KAIST 수리과학과

Constructive algorithm for path-width of matroids (and more)

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Historical backgrounds

Determining tree-width of graphs

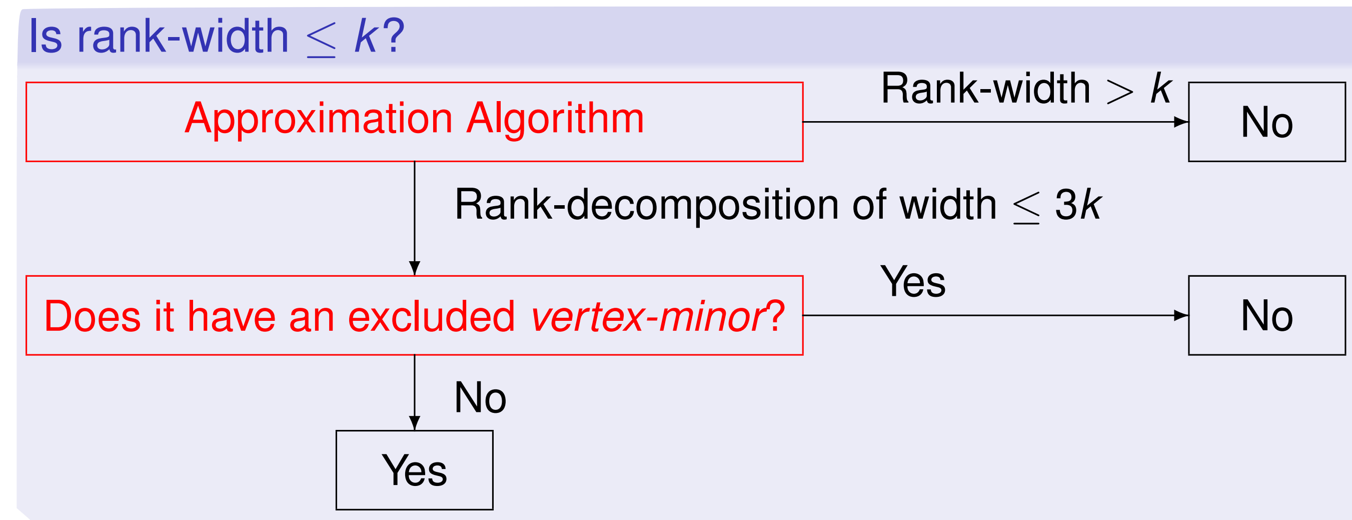
- 1987: $O(n^k)$ alg. (Arnborg, Corneil, and Proskurowski)
- 1995: $O(n^2)$ alg. (Robertson and Seymour) (GM XIII)
    - Step 1: find an **"approximate"** tree-decomp. of width$\leq$f(k)
    - **Step 2**: test forbidden minors for tree-width$\leq$k
    - 1994 "Self-reduction" technique (Fellows and Langston) → one can construct such an algorithm without knowing the complete list of forbidden minors
- **1996** (Bodlaender and Kloks)
    - **Dynamic Programming algorithm to do Step 2 without using forbidden minors**

Similar method for path-width of graphs (dynamic programming)

This finds a path-decomposition or a tree-decomposition as well.

Determining rank-width of graphs

- 2006 (O., Seymour): $O(n^9 \log n)$ algorithm to find an "**approximate**" rank-decomp. of width≤f(k)
  …Step 1 (Improved to $O(n^3)$ by O. 2008)
- 2005 (O.): #forbidden vertex-minors is finite for each k
- 2007 (Courcelle, O.): testing forbidden vertex-minors for graphs of bounded rank-width … **Step 2**
- 2008 (Hlineny, O.): constructing a rank-decomp of width≤k by using algorithms based on forbidden minors (using matroids)



Q1: **Bodlaender-Kloks type algorithm** to find a rank-decomposition of width≤k?

Do we have a …
Q2: **Constructive** algorithm to decide linear rank-width≤k for fixed k?

Our answer: Yes to Q1 and Q2

Solvable problems when rank-width is bounded (I)

Courcelle, Makowsky, and Rotics '00

Every graph problem expressible in
monadic second-order logic formula (with no edge-set variables)
is solvable in time $O(n^3)$
for graphs having rank-width at most k for fixed k.

CMR'00: Minimize $w(X)$ satisfying $\varphi(X)$ for graphs of bounded rank-width.
CMR'01: Counting the number of true assignments in polynomial time.
(assuming unit time for arithmetic operations on $\mathbb{R}$.)

Can I find a partition of vertices into three subsets such that each set has no edges inside? (graph 3-coloring problem)

$$\begin{aligned}\exists X_1 \exists X_2 \exists X_3 \forall v \forall w (v, w \in X_1 \Rightarrow \neg \operatorname{adj}(v, w))\\ \wedge \forall v \forall w (v, w \in X_2 \Rightarrow \neg \operatorname{adj}(v, w))\\ \wedge \forall v \forall w (v, w \in X_3 \Rightarrow \neg \operatorname{adj}(v, w)) \cdots\end{aligned}$$

Solvable prob

Many other proble
solved in polynom

- Finding a chr
- Deciding whe
- Given a mon
  there is a par
  satisfied for a

All of these algorithms

- need the rank-decomposition of width $\leq k$ as an input, and
- use the dynamic programming.

Arranging vectors — Alternative view of path-width of matroids

Arranging vectors: A problem in coding theory

- **Input**: n vectors in $\mathbb{F}^m$.
- **Goal**: Find the minimum k such that there is a **linear layout** $v_1, v_2, \ldots, v_n$ of the **vectors** such that
  $\dim(\langle v_1\rangle \cap \langle v_2,v_3,\ldots,v_n\rangle) \leq k$,
  $\dim(\langle v_1,v_2\rangle \cap \langle v_3,\ldots,v_n\rangle) \leq k$,
  …
  $\dim(\langle v_1,v_2,v_3,\ldots,v_{n-1}\rangle \cap \langle v_n\rangle) \leq k$.
- Minimum such k = "**Trellis State-Complexity** of a linear code" or "**Trellis-Width**" (coding theory) or "**Path-width**" (matroid theory) — matroid representable over $\mathbb{F}$

Computing trellis-width (or path-width)

- Deciding trellis-width$\leq$k is NP-complete (Kashyap07, 08)
- What if k is fixed? — Remark in Kashyap's paper (07)

4 Concluding Remarks

The main contribution of this paper was to show that the decision problem TRELLIS STATE-COMPLEXITY is NP-complete, thus settling a long-standing conjecture. Now, the situation is rather different if we consider a variation of the problem in which the integer w is *not* taken to be a part of the input to the problem. In other words, consider the following problem:

Problem: WEAK TRELLIS STATE-COMPLEXITY

Let $\mathbb{F}_q$ be a fixed finite field, and let w be a fixed positive integer.

Instance: An $m \times n$ generator matrix for a linear code $\mathcal{C}$ over $\mathbb{F}_q$.

Question: Is there a coordinate permutation of $\mathcal{C}$ that yields a code $\mathcal{C}'$ whose minimal trellis has state-complexity at most w?

There is good reason to believe that this problem is solvable in polynomial time.

For fixed k?

- **WQO(Well-quasi-ordering) Conjecture**: $\mathbb{F}$-representable matroids are well-quasi-ordered by the minor relation. (no infinite antichain w.r.t. minors)
- If true, then for every class X of $\mathbb{F}$-representable matroids closed under taking minors, there are **finitely** many $\mathbb{F}$-representable matroids $M_1, M_2, \ldots, M_n$ such that M is in X iff none of $M_1, M_2, \ldots, M_n$ is a minor of M.
- Theorem (Geelen, Gerards, Whittle; 2012+): WQO Conj is true for finite $\mathbb{F}$.
- Corollary: For each k and $\mathbb{F}$, there exists a **finite** list of matroids such that an $\mathbb{F}$-repre. matroid M has path-width$\leq$k iff none in the list is a minor of M.
- Enough to check whether an input matroid has some minors in the finite list (which can be done in poly time for matroids of bounded branch-width, shown by Hlineny.)
- **Trouble**: 1. No algorithm known to construct the list of forbidden minors.
  2. Even if you know the list, this **doesn't provide a linear ordering**!

Known algorithms for path-width/branch-width of matroids

- **STEP 1**: Find an approximate branch-decomposition (Hlineny 2006; $O(n^3)$ algorithm)
- **STEP 2**: Use the dynamic programming to test all forbidden minors for branch-width$\leq$k or path-width$\leq$k.
- Branch-width: (the size of each forbidden minor)$\leq(6^k-1)/5$ (Geelen, Gerards, Robertson, Whittle 2003)
- Path-width: (#forbidden minors) OPEN! No upper bound is known; Finite due to WQO.
- STEP 3: Use step 2 to construct a branch-decomp of width$\leq$k (Hlineny, Oum)
- For path-width, an efficient algorithm exists but we did not know how to construct.

What's new? (1/2)

- Theorem [Jeong, Kim, O.]
  $O(f(k)\, n^3)$-time algorithm to find
  a linear layout $v_1, v_2, \ldots, v_n$
  of the input **n vectors** in $\mathbb{F}^m$ such that
  $\dim(\langle v_1,v_2,\ldots,v_i\rangle \cap \langle v_{i+1},\ldots,v_n\rangle) \leq \mathbf{k}$ for all i,
  if it exists (when $\mathbb{F}$ is a finite field)

- Outcome: **Fixed Parameter Tractable to decide**
  trellis-width≤k or
  path-width≤k of $\mathbb{F}$-representable matroids

What's new (2/2)
Extension to subspaces

For example, if $V_i=\langle v_i\rangle$ for all ii, then matroid path-width

- Theorem [Jeong, Kim, O.]
  $O(f(k)\, n^3)$-time algorithm to find
  a linear layout $V_1, V_2, \ldots, V_n$
  of the input **n subspaces** of $\mathbb{F}^m$ such that
  $\dim((V_1+V_2+\ldots+V_i) \cap (V_{i+1}+\ldots+V_n)) \leq \mathbf{k}$ for all i,
  if it exists (when $\mathbb{F}$ is a finite field)

Corollary to Linear rank-width

- Cut-rank function $cutrk_G(X)$:=rank of X*(V-X) submatrix of the adjacency matrix of a graph G
- **Linear rank-width** of a graph G:= min k such that $\exists$ linear layout $v_1, v_2, \ldots, v_n$ of the vertices with $cutrk_G(\{v_1,v_2,\ldots,v_i\}) \leq k$ for all i.
- NP-complete to decide linear rank-width$\leq$k.
- THEOREM: For a fixed k, $O(n^3)$-time algorithm to decide linear rank-width$\leq$k. (NEW)

Corollary to Linear rank-width

If A is the adjacency matrix of G and I is the identity matrix, then

$$\begin{pmatrix} I & A \end{pmatrix}$$ over GF(2)

For a vertex v_i,

let V_i=span of the i-th and (i+n)-th column vectors

EASY FACT: $2 * \text{cutrk}_G(X) = \dim\left(\left(\Sigma\{V_i : v_i \in X\}\right) \cap \left(\Sigma\{V_i : v_i \notin X\}\right)\right)$

Path-width of {V1,V2,…,Vn} = 2*(linear rank-width of G)

Corollary to Linear clique-width

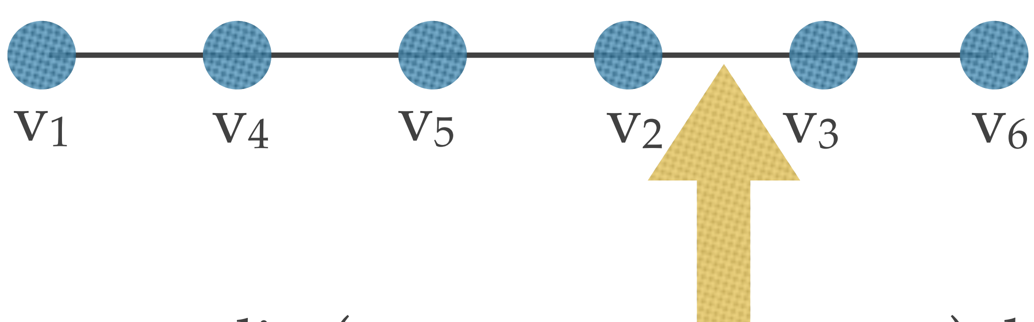
- Linear clique-width= “linearized version of clique-width”
- EASY FACT: If linear rank-width=k, then linear clique-width$\leq 2^k+1$.
- NP-complete to decide linear clique-width$\leq$k (when k is not fixed) (Fellows, Rosamond, Rotics, Szeider 2009)
- Corollary: The first approximation algorithm for linear clique-width.
  For a fixed k, $O(n^3)$-time algorithm to find a linear clique-width expression of width$\leq 2^k+1$ or confirms that linear clique-width>k.

Bodlaender-Kloks type algorithm for path-width of “subspaces”

Path-width vs Branch-width of (representable) matroids

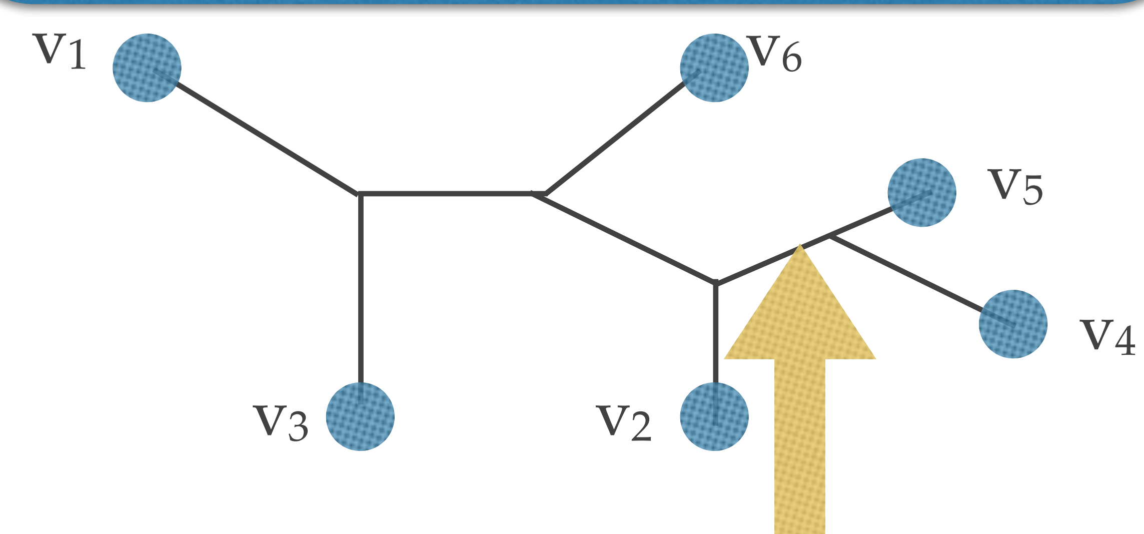
Input: $v_1, v_2, \ldots, v_n$: vectors in $\mathbb{F}^m$.
($\mathbb{F}$: finite field)
Output: Yes if $\exists$ permutation π of $\{1,2,\ldots,n\}$ s.t. …

Input: $v_1, v_2, \ldots, v_n$: vectors in $\mathbb{F}^m$.
($\mathbb{F}$: finite field)
Output: Yes if $\exists$ subcubic tree T with a bijection L:{leaves}→{vectors} s.t. …



$\dim(\langle v_1,v_4,v_5,v_2\rangle \cap \langle v_3,v_6\rangle) \le k$

Path-decomposition of width≤k

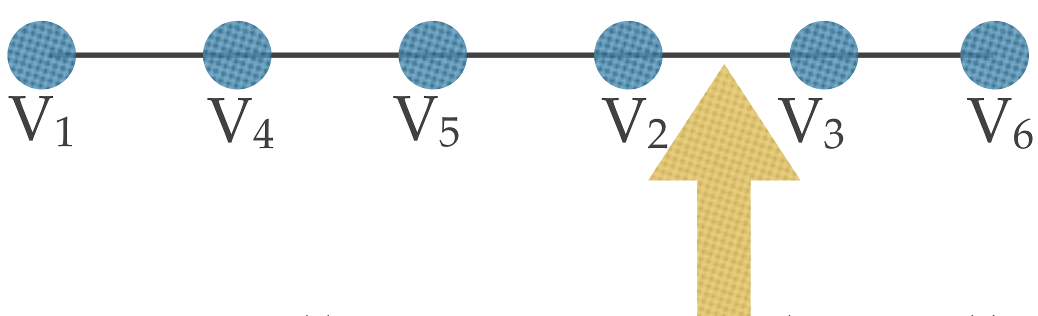


$\dim(\langle v_1,v_3,v_6,v_2\rangle \cap \langle v_4,v_5\rangle) \le k$

Branch-decomposition of width≤k

Path-width vs Branch-width of a subspace arrangement

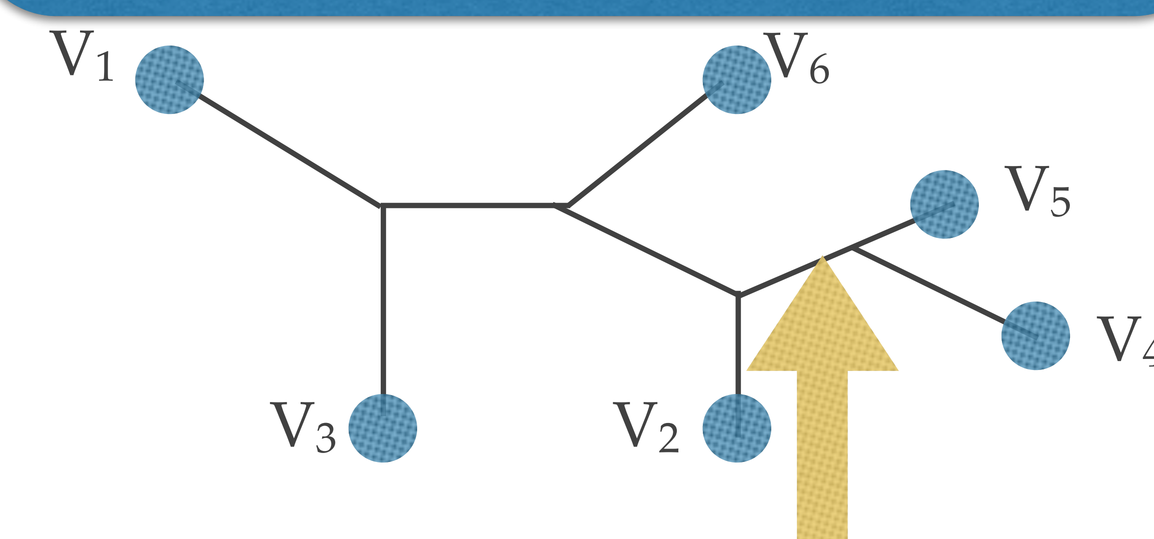
Input: $V_1, V_2, \ldots, V_n$: subspaces in $\mathbb{F}^m$.
($\mathbb{F}$: finite field)
Output: Yes if $\exists$ permutation π of $\{1,2,\ldots,n\}$ s.t. …



$\dim((V_1+V_4+V_5+V_2)\cap(V_3+V_6))\leq k$

Path-decomposition of width $\leq k$

Input: $V_1, V_2, \ldots, V_n$: subspaces in $\mathbb{F}^m$.
($\mathbb{F}$: finite field)
Output: Yes if $\exists$ subcubic tree T with a bijection L:{leaves}→{subspaces} s.t. …



$\dim((V_1+V_3+V_6+V_2)\cap(V_4+V_5))\leq k$

Branch-decomposition of width $\leq k$

Our algorithm

- Input: n subspaces
- Assume that we are given a branch-decomposition of width w.

We will discuss how to provide such a decomposition later

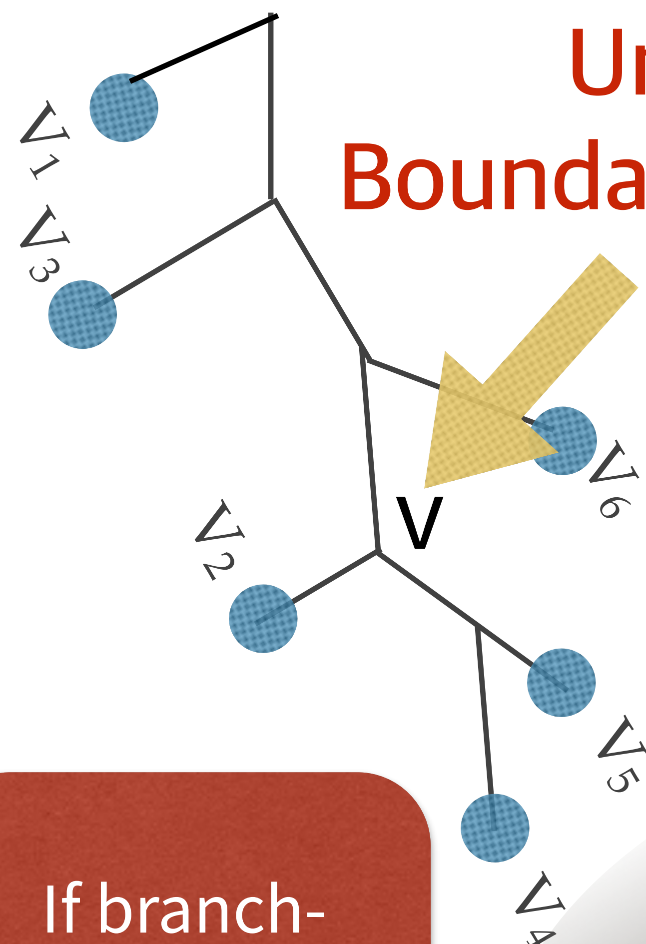
- Task: Do the dynamic programming to enumerate all “partial solutions” of width at most k.

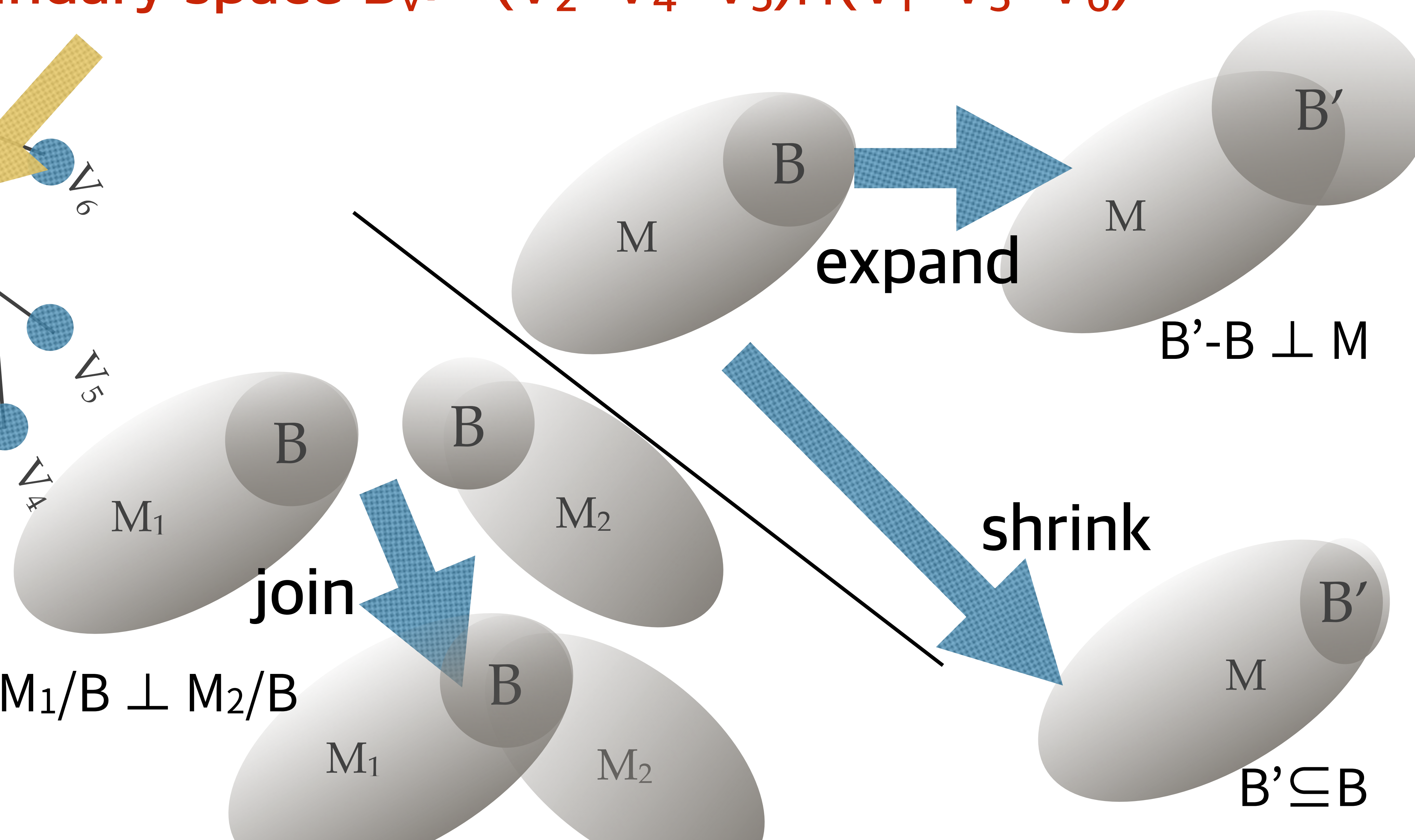
How to run dynamic-programming on a "branch-decomposition of subspaces"

Dynamic programming on a branch-decomposition

Underlying space: $M_v := V_2+V_4+V_5$

Boundary space $B_v := (V_2+V_4+V_5) \cap (V_1+V_3+V_6)$





If branch-width≤w, then we can keep dim (B)≤2w.

Path-decomposition: an alternative definition of path-width

- Let $\mathcal{V}=\{V_1, V_2, \ldots, V_n\}$ be a subspace arrangement (set of subspaces), $\langle\mathcal{V}\rangle = V_1+V_2+\ldots+V_n$.

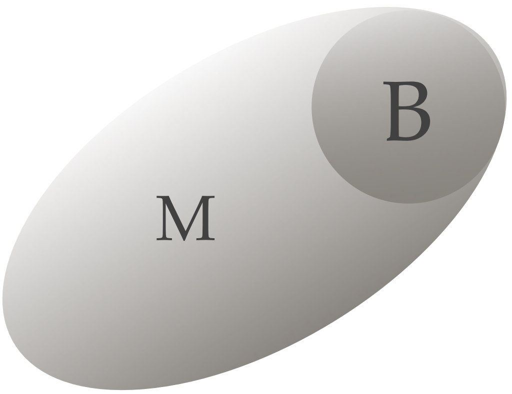
- A **path-decomposition** of $\mathcal{V}$
  := a **sequence** $(S_1,S_2,\ldots,S_m)$ of subspaces of $\langle\mathcal{V}\rangle$
  with an **injective function** $\mu:\{1,2,\cdots,n\}\to\{1,2,\ldots,m\}$ s.t. $V_i \subseteq S_{\mu(i)}$.

- **Width** of a path-decomposition := $\max (S_1+\ldots+S_i) \cap (S_{i+1}+\ldots+S_m)$.

- S_i is called a bag.

- THM: $\exists$ linear layout of width $\leq k \Leftrightarrow \exists$ path-dec. of width $\leq k$.

What do we keep?



For a path-decomposition $(S_1, S_2, \ldots, S_m)$

For each "gap", there is a triple (L, R, λ)

S_1 | S_2 | S_3 | S_4 | S_5 | S_6 | S_7 |

"Left subspace" shown on B

$L_3 = (S_1+S_2+S_3) \cap B$

"Right subspace" shown on B

$R_3 = (S_4+S_5+S_6+S_7) \cap B$

Extra connectivity not shown in B

$\lambda_3 = \dim(S_1+S_2+S_3) \cap (S_4+S_5+S_6+S_7) - \dim((S_1+S_2+S_3) \cap (S_4+S_5+S_6+S_7) \cap B)$

For (M,B), we only need to keep a sequence of (L, R, λ) in order to determine whether we have a path-decomposition

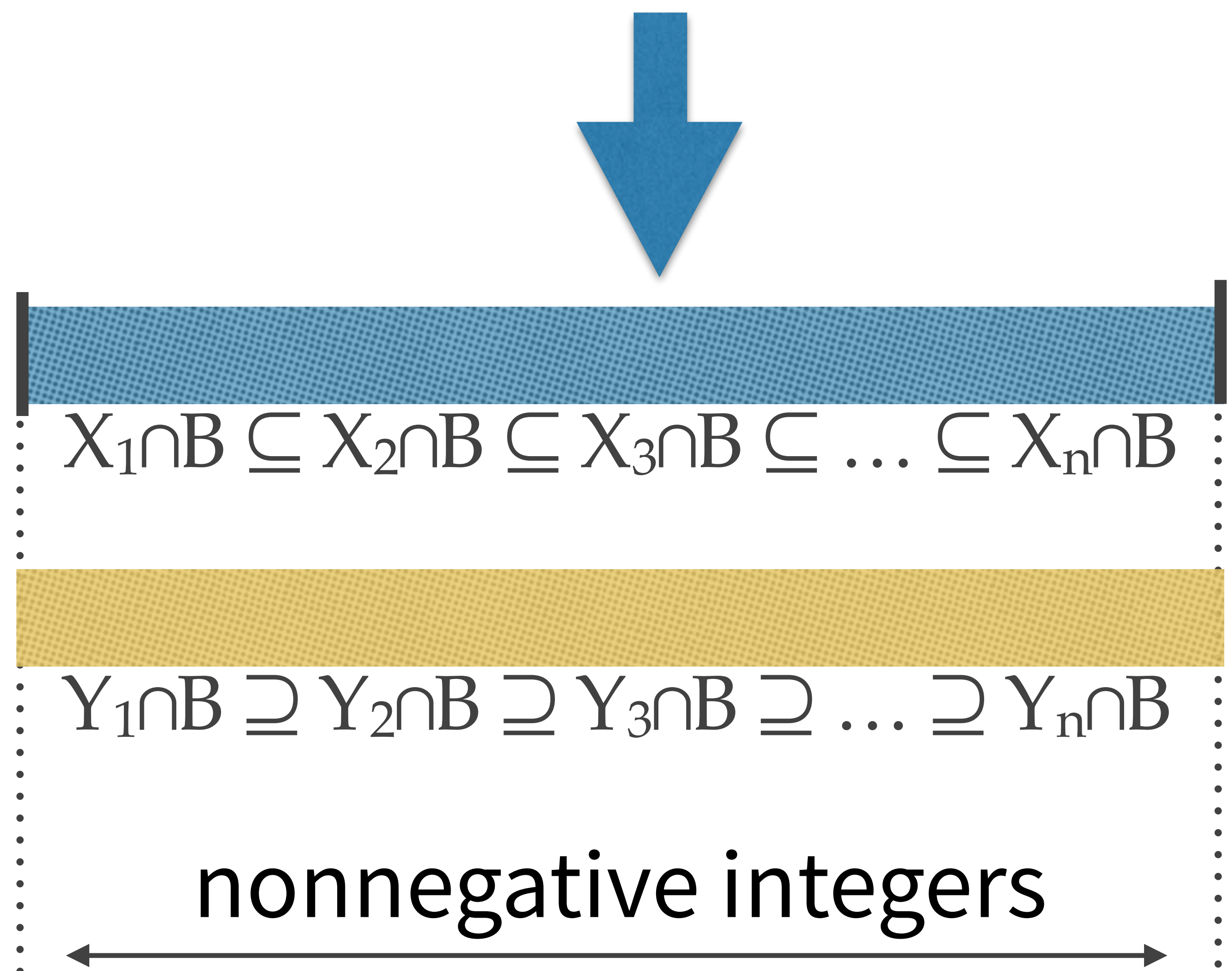
B-trajectory for a subspace B

- statistic:=triple (L,R,λ) of subspaces L, R of B and $\lambda \geq 0$.
- **B-trajectory**:=a finite sequence of statistics $\Gamma=a_1,a_2,\ldots,a_n$ such that $L(a_1)\subseteq L(a_2)\subseteq\ldots\subseteq L(a_n)$, $R(a_1)\supseteq R(a_2)\supseteq\ldots\supseteq R(a_n)$.
- Width of a B-trajectory:= max λ.
- The **canonical B-trajectory** of a <u>path-decomposition</u> $(S_1,S_2,\ldots,S_n)$

A <u>path-decomposition</u> $(S_1,S_2,\ldots,S_{n-1})$

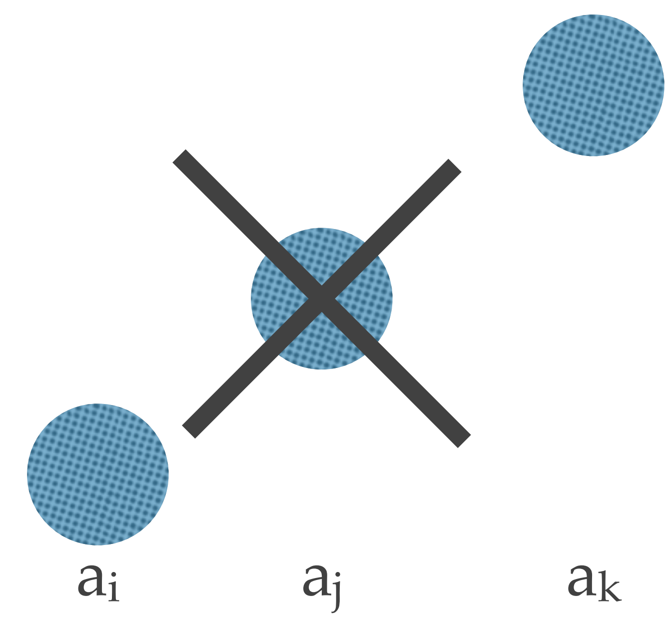
$X_i := S_1+S_2+\ldots+S_i$

$Y_i := S_{i+1}+\ldots+S_{n-1}$



How to compress B-trajectories? Typical sequences of Bodlaender and Kloks

- Reducing operation for a sequence of integers:
  In a sequence $a_1, a_2, \ldots, a_n$ of integers,
  remove a_j if a_j is between a_i and a_k and $i<j<k$.



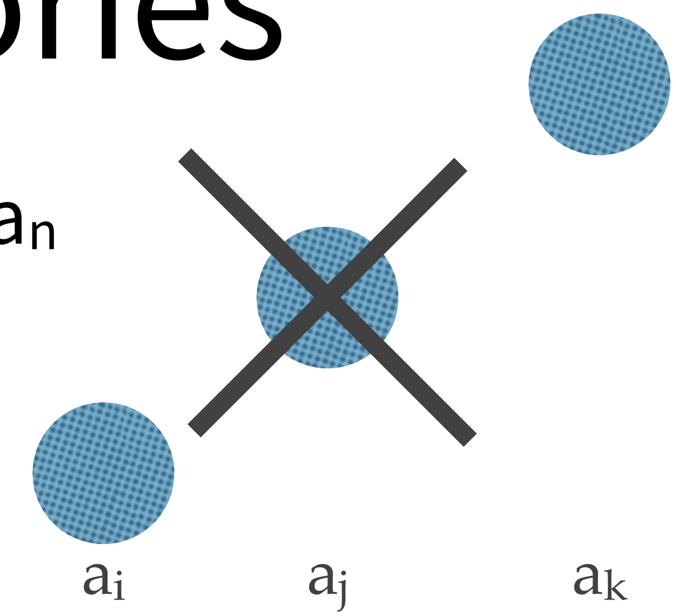
  - τ(1 2 5 3 9 2) = (1 9 2)
  - τ(1 3 8 4 5 6 3 7) = (1 8 3 7)

- A sequence is <u>typical</u> if no further reducing is possible.

- Lemma (Bodlaender and Kloks 1996):
  (#typical sequences consisting of $\{0,1,2,\ldots,k\}$) $\leq$ $(8/3)2^{2k}$.

How to compress B-trajectories? "Compact" B-trajectories

- Reducing operation for a B-trajectory $\boldsymbol{\Gamma}=a_1,a_2,\ldots,a_n$
  remove a_j if
  $L(a_i)=L(a_k)$, $R(a_i)=R(a_k)$, and
  $\lambda(a_j)$ is between $\lambda(a_i)$ and $\lambda(a_k)$ and $i<j<k$.



- A B-trajectory is <u>compact</u> if no further reducing is possible. The compactification $\tau(\boldsymbol{\Gamma})$ is a compact B-trajectory obtained by reducing from $\boldsymbol{\Gamma}$.

- Lemma: if $\dim(B)=w$ and $|\mathbb{F}|=q$, then
  (#compact B-trajectories of width$\leq$k) $\leq ((8/3)\ 2^{2k})^{2w+1}\ 2^{2(2w+1)q\text{^}w}$

What to store during dynamic programming? "Full Sets"

- For a subspace arrangement $\mathcal{V}=\{V_1,V_2,..,V_n\}$, the **full set** $FS_k(\mathcal{V},B)$:=set of all B-trajectories of width$\leq$k that are better than some B-trajectories, realizable in $\mathcal{V}$.

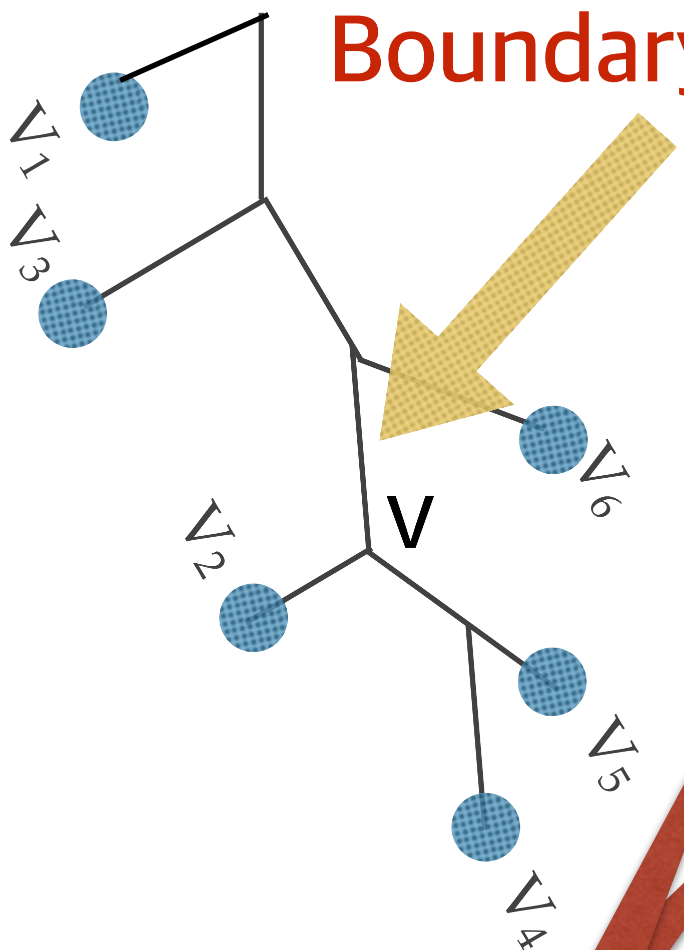
- $FS_k(\mathcal{V},\{0\})\neq\emptyset$ if and only if path-width$\leq$k.

We aim to compute $FS_k(\mathcal{V},\{0\})$ by the dynamic programming

Computing the Full Set

Underlying space: $M_v := V_2+V_4+V_5$

Boundary space $B_v := (V_2+V_4+V_5)\cap(V_1+V_3+V_6)$



At each leaf v, compute $FS(M_v, B_v)$ “easy”

At each internal node v

with two children x and y,

- compute $FS(M_x, B_x+B_y)$ from $FS(M_x, B_x)$
- compute $FS(M_y, B_x+B_y)$ from $FS(M_y, B_y)$
- compute $FS(M_v, B_x+B_y)$ from $FS(M_x, B_x+B_y)$ and $FS(M_y, B_x+B_y)$
- compute $FS(M_v,B_v)$ from $FS(M_v, B_x+B_y)$

Expand

Join

Shrink

O(n)-time to compute $FS_k(\mathcal{V},\{0\})$.

How to provide an initial "approximate" branch-decomposition

- Method 1: Iterative compression; O(n) overhead
  Modify the output for $V_1, \ldots, V_{i-1}$ of width $\leq k$ to be the branch-decomposition of width $\leq k+1$.
  TOTAL: $O(n^4)$ time (simpler but slower)

- Method 2: Use the algorithm by Hlineny and Oum (2008) that provides a branch-decomposition of width $\leq k$ in $O(n^3)$-time (faster)
  TOTAL: $O(n^3)$ time.

Concluding remarks

- By backtracking, we can construct a linear layout of width$\leq$k.
- Similar idea works for rank-width of graphs and branch-width of matroids representable over a fixed finite field.
- Two bottlenecks for a faster algorithm:
  - Finding an approximate branch-decomposition.
  - Preprocessing the approximate branch-decomposition to make it more useful for dynamic programming. (e.g. Precompute a basis for each boundary space B)

THANK YOU FOR YOUR ATTENTION!