From Matchings to Independent Sets

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UNIVERSITÀ DEL LITORALE

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OPERACIJO DELNO FINANCIRA EVROPSKA UNIJA
Evropski socialni sklad

From Matchings

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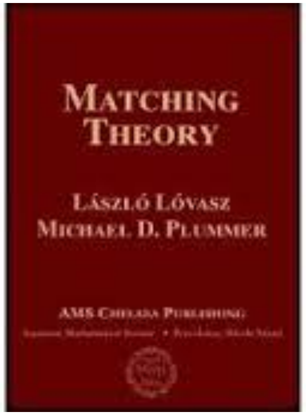

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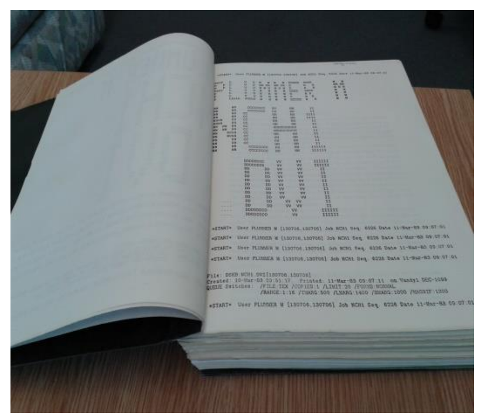

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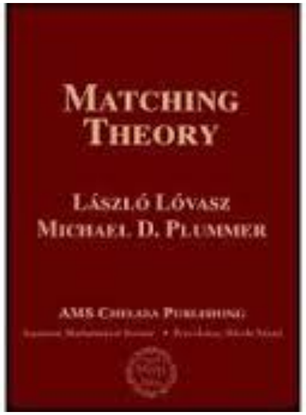

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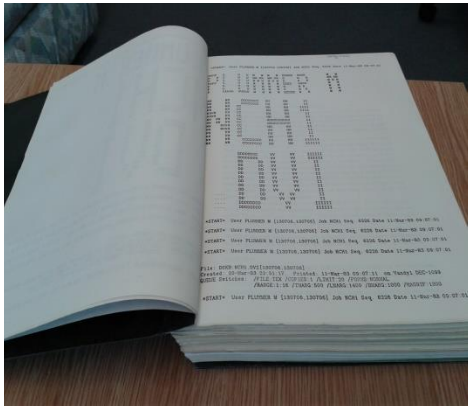

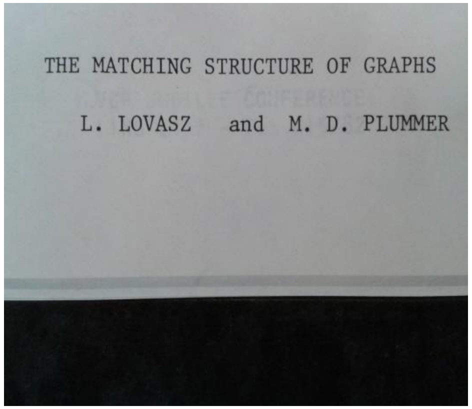

To independent sets

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Max Matching = Max Independent Set in Line Graphs

To independent sets

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To independent sets

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What makes the Maximum Independent Set problem easy in the class of line graphs?

Hereditary classes of graphs

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Examples:

- line graphs
- bipartite graphs
- planar graphs
- graphs of vertex degree at most 3
- forests (graphs every connected component of which is a tree; graphs without cycles)

Induced subgraph characterization of hereditary classes of graphs

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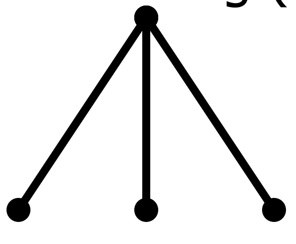
• bipartite graphs = Free(C_3,C_5,C_7,...)
• forests = Free(C_3,C_4,C_5,...)
• line graphs = Free(M), where M is a set of 9 graphs one of which is the claw

Dénes Kőnig (1916)



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Theorem. *Free(M)$\subseteq$Free(N) if and only if for each graph G$\in$N there is a graph H$\in$M such that H is an induced subgraph of G*

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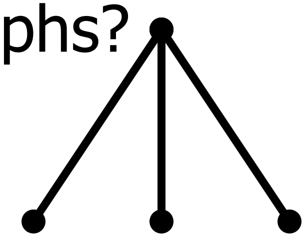
The absence of the claw $K_{1,3}=$  = claw

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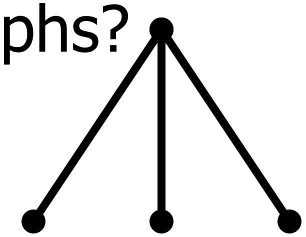
The absence of the claw $K_{1,3}$=  = claw

Line graphs $\subset$ Claw-free graphs

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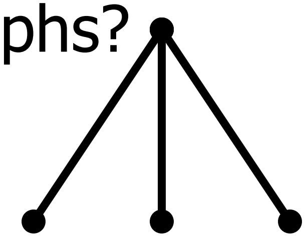
The absence of the claw $K_{1,3}$=  = claw

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Minty, George J. On maximal independent sets of vertices in claw-free graphs. *J. Combin. Theory Ser. B* **28** (1980), no. 3, 284–304.

Sbihi, Najiba Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. (French) *Discrete Math.* **29** (1980), no. 1, 53–76.

Why claw?

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and what other restrictions can make the Maximum Independent Set problem easy?

Complexity of the Maximum Independent Set problem in particular classes of graphs

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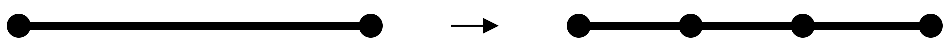

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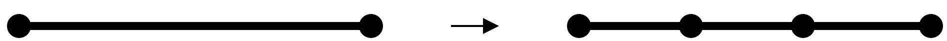
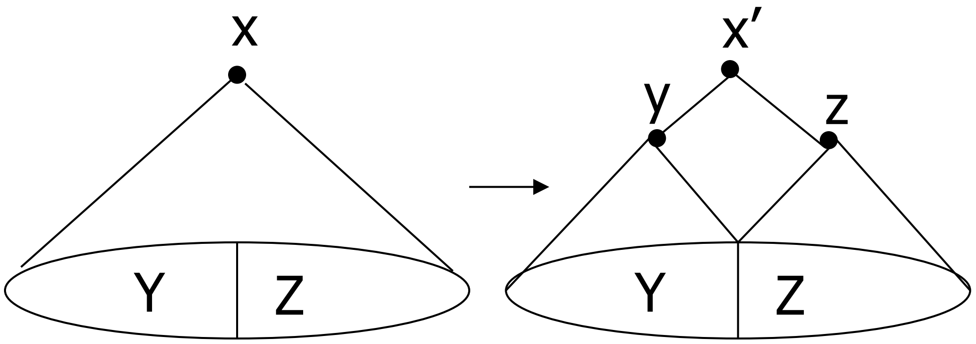


independence number (the size of a maximum independent set) $\alpha(G)+1 \quad = \quad \alpha(G')$

Vertex splitting



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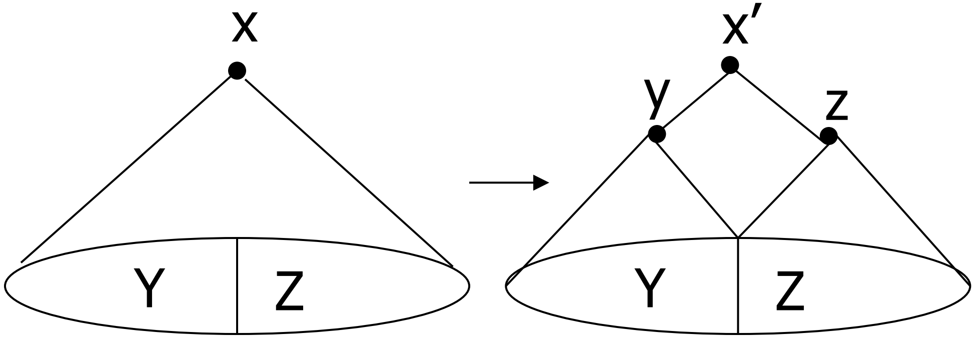

$$\alpha(G) \quad = \quad \alpha(G')-1$$

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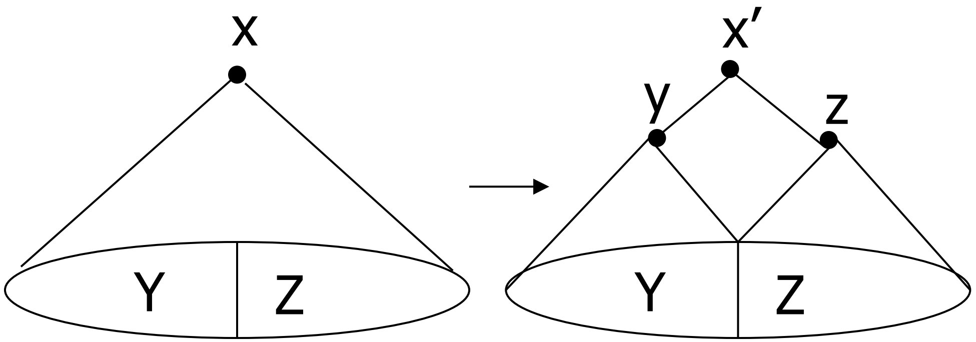

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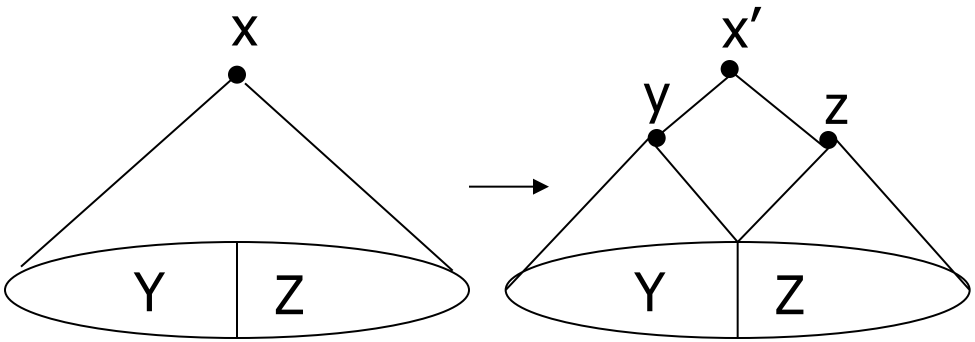

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If $|Y|=1$, then vertex splitting is equivalent to an edge subdivision

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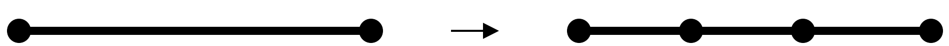
Fact. The Max Independent Set problem is NP-complete for graphs of vertex degree at most 3.

Fact. A double subdivision of an edge increases the independence number of the graph by exactly 1.



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Corollary. *The Max Independent Set problem is NP-complete for graphs of degree at most 3 in the class Free(C_3,C_4,C_5,…,C_k) for each fixed value of k.*

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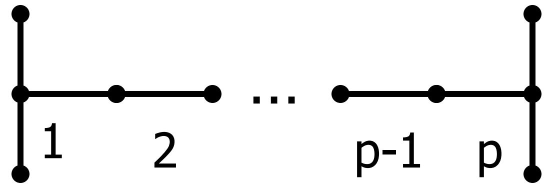

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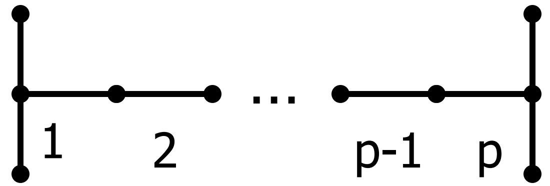

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If a graph belongs to S, it also belongs to each class of the sequence converging to S.

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Definition. A minimal limit class is called a *boundary* class.

Complexity of the Maximum Independent Set problem in particular classes of graphs

Theorem. *The maximum independent set problem is polynomial-time solvable in the class Free(M) defined by a finite set M of forbidden induced subgraphs if and only if Free(M) contains none of the boundary classes (M contains a graph from each of the boundary classes).*

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Theorem. *The class S is a boundary class for the maximum independent set problem.*

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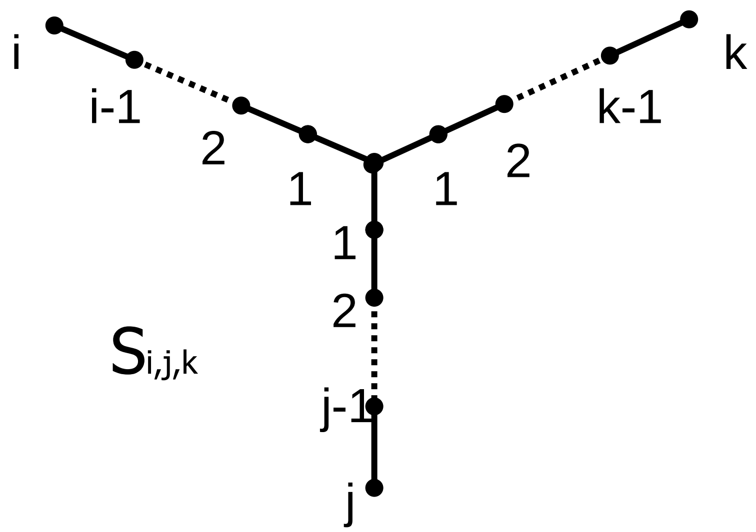
S is the class of graphs in which every connected component has the form $S_{i,j,k}$.



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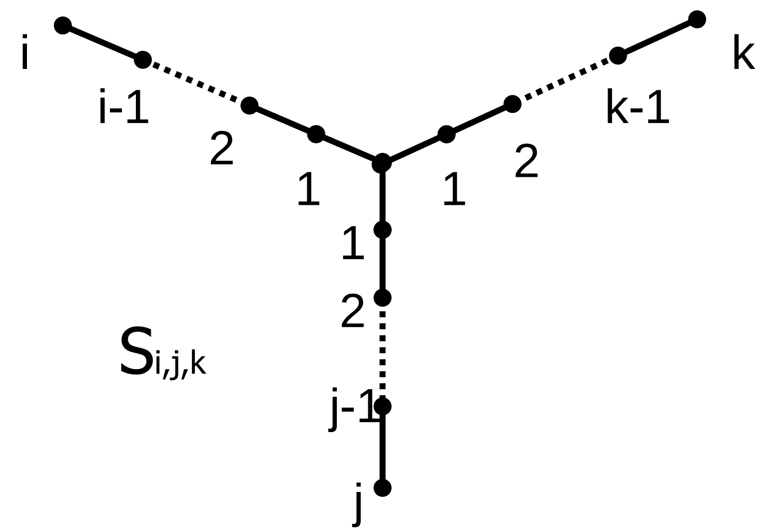

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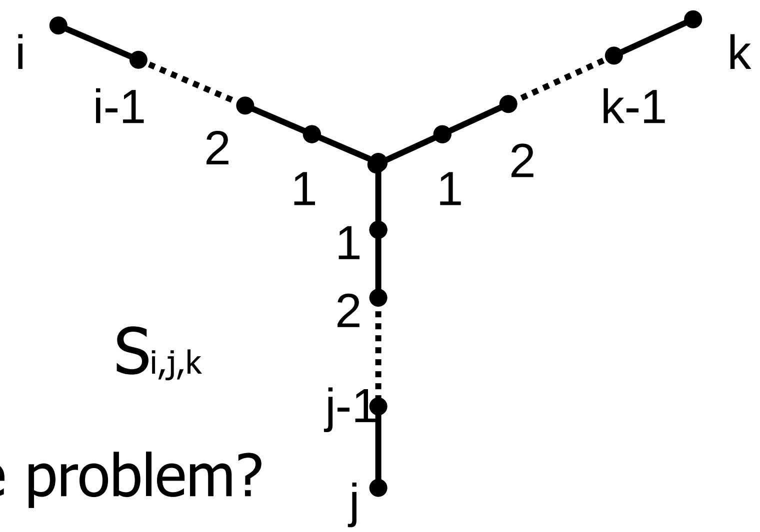

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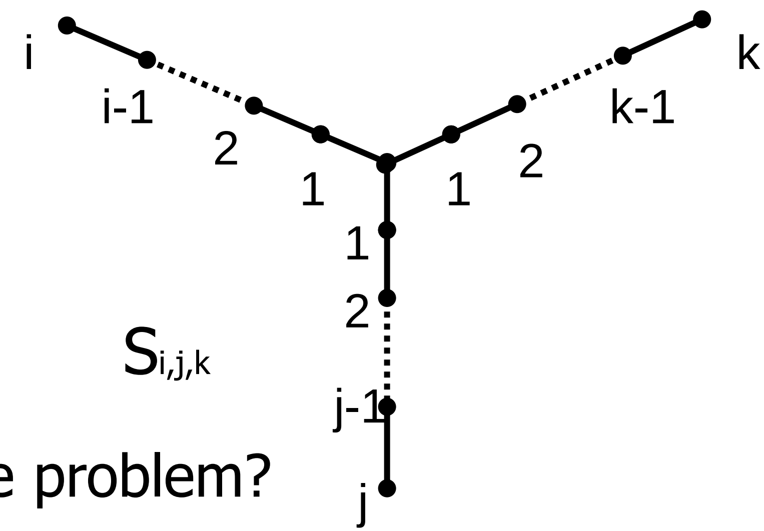

$S_{i,j,k}$

Is S the only boundary class for the problem?

Yes, if by forbidding any graph from S we obtain a class where the problem can be solved in polynomial time.

Complexity of the Maximum Independent Set problem in particular classes of graphs

The problem is solvable in polynomial time for

$S_{1,1,2}$-free graphs	Lozin, Vadim V.; Milanič, Martin A polynomial algorithm to find an independent set of maximum weight in a fork-free graph. *J. Discrete Algorithms* 6 (2008), no. 4, 595–604.

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(P_3+P_3)-free graphs — Lozin, Vadim V.; Mosca, Raffaele Maximum regular induced subgraphs in 2*P*3-free graphs. *Theoret. Comput. Sci.* 460 (2012), 26–33.

Complexity of the Maximum Independent Set problem in particular classes of graphs

The problem is solvable in polynomial time for

$S_{1,1,2}$-free graphs	Lozin, Vadim V.; Milanič, Martin A polynomial algorithm to find an independent set of maximum weight in a fork-free graph. *J. Discrete Algorithms* 6 (2008), no. 4, 595–604.
P_5-free graphs	Daniel Lokshtanov, Martin Vatshelle and Yngve Villanger Independent Set in P_5-Free Graphs in Polynomial Time, SODA 2014
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mP_2-free graphs	Farber, Martin; Hujter, Mihály; Tuza, Zsolt An upper bound on the number of cliques in a graph. *Networks* 23 (1993), no. 3, 207–210.

Conjecture

If M is a finite set, then the Maximum Independent Set problem is polynomial-time solvable for graphs in Free(M) if and only if M contains a graph from S, i.e. for any graph $G \in S$, the problem is polynomial-time solvable in Free(G).

Did you know that

The difference in the speed of clocks at different heights above the earth is now of considerable practical importance, with the advent of very accurate navigation systems based on signals from satellites. If one ignored the predictions of general relativity theory, the position that one calculated would be wrong by several miles!

Stephen Hawking *A brief history of time*

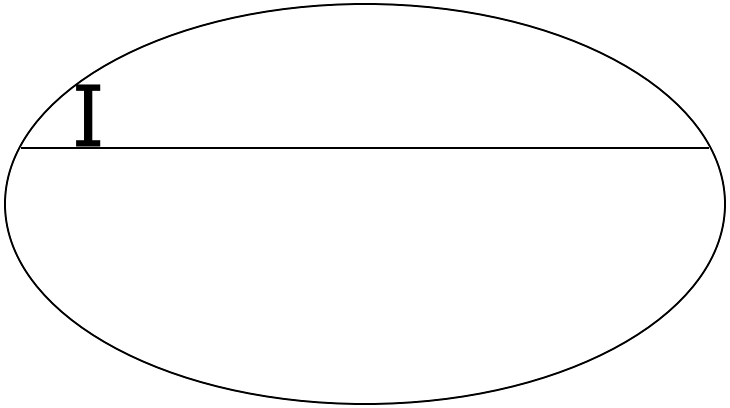
Algorithmic tools for Max Independent Set

- Modular decomposition
- Tree- and clique-width decompositions
- Separating cliques
- Augmenting graphs
- Graph Transformations

From augmenting paths to augmenting graphs

From augmenting paths to augmenting graphs

Let G=(V,E) be a graph and
I an independent set in G



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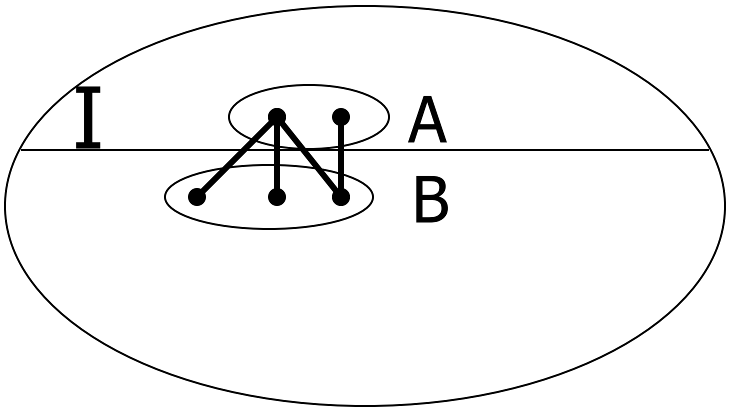

G=(V,E)

Let H be a bipartite subgraph of G with parts A and B such that

- $A \subseteq I$
- $B \subseteq V\text{-}I$
- the vertices of B do not have neighbours in I-A
- $|A|<|B|$

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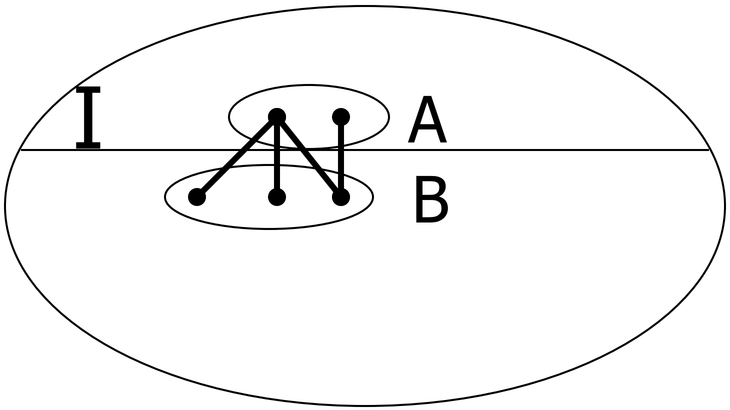

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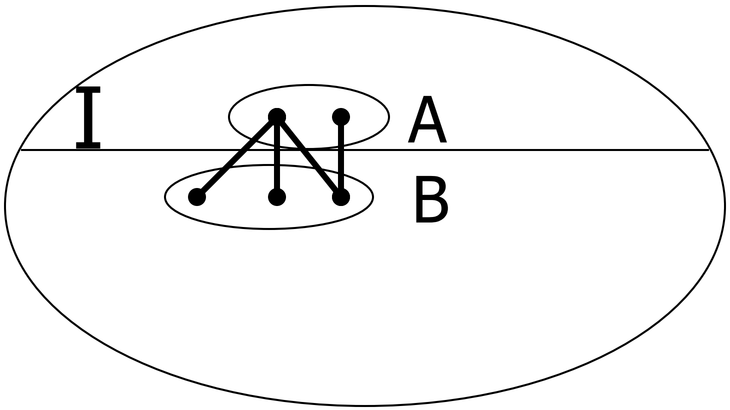

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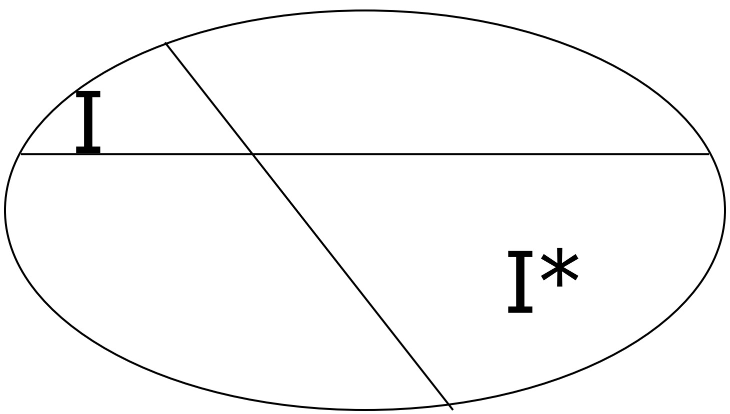
Then H is an *augmenting graph* for I

If there is an augmenting graph for I, then I is not maximum, because $I^*=(I-A)\cup B$ is a larger independent set.

From augmenting paths to augmenting graphs

If I is not maximum and I* is a larger independent set, then the bipartite graph with parts I-I* and I*-I is augmenting for I.



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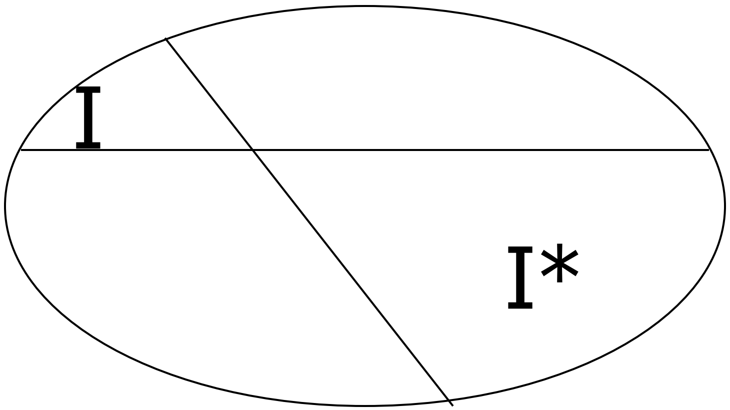

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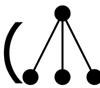
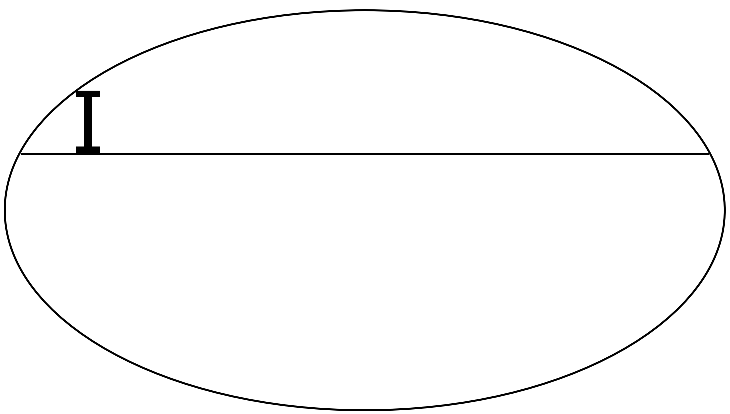
Theorem of augmenting graphs. *An independent set I is maximum if and only if there are no augmenting graphs for I.*

From augmenting paths to augmenting graphs

Line graphs $\subset$ Claw-free (-free) graphs



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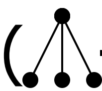
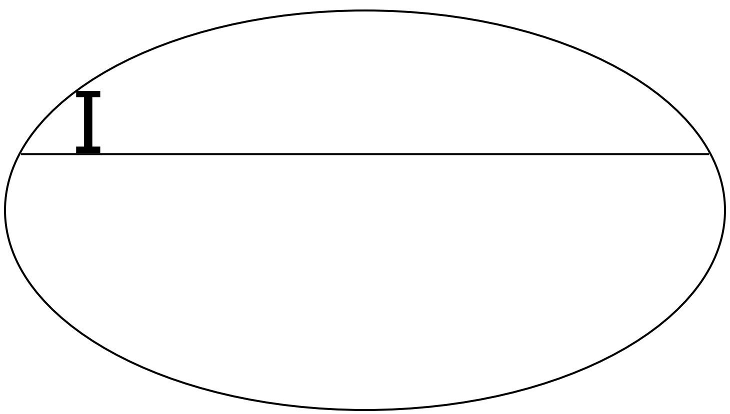
Line graphs $\subset$ Claw-free (-free) graphs



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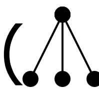
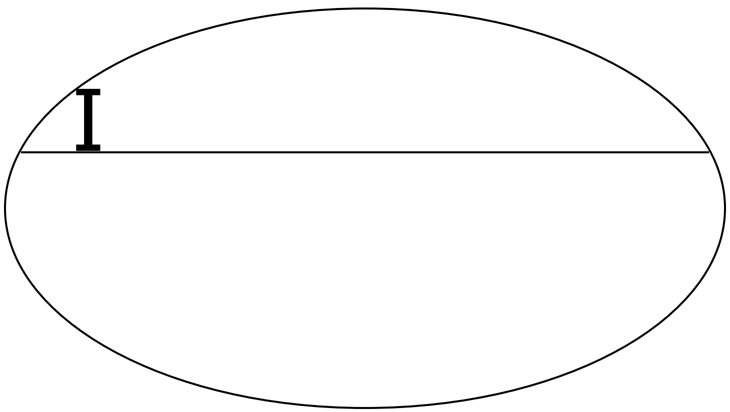
Every bipartite claw-free graph has vertex degree at most 2.

From augmenting paths to augmenting graphs

Line graphs $\subset$ Claw-free (-free) graphs



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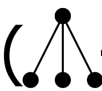
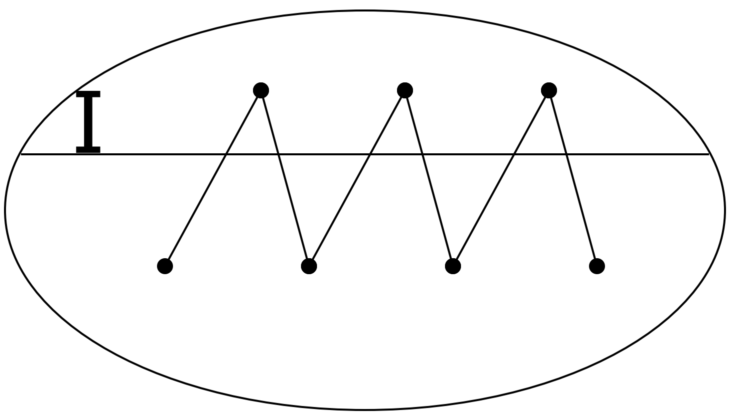
Line graphs $\subset$ Claw-free (-free) graphs



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From augmenting paths to augmenting graphs

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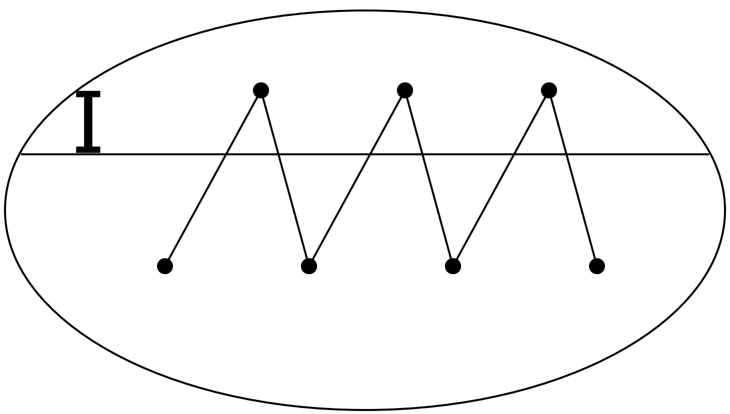

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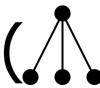
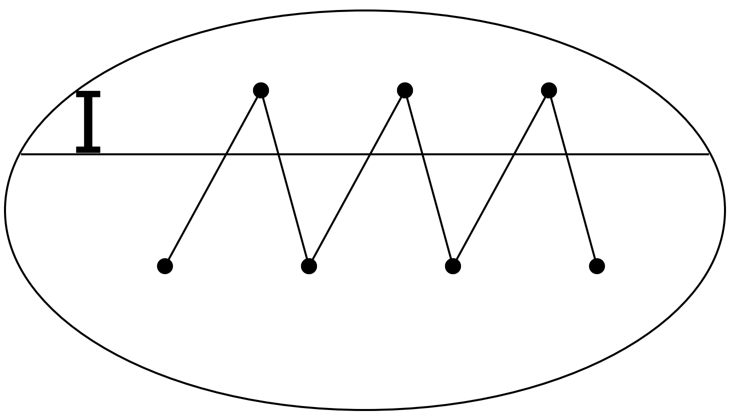
Theorem of augmenting paths.
An independent set I in a claw-free graph is maximum if and only if there are no augmenting paths for I.

From augmenting paths to augmenting graphs and back

Line graphs $\subset$ Claw-free (-free) graphs



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Theorem of augmenting paths (Berge’s lemma).
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Finding augmenting graphs

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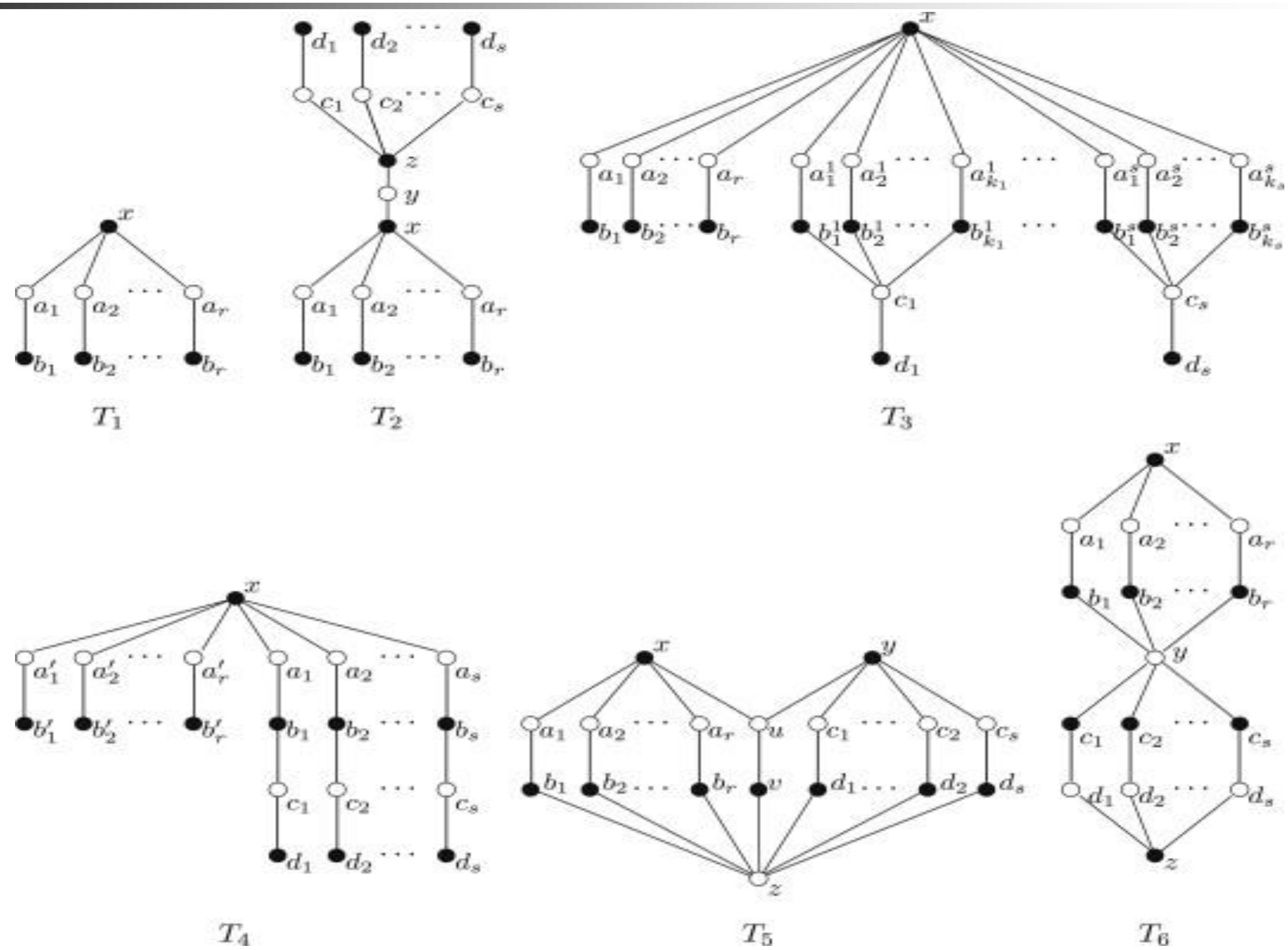
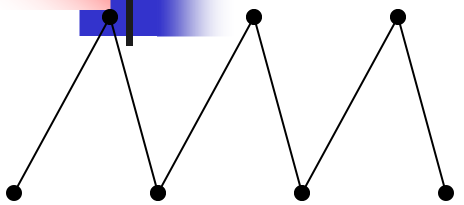
Structure of augmenting graphs



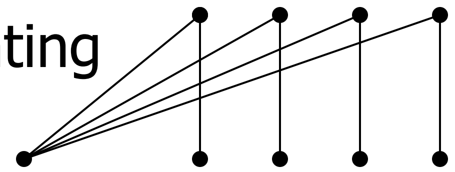
Combinatorics of augmenting graphs



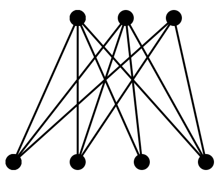
augmenting paths P_k

P

simple augmenting trees T_k



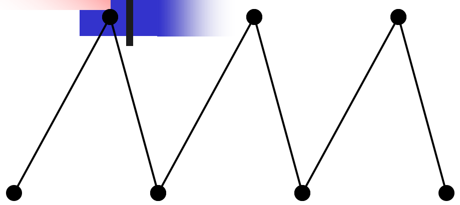
T



complete bipartite graphs $K_{k,k+1}$

K

Combinatorics of augmenting graphs



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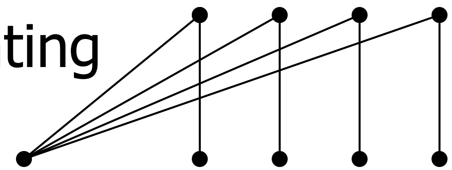

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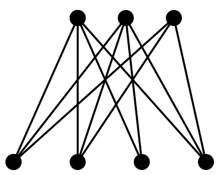

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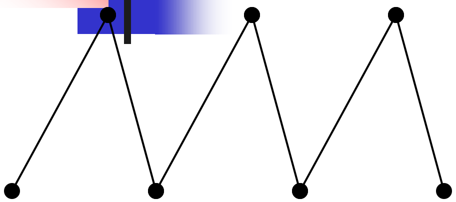
Theorem. *Let X be a hereditary class. If X contains infinitely many augmenting graphs, then it contains at least one of P, T or K.*

Combinatorics of augmenting graphs



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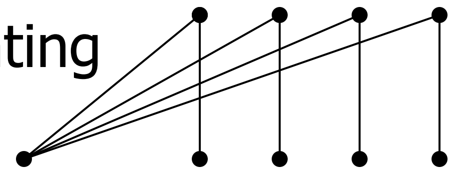

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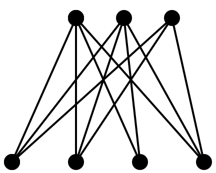

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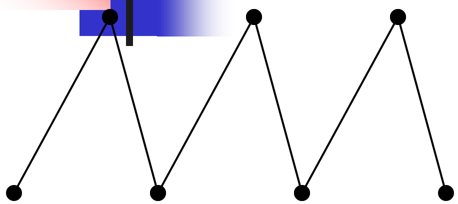
Combinatorics of augmenting graphs



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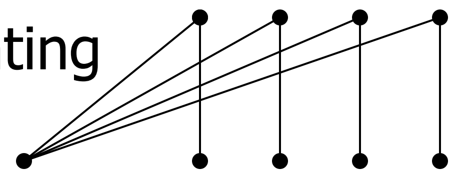

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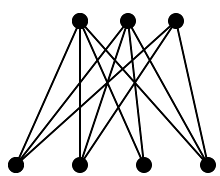

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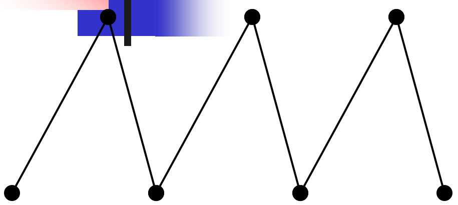

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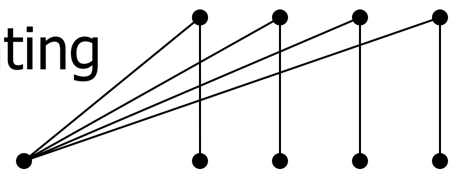

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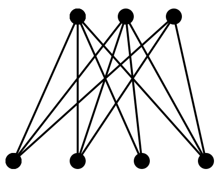

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Corollary. For any s,k,p, the maximum independent set problem for $(P_s,T_k,K_{p,p})$-free graphs can be solved in polynomial time.

Corollary. The maximum independent set problem can be solved in polynomial time in the class of $(S_{1,1,3},K_{p,p})$-free graphs.

If $p\geq 3$, the class of $(S_{1,1,3},K_{p,p})$-free graphs contains all claw-free graphs

Some open problems related to augmenting graphs

What is the complexity of detecting augmenting paths in general graphs?

What is the structure of $S_{1,2,2}$-free augmenting graphs?

Cycle shrinking for the Maximum Matching problem

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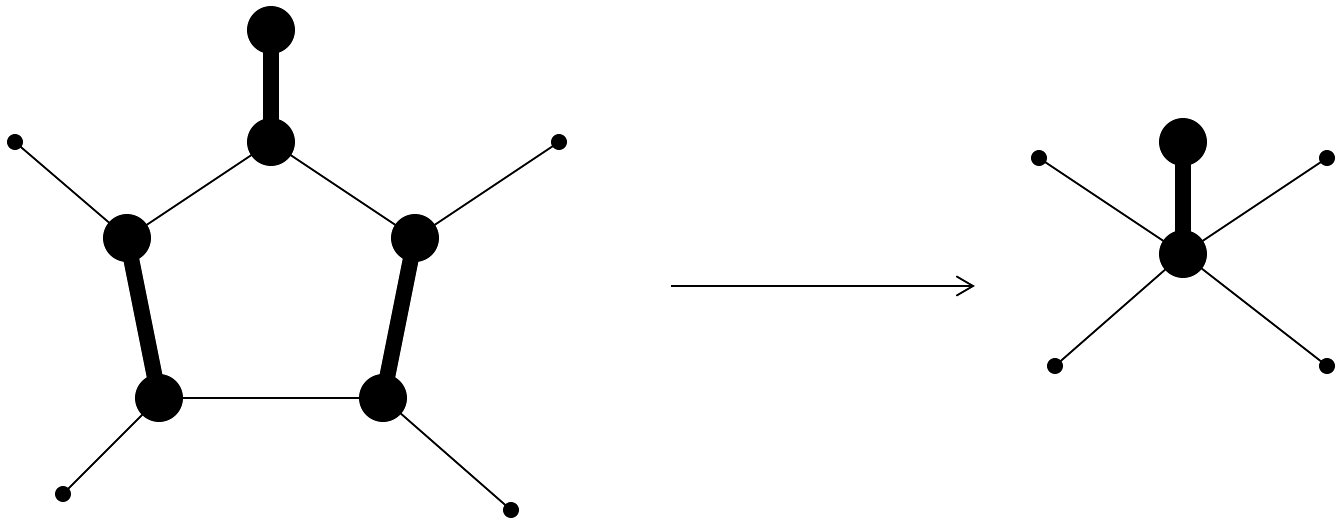


Even pair contraction for Vertex Coloring and Maximum Clique

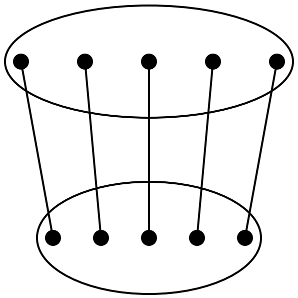
Definition. A pair of non-adjacent vertices is called an even pair if every induced path between them has an even number of edges.

Even pair contraction for Vertex Coloring and Maximum Clique

Definition. A pair of non-adjacent vertices is called an even pair if every induced path between them has an even number of edges.

Theorem. *Contraction of an even pair to a single vertex does not change the chromatic number of the graph and the clique number of the graph.*

Crown rule reduction for the Minimum Vertex Cover problem



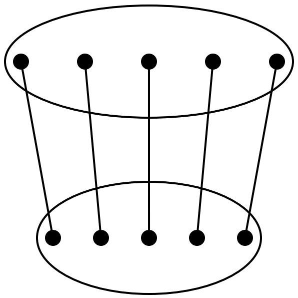
Crown

I an independent set

M a matching covering all vertices of N(I)

N(I) the neighbourhood of I

Crown rule reduction for the Minimum Vertex Cover problem



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$\tau(G)$ is the size of a minimum vertex cover of G

$$\tau(G\text{-Crown})=\tau(G)-|N(I)|$$

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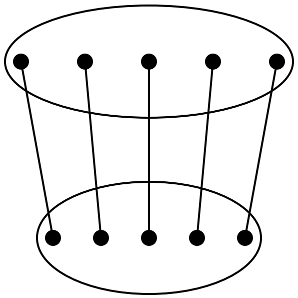

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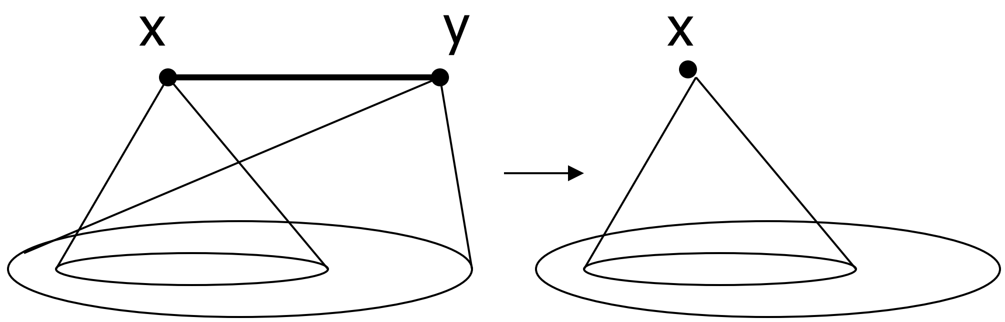
$$\tau(G) + \alpha(G)=|V(G)|$$

Graph Transformations

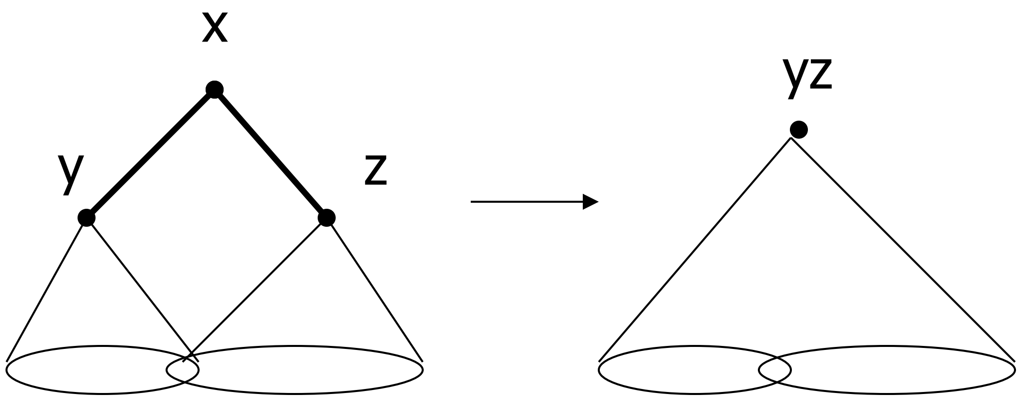
Neighbourhood reduction



$$\alpha(G) \quad = \quad \alpha(G')$$

Graph Transformations

Vertex folding

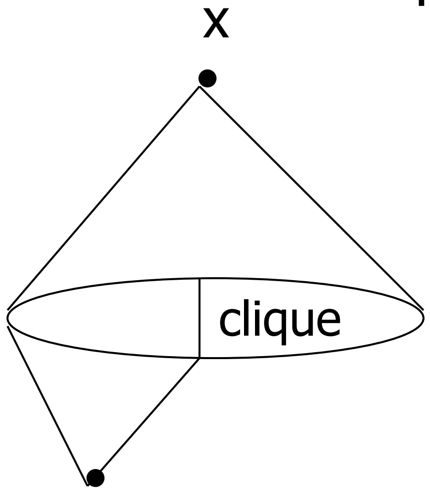


$$\alpha(G) = \alpha(G') + 1$$

Graph Transformations

Mirroring

Definition. Vertex u nonadjacent to x is a mirror of x if the neighbours of x non-adjacent to u form a clique.



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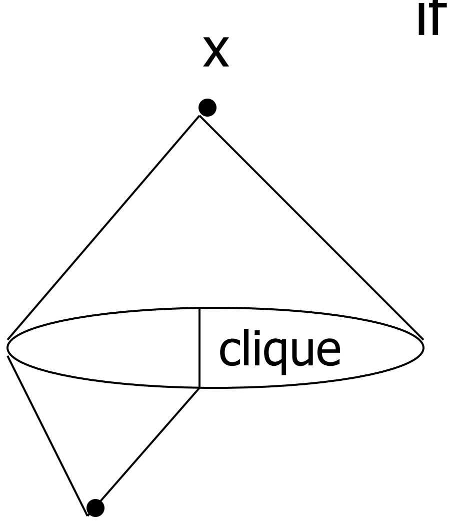


u is a mirror of x

Theorem. *If x does not belong to any maximum independent set of the graph, then so does u.*

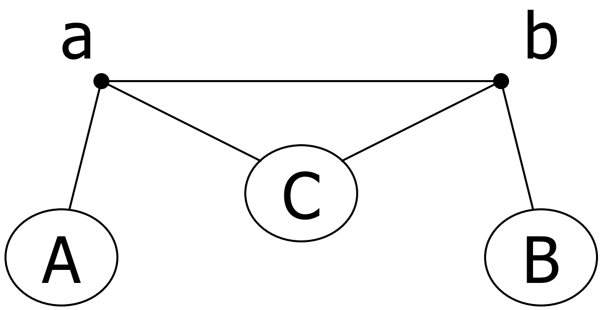
Graph Transformations

- Neighbourhood reduction
- Vertex folding
- Mirroring

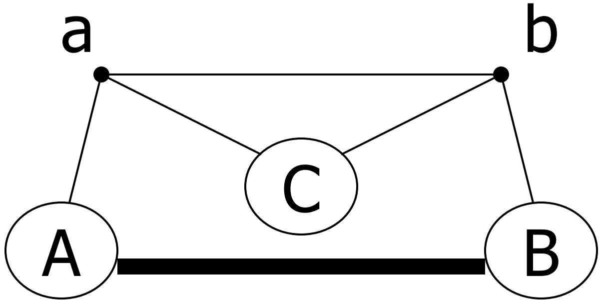
Fomin, Fedor V.; Grandoni, Fabrizio; Kratsch, Dieter

A measure & conquer approach for the analysis of exact algorithms. *J. ACM* 56 (2009), no. 5, Art. 25, 32 pp.

Graph Transformations



Graph Transformations



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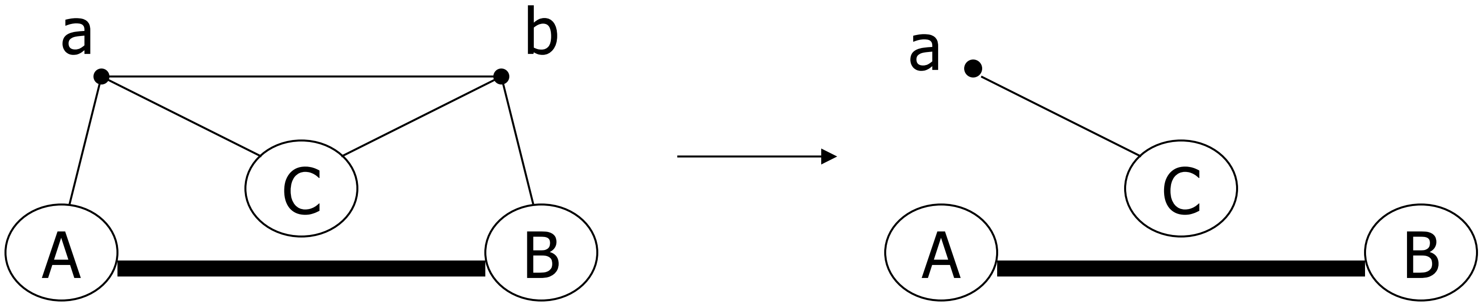
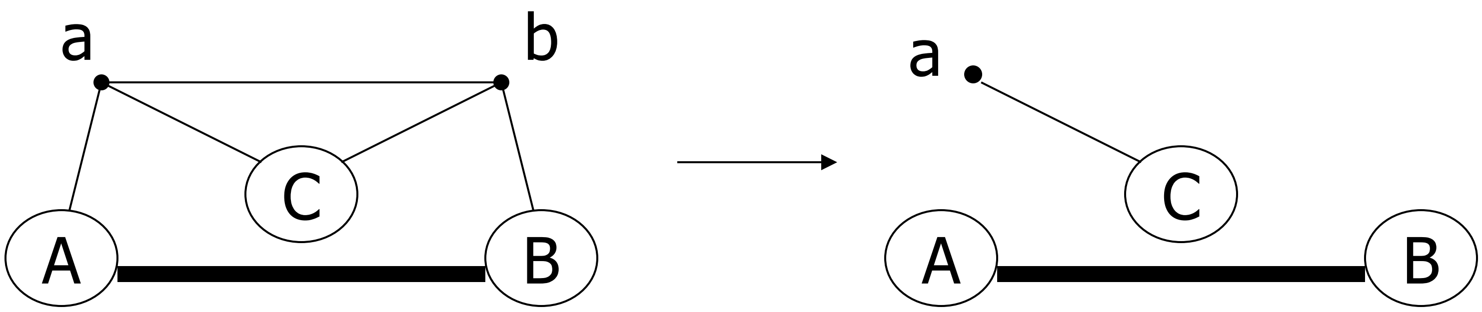


Graph Transformations



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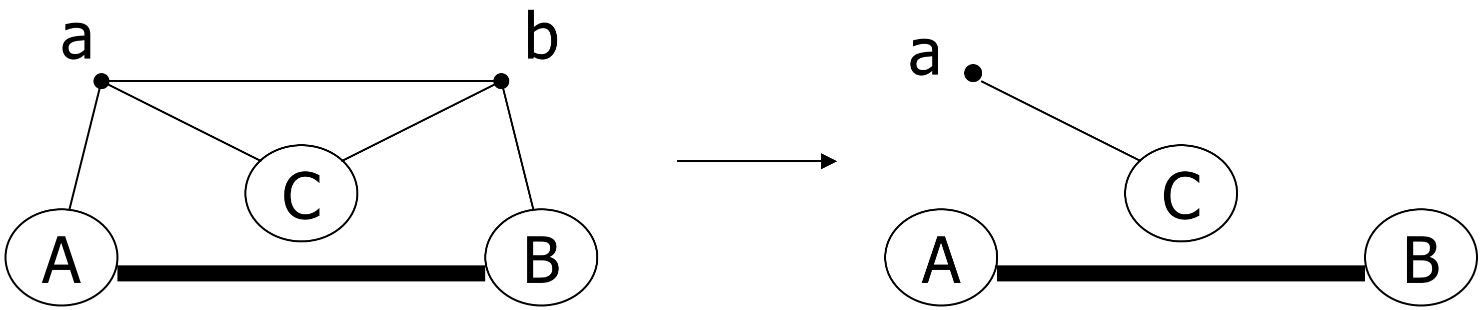

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Hertz, Alain; de Werra, Dominique A magnetic procedure for the stability number. *Graphs Combin.* 25(2009), 707–716.

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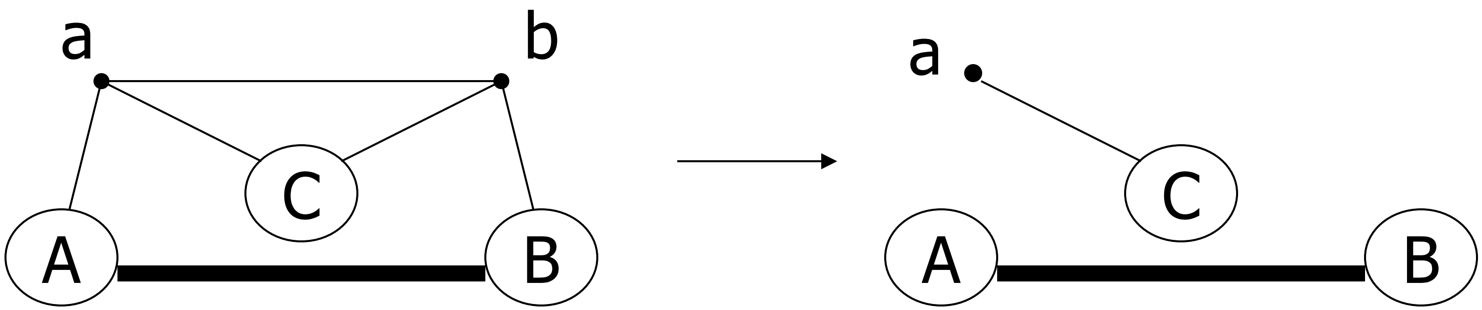

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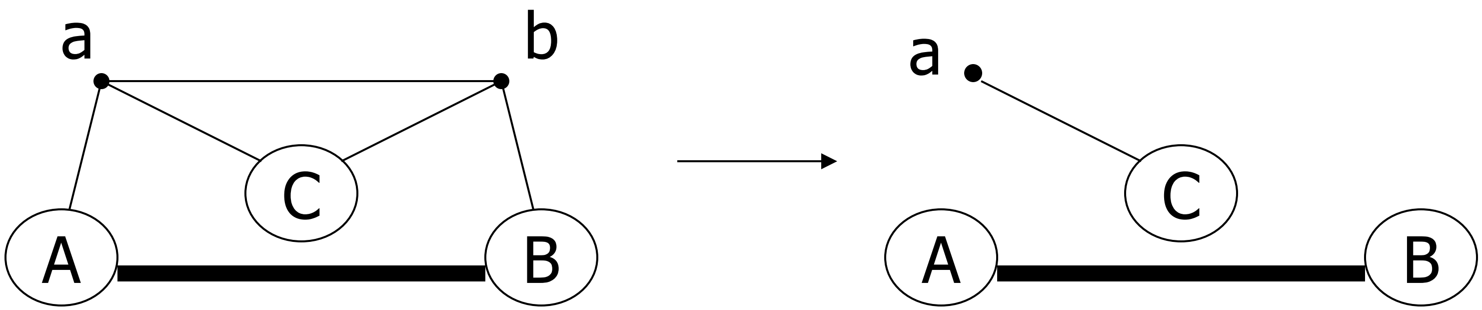

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Magnet generalizes the neighbourhood reduction and
the even pair contraction applied to the complement of the graph

Pseudo-Boolean optimization

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Definition. A *pseudo-Boolean function* $f(x_1, x_2, \ldots, x_n)$ is a real-valued function with Boolean variables.

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Pseudo-Boolean maximization

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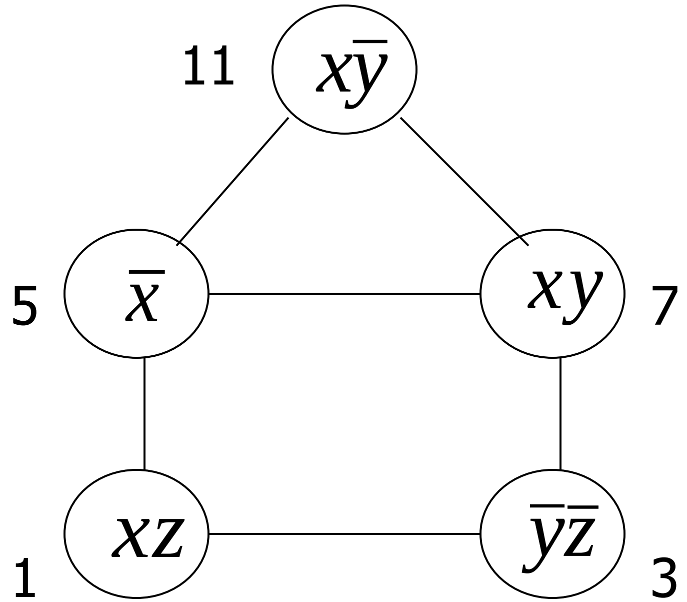
$$xz + 5\bar{x} + 11x\bar{y} + 7xy + 3\bar{y}\bar{z}$$ posiform

Conflict Graph



$$xz + 5\overline{x} + 11x\overline{y} + 7xy + 3\overline{y}\overline{z}$$

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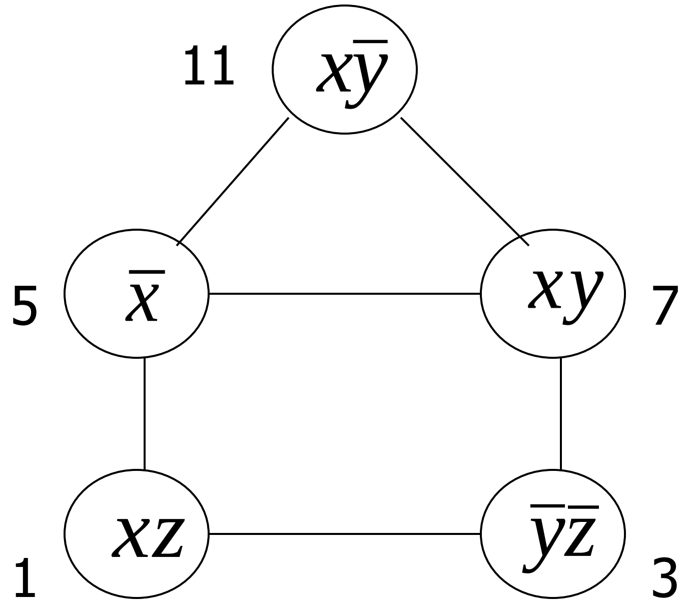

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The weight of a maximum independent set of the conflict graph coincides with the maximum of the posiform

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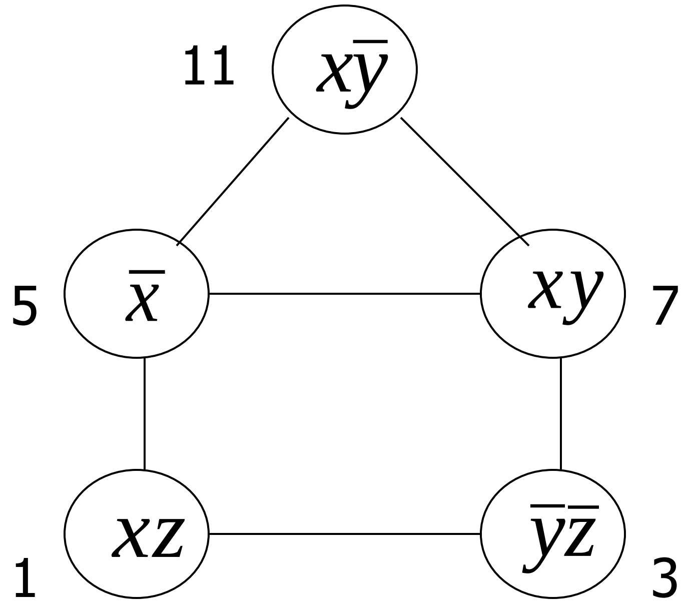

X=1
y=0
z=0

$$xz + 5\bar{x} + 11x\bar{y} + 7xy + 3\bar{y}\bar{z}$$

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Pseudo-Boolean identities and transformations of graphs that preserve the independence number

$$xy + \bar{x}y = y \qquad \longrightarrow \qquad \text{Magnet reduction}$$

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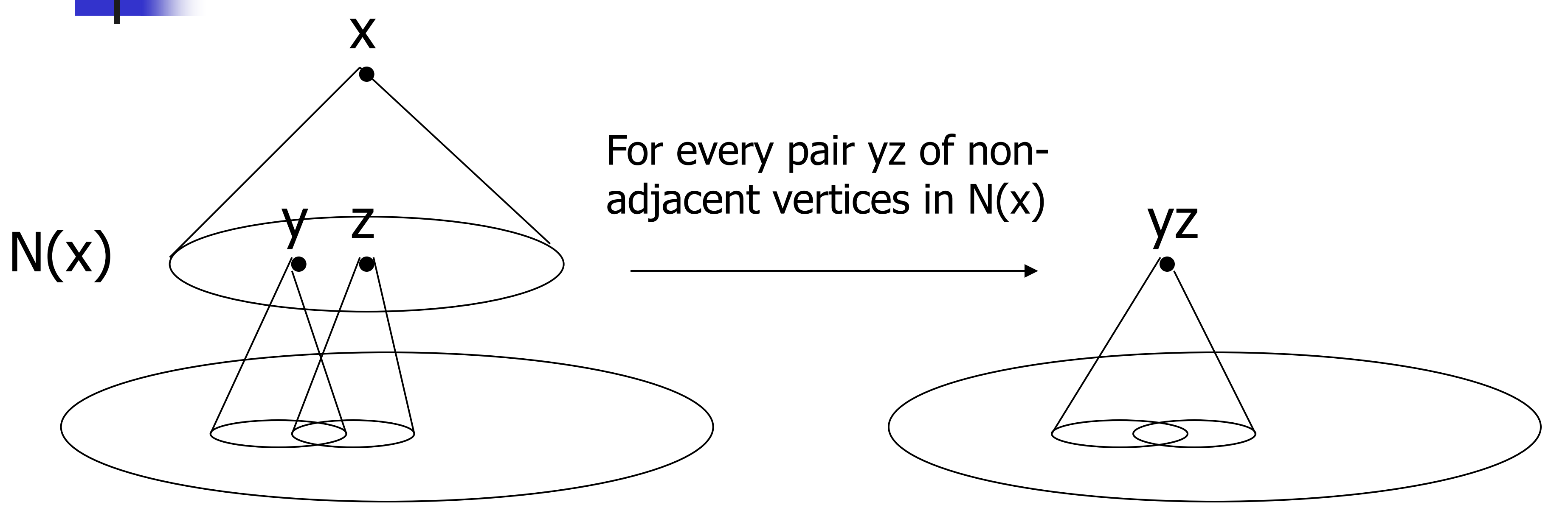
Boolean arguments $\longrightarrow$ Struction (STability RedUCTION)

Ebenegger, Ch.; Hammer, P. L.; de Werra, D. Pseudo-Boolean functions and stability of graphs. Annals of Discrete Math. 19 (1984) 83-97.

Struction



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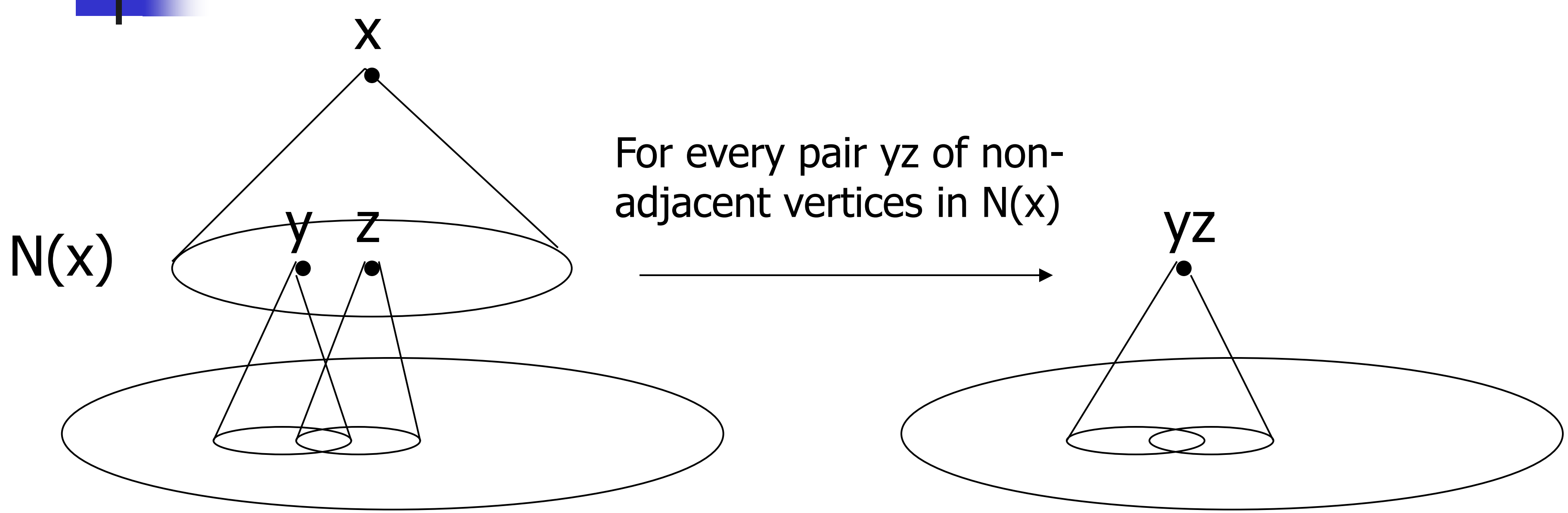
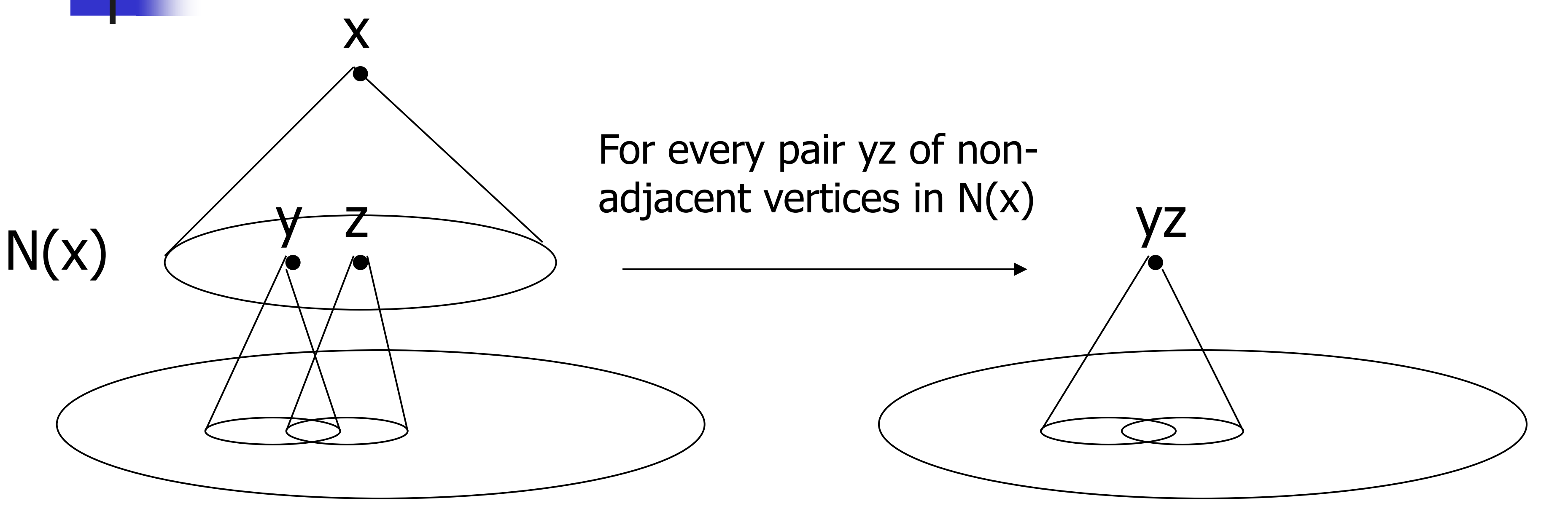


$$\alpha(G) \quad = \quad \alpha(G') + 1$$

Struction



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Struction generalizes vertex folding

Detour: Struction vs Resolution

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Davis, Martin; Putnam, Hilary A computing procedure for quantification theory. *J. Assoc. Comput. Mach.* **7** 1960 201–215.

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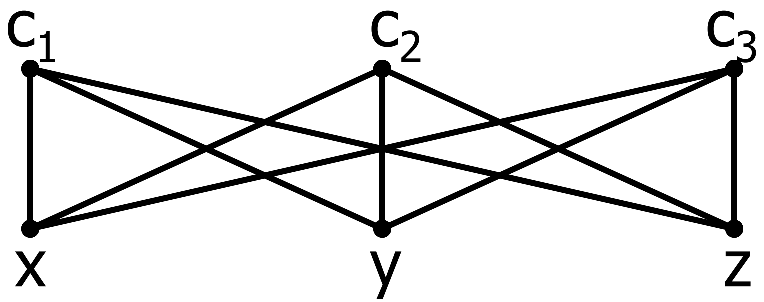
$$(x \vee y \vee \bar{z})(\bar{x} \vee y \vee z)(x \vee \bar{y} \vee z) \qquad F$$



incidence graph G_F

Detour: Struction vs Resolution

Planar SAT is NP-complete

D. Lichtenstein, Planar formulae and their uses, SIAM J. Comput. 11 (1982) 329-343.

Detour: Struction vs Resolution

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If all vertices of G_F have degree at most 3, the problem is NP-complete	C.A. Tovey, A simplified NP-complete satisfiability problem, Discrete Applied Mathematics, 8 (1984) 85-89.

Detour: Struction vs Resolution

$$(x \vee R)(\bar{x} \vee S) \longrightarrow (R \vee S)$$

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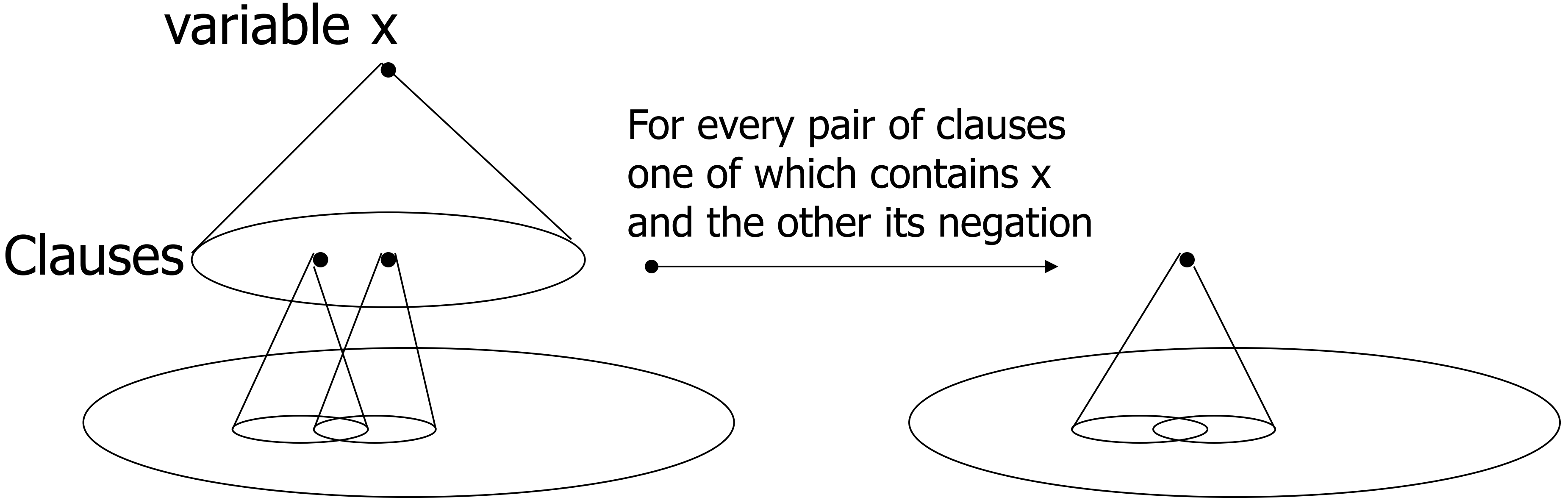

Struction revisited

Alexe, Gabriela; Hammer, Peter L.; Lozin, Vadim V.; de Werra, Dominique

Struction revisited. *Discrete Appl. Math.* 132 (2003), 27–46.

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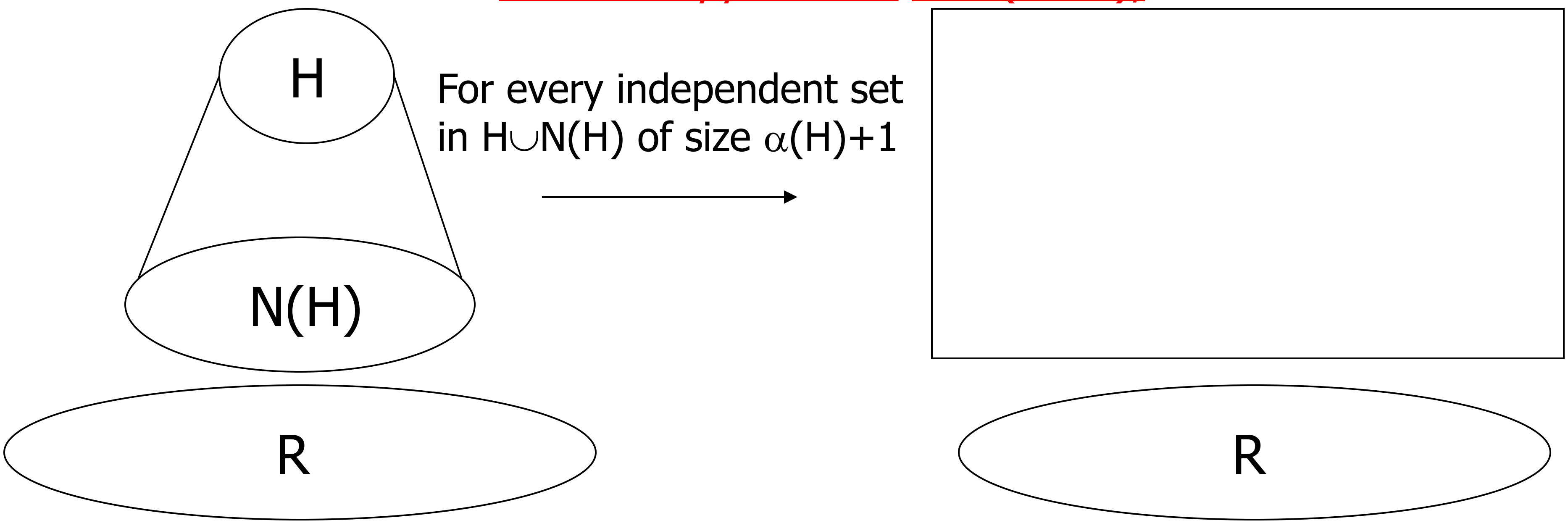

$\alpha(G) = \alpha(G') + \alpha(H)$

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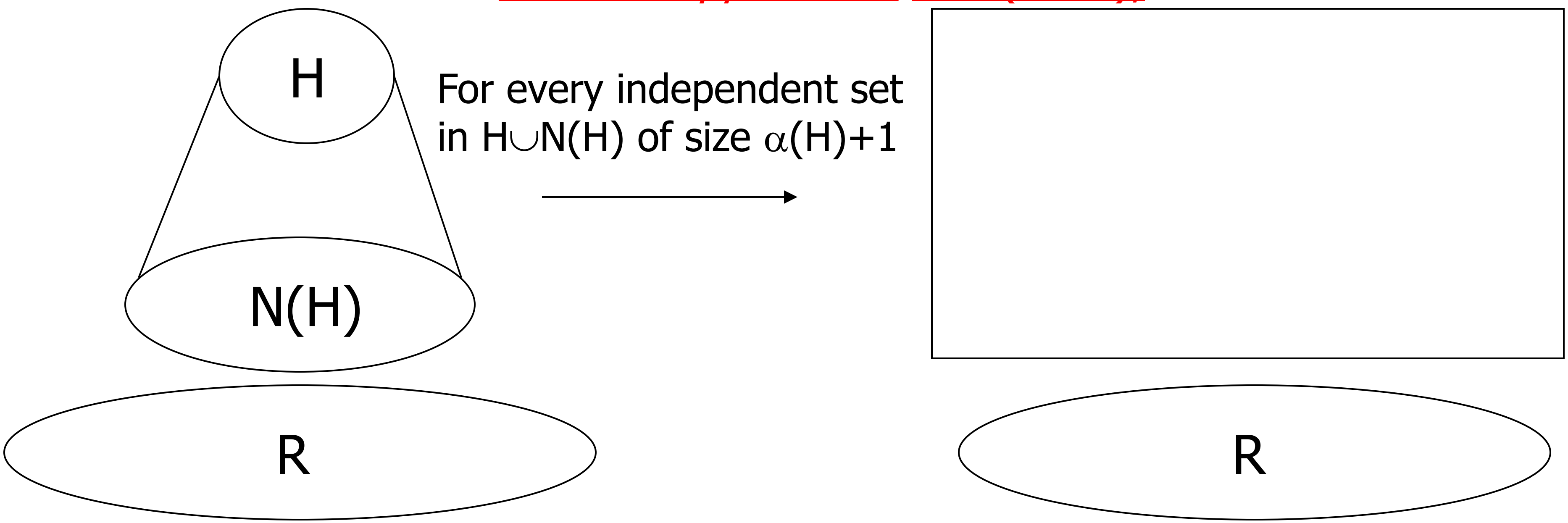

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Total struction $\supset$ Crown reduction

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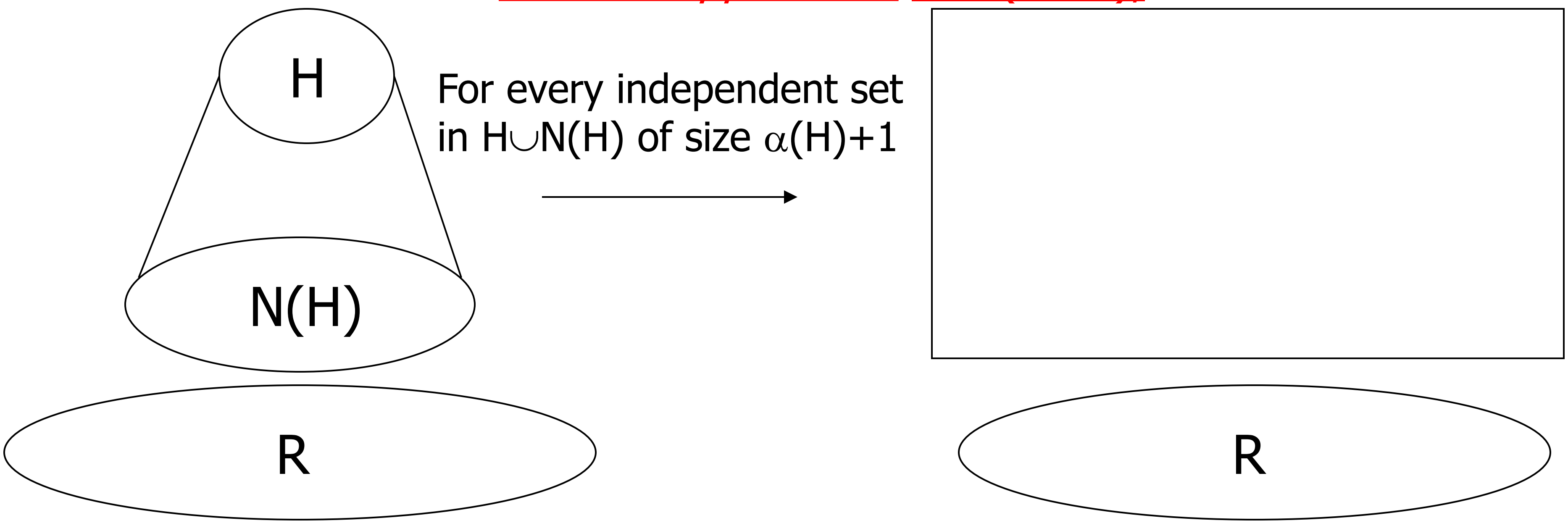

$$\alpha(G) \quad = \quad \alpha(G') + \alpha(H)$$

Total struction $\supset$ Crown reduction

Total struction + magnet $\supset$ cycle shrinking

From independent sets back to matchings

Is it possible to solve the maximum matching problem in polynomial time by means of graph transformations?

Thank you