation, determine whether the following matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is singular or nonsingular

$$\begin{pmatrix} n & 1 & 1 & \dots & 1 \\ 1 & n & 1 & \dots & 1 \\ 1 & 1 & n & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & n \end{pmatrix}.$$

2. Let $\mathcal{L} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0\right\}$. Find a linearly independent spanning set for a vector space $\mathcal{L}$.

3. Let $\mathcal{V} = \mathbb{R}^n$ and let $(a_1, a_2, ..., a_n)^\top$ be some fixed vector from $\mathcal{V}$ and let

$$\mathcal{M} = \{(x_1, x_2, ..., x_n)^\top \in \mathcal{V} \,|\, a_1x_1 + ... + a_nx_n = 0\}$$

be subspace of $\mathcal{V}$. Find a maximal linearly independent subset of $\mathcal{M}$.

4. Let $\mathbb{R}^+$ denote given vector space (set of all positive real numbers) over field $\mathbb{R}$, in which operations vector addition and scalar multiplication are defined on the following way

$$\text{vector addition: } \forall u, v \in \mathbb{R}^+ \quad u + v = uv;$$

$$\text{scalar multiplication: } \forall u \in \mathbb{R}^+, \quad \forall \alpha \in \mathbb{R} \quad \alpha u = u^\alpha.$$

Find minimal spanning set for $\mathcal{V}$. Explain your answer.

5. Let $\mathcal{M}$ and $\mathcal{N}$ denote subspaces of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, where

$$\mathcal{M} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a - 2b = 0,\ a + c + d = 0\right\} \quad \text{and}$$

$$\mathcal{N} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + c = 0,\ a - 2b + d = 0\right\}.$$

Find a linearly independent spanning set for $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} + \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$.

6. **(IMC 2016.)** Let k and n be positive integers. A sequence $(A_1, ..., A_k)$ of $n \times n$ real matrices is preferred by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.